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ERRATA

P. 73, lines 28–32: For "the series diverges, ..., $R(y) > \mu_2$." read "the series diverges, or converges without S_y being bounded, for $R(x) < \lambda_1$, $R(y) < \lambda_2$; converges conditionally with S_y bounded for $\lambda_1 < R(x) < \mu_1$, $\lambda_2 < R(y) < \mu_2$; and converges absolutely for $R(x) > \mu_1$, $R(y) > \mu_2$."

P. 78, line 4: For "x" and "y" read "x_0" and "y_0".

P. 116, line 4: For "$\mathfrak{J}$" read "$\mathfrak{S}$".

THE ALGEBRA OF RECURRING SERIES.*

BY MORGAN WARD.

1. Introduction. It is well known that if the function

$$F(x) \equiv x^3 - Px^2 + Qx - R, \qquad R \neq 0 \tag{1.1}$$

is irreducible in the field $\mathfrak{F}$ of its coefficients, then the properties of those solutions of the linear difference equation

$$\Omega_{n+3} = P\Omega_{n+2} - Q\Omega_{n+1} + R\Omega_n \tag{1.2}$$

which lie in $\mathfrak{F}$ are ultimately based on the algebra of the field $\mathfrak{F}(\alpha)$ obtained by adjoining to $\mathfrak{F}$ a root α of $F(x) = 0$.[1]

The object of this paper is to develop a general method for obtaining formal properties of the solutions of (1.2) from simple algebraic identities in $\mathfrak{F}(\alpha)$. The process is as follows:

We set up a one-to-one correspondence between the field $\mathfrak{F}(\alpha)$ and a certain class of square matrices of order three with elements lying in $\mathfrak{F}$. We then group these matrices into sets which are particular solutions of a matric difference equation of order one. Finally, we associate with each set a number of particular solutions of the scalar difference equation (1.2). Our method then consists of translating identities in $\mathfrak{F}(\alpha)$ into identities between matrices, and these in turn into relations between solutions of (1.2).[2] The treatment is simple and direct, and leads to a number of interesting formulas.[2]

The method may easily be extended to a difference equation of any order whose characteristic function is irreducible. We confine ourselves to the case of a third order equation, both for its interest in view of Lucas' claim to have discovered a remarkable connection between (1.2) and the theory of the elliptic functions,[3] and for simplicity of notation.

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* Received January 7, 1930.

[1] See for example, Bell, Tohoku Mathematical Journal, vol. 24, Numbers 1, 2 (1924), page 168. This paper gives a concise exposition of the algebraic theory of (1.2). We shall refer to it as "Bell", giving page reference. For the elementary theory of the linear difference equation of order r, see Bachmann, Niedere Zahlentheorie. The equation of order three is treated with considerable detail by Draeger, Thesis, Jena 1919. For other references, see Dickson's History, vol. I.

[2] See Section 5.

[3] In this connection, see Bell, p. 168.

2. **Basic definitions.** The most general solution of the difference equation (1.2) lying in the field $\mathfrak{F}$ of its coefficients is given by

$$(2.1)\quad \begin{aligned}\Omega_n = (K_0 + K_1\alpha + K_2\alpha^2)\alpha^n + (K_0 + K_1\beta + K_2\beta^2)\beta^n \\ + (K_0 + K_1\gamma + K_2\gamma^2)\gamma^n, \qquad (n = 0, \pm 1, \cdots).\end{aligned}$$

Here α, β, γ are the roots of the irreducible equation $F(x) = 0$, and K_0, K_1, K_2 are arbitrary elements of $\mathfrak{F}$.

If

$$\Omega_n = A_n, \qquad (n = 0, \pm 1, \cdots)$$

is any particular solution of (1.2) obtained by giving the constants K_0, K_1, K_2 in (2.1) definite values, A_0, A_1, A_2 are called the *initial values* of the *sequence* $(A)_n$. We write

$$(A)_n \sim [A_0, A_1, A_2].$$

Any sequence $(A)_n$ is completely determined as soon as we have specified its initial values.

There are four particular solutions of (1.2) of sufficient importance to have a special notation; we shall invariably write $(X)_n$, $(Y)_n$, $(Z)_n$, $(S)_n$ for the sequences defined by[4]

$$(2.2)\quad \begin{aligned}&(X)_n \sim [1, 0, 0]; \qquad &&(Y)_n \sim [0, 1, 0]; \\ &(Z)_n \sim [0, 0, 1]; &&(S)_n \sim [3, P, P^2 - 2Q].\end{aligned}$$

Finally, we have the well known formulas

$$(2.3)\quad \begin{aligned}&P = \alpha + \beta + \gamma, \qquad &&Q = \alpha\beta + \beta\gamma + \gamma\alpha, \qquad &&R = \alpha\beta\gamma; \\ &S_n = \alpha^n + \beta^n + \gamma^n, &&R^n S_{-n} = \alpha^n\beta^n + \beta^n\gamma^n + \gamma^n\alpha^n.\end{aligned}$$

3. **Introduction of matrices.** Let $\boldsymbol{M}_n$ denote the square matrix of order three

$$(3.1)\quad \boldsymbol{M}_n = \begin{pmatrix} X_n, & Y_n, & Z_n \\ X_{n+1}, & Y_{n+1}, & Z_{n+1} \\ X_{n+2}, & Y_{n+2}, & Z_{n+2} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots).$$

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[4] For properties of the first three, see Bell's paper. The solution

$$(Z)_n : 0, 0, 1, P, P^2 - Q, P^3 - 2PQ + R, \cdots$$

is important in Combinatory Analysis; in fact, Z_{n+2}, $n > 0$ is the homogeneous product sum of α, β, γ of degree n. See MacMahon, Combinatory Analysis, Cambridge, (1915), vol. I, p. 3. S_n is the familiar sum of the nth powers of the roots of $F(x) = 0$.

Then by direct calculation from (2.2) and (1.2), we find that

$$(3.2)\quad \boldsymbol{M}_0=\begin{pmatrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 1\end{pmatrix}, \boldsymbol{M}_1=\begin{pmatrix}0, & 1, & 0\\ 0, & 0, & 1\\ R, & -Q, & P\end{pmatrix}, \boldsymbol{M}_2=\begin{pmatrix}0, & 0, & 1\\ R, & -Q, & P\\ PR, & R-PQ, & P^2-Q\end{pmatrix}.$$

Thus $\boldsymbol{M}_0=\boldsymbol{I}$, the identity matrix. We shall often write $\boldsymbol{M}$ for $\boldsymbol{M}_1$, omitting the subscript one.

The following properties of the matrices (3.1) are easily proved by induction:

$$(3.3)\quad \begin{aligned}\boldsymbol{M}_{n+1} &= \boldsymbol{M}\cdot\boldsymbol{M}_n = \boldsymbol{M}^{n+1},\\ \boldsymbol{M}_n\cdot\boldsymbol{M}_m &= \boldsymbol{M}_m\cdot\boldsymbol{M}_n = \boldsymbol{M}_{m+n},\\ \boldsymbol{M}_{n+3} &= P\boldsymbol{M}_{n+2}-Q\boldsymbol{M}_{n+1}+R\boldsymbol{M}_n,\\ \boldsymbol{M}_n &= X_n\boldsymbol{M}_0+Y_n\boldsymbol{M}_1+Z_n\boldsymbol{M}_2.\end{aligned}\qquad (m, n=0, \pm 1, \cdots).$$

The first of these formulas shows that $\boldsymbol{M}_n$ is a particular solution of the matric difference equation of order one

$$(3.4)\qquad \boldsymbol{\Omega}_{n+1}=\boldsymbol{M}\cdot\boldsymbol{\Omega}_n$$

and the third formula shows that $\boldsymbol{M}_n$ is a particular solution of (1.2). We shall hereafter refer to the matrices (3.1) as the *sequence* $(\boldsymbol{M})_n$.

Combining the second and fourth of the formulas (3.3) gives the more general formula

$$\boldsymbol{M}_{n+m}=X_n\boldsymbol{M}_m+Y_n\boldsymbol{M}_{m+1}+Z_n\boldsymbol{M}_{m+2},\quad (m, n=0, \pm 1, \cdots).$$

By transforming $\boldsymbol{M}$ to the diagonal form and applying (3.3), (2.3), we can prove[5]

THEOREM 3.1. *The characteristic function of the matrix* $\boldsymbol{M}_n$ *is* $\lambda^3-S_n\lambda^2+R^nS_{-n}\lambda-R^n$.

The most general solution of (3.4) is

$$\boldsymbol{\Omega}_n=\boldsymbol{M}_n\cdot\boldsymbol{\Omega}_0,\qquad (n=0, \pm 1, \cdots)$$

where the elements of the matrix $\boldsymbol{\Omega}_0$ are arbitrary. Let

$$\boldsymbol{\Omega}_n=\boldsymbol{P}_n$$

[5] We may note in passing a useful formula immediately obtainable from Theorem 3.1 and (3.1); namely,

$$S_n=X_n+Y_{n+1}+Z_{n+2},\qquad (n=0, \pm 1, \cdots).$$

be a particular solution obtained by letting

$$(3.41)\qquad \boldsymbol{\Omega}_0 = \boldsymbol{P}_0 = \begin{pmatrix} U_0, & V_0, & W_0 \\ U_1, & V_1, & W_1 \\ U_2, & V_2, & W_2 \end{pmatrix},$$

where $U_0, \cdots, W_2$ are fixed elements of $\mathfrak{F}$. Then from (3.3),

$$(3.5)\qquad \begin{aligned} \boldsymbol{P}_{m+n} &= \boldsymbol{M}_n \cdot \boldsymbol{P}_m, \\ \boldsymbol{P}_{n+3} &= P\boldsymbol{P}_{n+2} - Q\boldsymbol{P}_{n+1} + R\boldsymbol{P}_n, \end{aligned} \qquad (m, n = 0, \pm 1, \cdots).$$

From (3.5), we obtain the following theorem:

THEOREM 3.2. *If the sequence of matrices* $(\boldsymbol{P})_n$ *is a particular solution of the matric difference equation* (3.4), *where the value of* $\boldsymbol{P}_0$ *is given by* (3.41), *then*

$$\boldsymbol{P}_n = \begin{pmatrix} U_n, & V_n, & W_n \\ U_{n+1}, & V_{n+1}, & W_{n+1} \\ U_{n+2}, & V_{n+2}, & W_{n+2} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots),$$

where $(U)_n \sim [U_0, U_1, U_2]$, $(V)_n \sim [V_0, V_1, V_2]$, $(W)_n \sim [W_0, W_1, W_2]$ *are particular solutions of the scalar difference equation* (1.2).

It is easily shown that the converse of this theorem is also true.

4. **Associated fields.** We shall now establish an isomorphism between $\mathfrak{F}(\alpha)$ and a certain class of matrices with elements in $\mathfrak{F}$.

THEOREM 4.1. *The class* $\mathfrak{M}$ *of all matrices of the form*

$$\boldsymbol{P} = U\boldsymbol{I} + V\boldsymbol{M} + W\boldsymbol{M}^2$$

where U, V, W *are any elements of* $\mathfrak{F}$ *forms a field which is simply isomorphic with the field* $\mathfrak{F}(\alpha)$ *obtained by adjoining a root* α *of* $F(x) = 0$ *to* $\mathfrak{F}$.

Proof. It is clear from formulas (3.2), (3.3), that any matrix $\boldsymbol{P}$ of $\mathfrak{M}$ can vanish when and only when U, V and W vanish.

$\mathfrak{M}$ is obviously closed under addition and subtraction; by (3.3), $\boldsymbol{M}^3 = P\boldsymbol{M}^2 - Q\boldsymbol{M} + R\boldsymbol{I}$; consequently, $\mathfrak{M}$ is also closed under multiplication. Furthermore, multiplication is commutative, and distributive with respect to addition.

Any element π of the field $\mathfrak{F}(\alpha)$ may be put in the unique canonical form

$$\pi = U + V\alpha + W\alpha^2$$

where U, V, W are elements of $\mathfrak{F}$. Set $\mathfrak{M}$ and $\mathfrak{F}(\alpha)$ into one-to-one correspondence by pairing the elements π and $\boldsymbol{P}$ for which U, V, W have the same values; we write in this case $\pi \sim \boldsymbol{P}$.

Then if $\pi_1 \sim \boldsymbol{P}_1$, $\pi_2 \sim \boldsymbol{P}_2$, $\pi_3 \sim \boldsymbol{P}_3$, it is easily verified that

$$\pi_1 \pm \pi_2 \sim \boldsymbol{P}_1 \pm \boldsymbol{P}_2;\ \pi_1 \cdot \pi_2 \sim \boldsymbol{P}_1 \cdot \boldsymbol{P}_2;\ \pi_1 (\pi_2 \pm \pi_3) \sim \boldsymbol{P}_1 (\boldsymbol{P}_2 \pm \boldsymbol{P}_3).$$

Furthermore, if $\pi\pi' = 1$, $\pi \sim \boldsymbol{P}$, $\pi' \sim \boldsymbol{P}'$, then $\boldsymbol{P} \cdot \boldsymbol{P}' = \boldsymbol{I}$.

Hence $\mathfrak{M}$ forms a field simply isomorphic with $\mathfrak{F}(\alpha)$.

As a corollary to this theorem, we have

THEOREM 4.11. *The characteristic equation of any matrix* $\boldsymbol{P}$ *of* $\mathfrak{M}$ *is the same as the equation which the corresponding element* π *of* $\mathfrak{F}(\alpha)$ *satisfies in* $\mathfrak{F}$.

We shall use the notations Det $\boldsymbol{P}$ and Adj $\boldsymbol{P}$ for the determinant and adjoint of any matrix $\boldsymbol{P}$, and $N(\pi)$ for the norm of any number π of $\mathfrak{F}(\alpha)$. It is easily seen from Theorem 4.11 that

(4.12) If $\boldsymbol{P} \sim \pi$, then $\operatorname{Det} \boldsymbol{P} = N(\pi)$;

(4.13) If $\boldsymbol{P} \sim \pi \neq 0$, then $\operatorname{Adj} \boldsymbol{P} \sim N(\pi)/\pi$.

THEOREM 4.2. *The necessary and sufficient condition that any matrix of order three with elements in* $\mathfrak{F}$ *be commutative with* $\boldsymbol{M}$ *is that it lies in the field* $\mathfrak{M}$.

Proof. The sufficiency of the condition follows from Theorem 4.1. To establish the necessity, suppose that $\boldsymbol{L}$ is a matrix of order three over $\mathfrak{F}$ commutative with $\boldsymbol{M}$. Then

$$\boldsymbol{L} \cdot \boldsymbol{M} = \boldsymbol{M} \cdot \boldsymbol{L}; \quad \boldsymbol{L} \cdot \boldsymbol{M}^2 = \boldsymbol{M}^2 \cdot \boldsymbol{L}. \tag{4.2}$$

There exists a non-singular matrix $\boldsymbol{T}$ transforming $\boldsymbol{M}$ into the diagonal form $\boldsymbol{M}^*$. By (4.2), $\boldsymbol{T}^{-1} \cdot \boldsymbol{L} \cdot \boldsymbol{T} = \boldsymbol{L}^*$ must also be in the diagonal form. Let α, β, γ; α', β', γ' be the diagonal elements in $\boldsymbol{M}^*$; $\boldsymbol{L}^*$, and consider the traces of $\boldsymbol{L}^*$, $\boldsymbol{M}^* \cdot \boldsymbol{L}^*$, $(\boldsymbol{M}^*)^2 \cdot \boldsymbol{L}^*$. They are the same as the traces of $\boldsymbol{L}$, $\boldsymbol{M} \cdot \boldsymbol{L}$, $\boldsymbol{M}^2 \cdot \boldsymbol{L}$. Hence

$$\alpha' + \beta' + \gamma' = I, \quad \alpha\alpha' + \beta\beta' + \gamma\gamma' = J, \quad \alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma' = K,$$

where I, J, K are elements of $\mathfrak{F}$. Solving these equations for α', β', γ' we find that

$$\alpha' = U + V\alpha + W\alpha^2, \quad \beta' = U + V\beta + W\beta^2, \quad \gamma' = U + V\gamma + W\gamma^2$$

where U, V, W are elements of $\mathfrak{F}$. Thus

$$\boldsymbol{L}^* = U\boldsymbol{I} + V\boldsymbol{M}^* + W\boldsymbol{M}^{*2},$$
$$\boldsymbol{L} = \boldsymbol{T} \cdot \boldsymbol{L}^* \cdot \boldsymbol{T}^{-1} = U\boldsymbol{I} + V\boldsymbol{M} + W\boldsymbol{M}^2.$$

THEOREM 4.3. *Let $(\boldsymbol{P})_n$ denote the sequence of matrices defined in Theorem 3.2. Then a necessary and sufficient condition that $(\boldsymbol{P})_n$ should ie in $\mathfrak{M}$ is that the sequences $(U)_n$, $(V)_n$, $(W)_n$ be connected by the relations*

$$\begin{aligned} V_n &= W_{n+1} - P\,W_n, \\ U_n &= W_{n+2} - P\,W_{n+1} + Q\,W_n = R\,W_{n-1}, \end{aligned} \qquad (n = 0, \pm 1, \cdots). \tag{4.3}$$

Proof. We easily find that

$$\boldsymbol{P}_n \cdot \boldsymbol{M} = \boldsymbol{M} \cdot \boldsymbol{P}_n$$

when and only when the relations (4.3) hold. The result now follows from Theorem 4.2.

Let $(\boldsymbol{P})_n$ now denote a sequence of matrices whose elements satisfy the relations[6] (4.3). Then

$$\boldsymbol{P}_n = I\boldsymbol{M}_0 + J\boldsymbol{M}_1 + K\boldsymbol{M}_2$$

where I, J, K lie in $\mathfrak{F}$. By comparing the elements in the first row of both sides of this identity, we find from (3.2) that

$$I = U_n, \qquad J = V_n, \qquad K = W_n$$

so that

$$\boldsymbol{P}_n = U_n\boldsymbol{M}_0 + V_n\boldsymbol{M}_1 + W_n\boldsymbol{M}_2, \qquad (n = 0, \pm 1, \cdots).$$

5. **Derivation of formulas.** We are now in a position to illustrate the method of translating identities in $\mathfrak{F}(\alpha)$ into relations between solutions of (1.2). With the notation of Theorem 4.1, we write $\pi \sim \boldsymbol{P}_0$ for

$$\pi = U_0 + V_0\,\alpha + W_0\,\alpha^2, \qquad \boldsymbol{P}_0 = U_0\boldsymbol{M}_0 + V_0\boldsymbol{M}_1 + W_0\boldsymbol{M}_2.$$

In particular, $\alpha \sim \boldsymbol{M}_1$, and by Theorem 4.1, (3.11), Theorem 3.3,

$$\alpha^n \sim \boldsymbol{M}_n, \qquad \alpha^n \cdot \pi \sim \boldsymbol{P}_n \tag{5.1}$$

where the elements of $\boldsymbol{P}_n$ satisfy the conditions (4.3).

Let us start with the following trivial identities in $\mathfrak{F}(\alpha)$.

$$\begin{aligned} &\text{I.} \quad \alpha^{n+m} \cdot \pi = \pi \cdot \alpha^{n+m} = (\alpha^n \cdot \pi)\,\alpha^m = \alpha^m(\alpha^n \cdot \pi), \\ &\text{II.} \quad (\pi \cdot \alpha^{n+m})\,\pi = \pi(\alpha^{n+m} \cdot \pi) = (\alpha^m \cdot \pi) \cdot (\alpha^n \cdot \pi). \end{aligned}$$

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[6] It is perhaps worth noting that on account of the linearity of (1.2), (4.3) will hold for all values of n if it holds for $n = 0, 1, 2$; i. e. $(\boldsymbol{P})_n$ lies in $\mathfrak{M}$ if $\boldsymbol{P}_0$ lies in $\mathfrak{M}$.

The corresponding matrix identities in $\mathfrak{M}$ are

$$\text{I}'.\quad \boldsymbol{P}_{n+m} = \boldsymbol{P}_0\cdot\boldsymbol{M}_{n+m} = \boldsymbol{P}_n\cdot\boldsymbol{M}_m = \boldsymbol{M}_m\cdot\boldsymbol{P}_n,$$
$$\text{II}'.\quad \boldsymbol{P}_{n+m}\cdot\boldsymbol{P}_0 = \boldsymbol{P}_0\cdot\boldsymbol{P}_{m+n} = \boldsymbol{P}_m\cdot\boldsymbol{P}_n.$$

By equating corresponding elements on both sides of I′ and II′, we obtain a number of formulas involving $(U)_n$, $(V)_n$, $(W)_n$, $(X)_n$, $(Y)_n$, $(Z)_n$; for instance, from I′ we obtain

$$\begin{aligned} U_{n+m} &= U_0X_{n+m}+V_0X_{n+m+1}+W_0X_{n+m+2} \\ &= U_nX_m+V_nX_{m+1}+W_nX_{m+2} \\ &= X_mU_n+Y_mU_{n+1}+Z_mU_{n+2}. \end{aligned} \tag{5.2}$$

From II′, we obtain

$$\begin{aligned} U_{n+m}U_0+V_{n+m}U_1+W_{n+m}U_2 &= U_0U_{m+n}+V_0U_{m+n+1}+W_0U_{m+n+2} \\ &= U_mU_n+V_mU_{n+1}+W_mU_{n+2}. \end{aligned}$$

If we introduce the number

$$\pi' = N(\pi)/\pi = U_0'+V_0'\alpha+W_0'\alpha^2,$$

we obtain another class of formulas. For by (4.13),

$$\text{Adj}\,\boldsymbol{P} = \boldsymbol{P}_0' = U_0'\boldsymbol{M}_0+V_0'\boldsymbol{M}_1+W_0'\boldsymbol{M}_2,$$

and if we let $\boldsymbol{P}_n' = \boldsymbol{M}_n\cdot\boldsymbol{P}_0'$, we find that

$$\text{Adj}\,\boldsymbol{P}_n = R^n\boldsymbol{P}_n'.$$

Hence,

$$V_{n+1}W_{n+2}-W_{n+1}V_{n+2} = R^nU_{-n}', \qquad W_{n+2}U_{n+1}-U_{n+2}W_{n+1} = R^nV_n',$$

and so on.

The identity

$$\begin{aligned} \text{III}.\quad (\alpha^m\cdot\pi')\cdot(\alpha^n\cdot\pi) &= (\alpha^m\cdot\pi)\cdot(\alpha^n\cdot\pi') = \pi\cdot(\alpha^{m+n}\cdot\pi') \\ &= \pi'\cdot(\alpha^{n+m}\cdot\pi) \qquad = N(\pi)\,\alpha^{n+m} \end{aligned}$$

gives us

$$\text{III}'.\quad \boldsymbol{P}_m'\cdot\boldsymbol{P}_n = \boldsymbol{P}_m\cdot\boldsymbol{P}_n' = \boldsymbol{P}_0\cdot\boldsymbol{P}_{m+n}' = \boldsymbol{P}_{m+n}\cdot\boldsymbol{P}_0' = N(\pi)\boldsymbol{M}_{n+m}.$$

From III′, we obtain formulas of the type

$$\begin{aligned} U_m'U_n+V_m'U_{n+1}+W_m'U_{n+2} &= U_mU_n'+V_mU_{n+1}'+W_mU_{n+2}' \\ = U_0U_{m+n}'+V_0U_{m+n+1}'+W_0U_{m+n+2}' &= U_{m+n}U_0'+V_{m+n}U_1'+W_{m+n}U_2' \\ &= N(\pi)X_{n+m}, \quad (m, n = 0, \pm1, \cdots). \end{aligned}$$

6. Extension of method. We shall conclude by extending the method so as to apply to an important class of matrices not lying in $\mathfrak{M}$.

Let $(\boldsymbol{S})_n$ denote the sequence of matrices

$$\boldsymbol{S}_n = \begin{pmatrix} S_n, & S_{n+1}, & S_{n+2} \\ S_{n+1}, & S_{n+2}, & S_{n+3} \\ S_{n+2}, & S_{n+3}, & S_{n+4} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots)$$

where S_n is given by (2.3).

By the converse to Theorem 3.2, $\boldsymbol{S}_m = \boldsymbol{M}_m \cdot \boldsymbol{S}_0$, so that

$$\text{(6.1)} \qquad \boldsymbol{M}_n \cdot \boldsymbol{S}_m = \boldsymbol{S}_{m+n}.$$

However, $\boldsymbol{M} \cdot \boldsymbol{S}_0 \neq \boldsymbol{S}_0 \cdot \boldsymbol{M}$, so that $(\boldsymbol{S})_n$ does not lie in $\mathfrak{M}$.

Define a new matrix $\boldsymbol{T}_n$ by

$$\text{(6.2)} \qquad \boldsymbol{T}_n = \boldsymbol{P}_0 \cdot \boldsymbol{S}_n.$$

If we write T_n for

$$U_0 S_n + V_0 S_{n+1} + W_0 S_{n+2}, \qquad (n = 0, \pm 1, \cdots)$$

we find from (6.2) that

$$\boldsymbol{T}_n = \begin{pmatrix} T_n, & T_{n+1}, & T_{n+2} \\ T_{n+1}, & T_{n+2}, & T_{n+3} \\ T_{n+2}, & T_{n+3}, & T_{n+4} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots).$$

From (6.2) and (6.1)

$$\boldsymbol{T}_{m+n} = \boldsymbol{P}_0 \cdot \boldsymbol{S}_{m+n} = \boldsymbol{P}_0 \cdot \boldsymbol{M}_n \cdot \boldsymbol{S}_m = \boldsymbol{P}_n \cdot \boldsymbol{S}_m,$$

giving the useful formula

$$\text{(6.3)} \qquad T_{m+n} = U_n S_m + V_n S_{m+1} + W_n S_{m+2}.$$

Formula (6.3) applies to any sequence satisfying (1.2). For if

$$(T)_n \sim [T_0, T_1, T_2],$$

the three equations

$$T_i = U_0 S_i + V_0 S_{i+1} + W_0 S_{i+2}, \qquad (i = 0, 1, 2)$$

will determine U_0, V_0, W_0. The six remaining elements of $\boldsymbol{P}_0$, $U_1, \cdots, W_2$ are then completely determined by the relations (4.3), and the demonstration given applies.

There is an interesting consequence of formula (6.3). We may use (4.3) to express (6.3) in the form

$$\text{(6.31)} \quad T_{m+n} = (W_{n+2} - PW_{n+1} + QW_n) S_m + (W_{n+1} - PW_n) S_{m+1} + W_n S_{m+2}.$$

On interchanging m and n in (6.31) and rearranging the terms, we obtain

(6.32) $$T_{m+n} = (S_{n+2} - PS_{n+1} + QS_n)W_m + (S_{n+1} - PS_n)W_{m+1} + S_n W_{m+2}.$$

Since (6.32) may be derived from (6.31) by simply interchanging the S and the W, we have a parallelism between the expression for T_{m+n} in terms of S_m, S_{m+1}, S_{m+2} and in terms of W_m, W_{m+1}, W_{m+2}. There is a similar parallelism between the expression for T_{m+n} in terms of T_m, T_{m+1}, T_{m+2} and in terms of Z_m, Z_{m+1}, Z_{m+2}. For from (5.2), taking $(U)_n = (T)_n$,

$$T_{m+n} = X_n T_m + Y_n T_{m+1} + Z_n T_{m+2}.$$

On replacing X_n and Y_n by their expressions in terms of Z_n from (4.3),[7] we obtain two formulas analogous to (6.31) and (6.32); namely,[8]

$$T_{m+n} = (Z_{n+2} - PZ_{n+1} + QZ_n)T_m + (Z_{n+1} - PZ_n)T_{m+1} + Z_n T_{m+2},$$
$$T_{m+n} = (T_{n+2} - PT_{n+1} + QT_n)Z_m + (T_{n+1} - PT_n)Z_{m+1} + T_n Z_{m+2}.$$

[7] Bell, p. 173 formula (12). The matrix $\boldsymbol{M}_n$ is of course a special form of $\boldsymbol{P}_n$.

[8] If we take $(T)_n = (Z)_n$, $n = n+1$, $m = m-1$, the last two formulas become equation (34) in section 7 of Bell, p. 179.

A PROOF OF THE ASYMPTOTIC SERIES FOR $\log \Gamma(z)$ AND $\log \Gamma(z+a)$.*

By Charles H. Rowe.

The purpose of this note is to show how Stirling's series for $\log \Gamma(z)$ and the similar series for $\log \Gamma(z+a)$ can be established by reasoning which is of a very elementary nature and which uses only properties of the gamma-function that are almost immediate consequences of its definition as an infinite product.

1. We shall employ the usual notation $O(z^{-n})$ to denote any function $f(z)$ of the complex variable z such that $|z^n f(z)|$ is bounded when z tends to infinity along a specified curve. When the way in which z tends to infinity is restricted only by the requirement that z should remain in the interior of a certain region which extends to infinity, we shall agree to interpret the statement that $f(z) = O(z^{-n})$ as meaning that there exist positive constants K and R such that $|z^n f(z)| < K$ provided that z lies within the region and that $|z| > R$.

The statement that a function $F(z)$ admits the asymptotic expansion

$$a_1 z^{-1} + a_2 z^{-2} + \cdots$$

as z tends to infinity along a specified curve means that, for each fixed value of the integer n, $z^n R_n(z)$ tends to zero as z tends to infinity, where $R_n(z)$ is the difference between $F(z)$ and the sum of the first n terms of the expansion. We shall first show that, in order to prove the validity of an asymptotic expansion, it is sufficient to establish a less stringent inequality for $R_n(z)$. This will be clear from the following proposition:

Let z tend to infinity inside a certain region of the plane (which may as a particular case reduce to a curve), let $R_n(z)$ denote

$$F(z) - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n},$$

where $a_1, a_2, \cdots$ are constants, and let $\lambda(n)$ be any positive function of n that tends to infinity with n. Then, if $R_n(z) = O(z^{-\lambda(n)})$ for each value of the positive integer n, we may infer that $R_n(z) = O(z^{-n-1})$.

To prove this, let us take any value of n and find an integer m greater than n so that $\lambda(m) \geqq n+1$. Then we have

$$z^{n+1} R_n(z) = a_{n+1} + a_{n+2} z^{-1} + \cdots + a_m z^{-m+n+1} + z^{n+1} R_m(z),$$

* Received February 5, 1930.

and therefore, if $|z|$ exceeds unity,

$$|z^{n+1} R_n(z)| \leqq |a_{n+1} + a_{n+2} z^{-1} + \cdots + a_m z^{-m+n+1}| + |z^{\lambda(m)} R_m(z)|.$$

Since the right-hand member of this inequality is bounded when $|z|$ is large enough, it follows that $R_n(z) = O(z^{-n-1})$. If for a particular value of n it happens that a_{n+1} is zero, we may then conclude that $R_n(z) = O(z^{-n-r})$ where a_{n+r} is the first non-zero coefficient after a_n.

2. Stirling's series is usually established subject to the condition that z should remain in a region of the plane defined by inequalities of the form $-\pi + \delta \leqq \arg z \leqq \pi - \delta$, where δ is an arbitrarily small positive constant. We shall impose a somewhat less restrictive condition on the way in which z tends to infinity. Let D denote the region that remains when we remove from the plane all points $x + iy$ on the left of the imaginary axis at which $|y|^k < a|x|$, where k and a are two arbitrary positive constants. We shall see that Stirling's series is valid if z remains inside D no matter how large the constant k is. We may remark that when k is equal to unity the region D reduces to a region of the type that we mentioned first; and we may assume without loss of generality that $k \geqq 1$.

We shall now express in the form of a general theorem the essentials of the method that we are going to apply to the gamma-function:

If $F(z)$ is a function of the complex variable z such that the difference $F(z+1) - F(z)$ can be expanded in a convergent series of the form

$$F(z+1) - F(z) = a_2 z^{-2} + a_3 z^{-3} + \cdots \tag{1}$$

when $|z|$ is large enough, then $F(z)$ admits an asymptotic expansion of the form

$$F(z) = p(z) + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n} + O(z^{-n-1}) \tag{2}$$

which is valid in the region D, $p(z)$ being a periodic function of period unity.[1]

Before proving this theorem it will be convenient to establish the following lemma:

If z remains inside the region D, the inequality

$$\varphi(z+1) - \varphi(z) = O(z^{-n-2}),$$

where n is positive, implies that there exists a function $p(z)$ of which unity is a period such that

$$\varphi(z) = p(z) + O(z^{-n/k}).$$

[1] Of course, novelty is not claimed for this result, except perhaps for its extension to regions of the type D.

k being the number that we used in the definition of the region D.[2]

If the point z lies in D, so do all the points $z+1$, $z+2, \cdots$, and we can ensure that their distances from the origin all exceed an assigned limit by making $|z|$ large enough. It is thus clear that, if z is a sufficiently distant point of D, there is a constant K such that $|\varphi(z+r+1)-\varphi(z+r)| < K|z+r|^{-n-2}$ when r is any positive integer or zero. The series

$$\psi(z) = \sum_{r=0}^{\infty} \{\varphi(z+r+1)-\varphi(z+r)\} \tag{3}$$

therefore converges; and if ϱ denotes the distance from the point z to the nearest point on the negative half of the real axis (so that $|z+r| \geqq \varrho$ when $r \geqq 0$), we have

$$|\psi(z)| < K \sum_{r=0}^{\infty} |z+r|^{-n-2} < K\varrho^{-n} \sum_{r=0}^{\infty} |z+r|^{-2}.$$

Now it will be seen that, in the region D, $\varrho^{-1} = O(z^{-1/k})$ and therefore $\varrho^{-n} = O(z^{-n/k})$; also, $\sum |z+r|^{-2}$ is bounded in D except near the origin.[3] It follows that $\psi(z) = O(z^{-n/k})$ in D. To complete the proof of our lemma we need only remark that the function $p(z)$ defined by writing $\varphi(z) = p(z) - \psi(z)$ is equal, in virtue of (3), to the limit of $\varphi(z+r)$ as the positive integer r tends to infinity, so that unity is clearly a period of $p(z)$.

We may now proceed with the proof of the theorem. Corresponding to any value of the positive integer n we can determine uniquely the n constants $a_1, a_2, \cdots, a_n$ in the expression

$$Q_n(z) = F(z) - a_1 z^{-1} - \cdots - a_n z^{-n} \tag{4}$$

so that in the expansion in negative powers of z of $Q_n(z+1) - Q_n(z)$ the coefficients of $z^{-2}, z^{-3}, \cdots, z^{-n-1}$ are all zero. It will be seen that the value thus found for any of these constants, say a_r, is the same whatever value n has, provided of course that $n \geqq r$. We thus determine a sequence of numbers $a_1, a_2, \cdots$, and (4) then defines a function $Q_n(z)$ such that,

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[2] We can as a matter of fact infer that $\varphi(z) = p(z) + O(z^{-(n+1)/k})$, but the less precise inequality that we give will suffice for our purpose in virtue of the proposition of the preceding paragraph.

[3] This series is the sum of the inverse squares of the distances of the point z from the points $0, -1, -2, \cdots$, and of these points, those that are on the left of the ordinate through z correspond to terms whose sum is less than

$$\frac{1}{\varrho^2} + \frac{1}{\varrho^2+1^2} + \frac{1}{\varrho^2+2^2} + \cdots,$$

and the remaining terms, if there are any, have a sum that also is less than this same quantity. The series under consideration is thus bounded if ϱ^{-1} is bounded.

for each value of n, $|Q_n(z+1)-Q_n(z)|$ remains less than a constant multiple of $|z^{-n-2}|$ when $|z|$ exceeds a certain value. It follows from our lemma that there exists a function $p_n(z)$, of which unity is a period, such that when z is in D

$$Q_n(z)=p_n(z)+O(z^{-n/k}), \tag{5}$$

and it is easy to see that $p_n(z)$ does not depend on n because (4) and (5) show that the periodic function $p_{n+1}(z)-p_n(z)$ tends to zero when z tends to infinity in D, and this cannot be true unless $p_n(z)$ and $p_{n+1}(z)$ are identical. Writing $p(z)$ instead of $p_n(z)$ in (5) we have

$$F(z)=p(z)+a_1z^{-1}+\cdots+a_nz^{-n}+O(z^{-n/k}).$$

Since n/k tends to infinity with n, the conditions that we gave in the preceding paragraph for the validity of the asymptotic expansion (2) are satisfied.

3. We shall now apply this theorem to establish the validity of Stirling's series for $\log\Gamma(z)$ in the region D. Consider the function $J(z)$ defined by the equation

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log 2\pi+J(z),$$

the logarithms being determined throughout D by the condition of being real when z is real and positive. Using the formula $\Gamma(z+1)=z\Gamma(z)$, we find that

$$J(z+1)-J(z)=1-\left(z+\frac{1}{2}\right)\log\left(1+\frac{1}{z}\right), \tag{6}$$

and hence $J(z+1)-J(z)$ can be expanded in a convergent series of the form (1). We can therefore determine the constants $a_1, a_2, \cdots$, so that

$$J(z)=p(z)+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}+O(z^{-n-1}) \tag{7}$$

in the interior of D, where $p(z)$ is a function of which unity is a period.

We shall first show that the function $p(z)$ reduces to a constant. From the fact that $\Gamma(z+n)/\{n^z\Gamma(n)\}$ tends to unity when the positive integer n tends to infinity, z being fixed, we infer easily that $J(z+n)-J(n)$ tends to zero, and therefore, in virtue of (7), that $p(z+n)-p(n)$ also tends to zero. Since unity is a period of $p(z)$, this cannot be so unless $p(z)$ reduces to a constant C.

The calculation of the constant C forms an essential part of any treatment of Stirling's series, and there are several methods that may be used. The following method is perhaps less familiar than some of the others.

From the formula $\Gamma(z)\,\Gamma(-z) = -\pi/(z\sin\pi z)$ we infer that $\Gamma(iy)\,\Gamma(-iy) = \pi/(y\sinh\pi y)$ and hence that

$$\log\Gamma(iy) + \log\Gamma(-iy) = -\pi y + \log(2\pi/y) - \log(1-e^{-2\pi y}).$$

On the other hand, the formula (7), when $p(z)$ has been replaced by C, shows that as the real positive number y tends to infinity we have, for any value of n,

$$\log\Gamma(iy) + \log\Gamma(-iy) = -\pi y + \log(2\pi/y) + C + O(y^{-n-1}).$$

A comparison of these two results leads at once to the value zero for the constant C.

In order to calculate the numbers $a_1, a_2, \cdots$, we must, in accordance with our proof of the theorem, expand in negative powers of z the function

$$1 - \left(z+\frac{1}{2}\right)\log\left(1+\frac{1}{z}\right) - \sum_{s=1}^{n} a_s\,\{(z+1)^{-s} - z^{-s}\}$$

and equate to zero the coefficients of $z^{-2}, z^{-3}, \cdots, z^{-n-1}$. If we equate to zero the coefficient of z^{-r+1}, having taken a value of n greater than $r-3$, we find that

$$\frac{r-2}{2r(r-1)} = a_1 - \binom{r-2}{1}a_2 + \cdots + (-1)^{r-1}\binom{r-2}{r-3}a_{r-2},$$

which may be written

$$\frac{r-2}{2} = \binom{r}{2}A_2 + \binom{r}{3}A_3 + \cdots + \binom{r}{r-1}A_{r-1},$$

where

$$A_{s+1} = (-1)^{s+1}s(s+1)\,a_s$$

for $s > 0$. If we also write $A_1 = -\frac{1}{2}$, $A_0 = 1$, we see that the numbers $A_0, A_1, \cdots$ are determined by the equations

$$A_0 = 1, \qquad \sum_{s=0}^{r-1}\binom{r}{s}A_s = 0 \qquad (r>1).$$

Now these equations are identical with the equations that determine the numbers B_r of Bernoulli which are defined by means of the formula

$$\frac{t}{e^t-1} = \sum_{r=0}^{\infty}\frac{B_r t^r}{r!},$$

and therefore $A_r = B_r$. Remembering that $B_r = 0$ if r is odd and greater than unity, we see that $a_r = 0$ if r is even, and $a_r = B_{r+1}/\{r(r+1)\}$ if r is odd. The required asymptotic expansion for $J(z)$ is thus

$$(8) \quad J(z) = \frac{B_2}{1\cdot 2z} + \frac{B_4}{3\cdot 4z^3} + \cdots + \frac{B_{2n}}{(2n-1)\,2n\,z^{2n-1}} + O\left(\frac{1}{z^{2n+1}}\right),$$

and we have seen that this expansion is valid in the interior of the region D.

4. We shall now establish the asymptotic expansion for $\log \Gamma(z+a)$ where a is any constant. If $H(z, a)$ is the function defined by the equation

$$\log \Gamma(z+a) = \left(z+a-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + H(z, a),$$

or by its equivalent

$$(9) \qquad H(z, a) = J(z+a) + \left(z+a-\frac{1}{2}\right)\log\left(1+\frac{a}{z}\right) - a,$$

we verify at once that $H(z+1, a) - H(z, a)$ is equal to

$$1 - \left(z+a+\frac{1}{2}\right)\log\left(1+\frac{1}{z}\right) + \log\left(1+\frac{a}{z}\right)$$

and therefore admits a convergent expansion of the form (1) when $|z|$ is sufficiently large. We can therefore determine numbers $P_1(a), P_2(a), \cdots$ such that when z lies in D

$$(10) \quad H(z, a) = p(z, a) + P_1(a)z^{-1} + \cdots + P_n(a)z^{-n} + O(z^{-n-1}),$$

where $p(z, a)$, regarded as a function of z, admits the period unity. We infer from (8), (9) and (10) that $p(z, a) = O(z^{-1})$ in D, and this can be true only if $p(z, a)$ vanishes identically.

In order to determine the coefficients $P_r(a)$ we first remark that they are polynomials in a, as we see at once on considering the equations by which they are defined. We then compare the formula

$$H(z, a+1) - H(z, a) = \sum_{r=1}^{n} \{P_r(a+1) - P_r(a)\}\, z^{-r} + O(z^{-n-1})$$

with the formula

$$H(z, a+1) - H(z, a) = \log\left(1+\frac{a}{z}\right),$$

which follows from (9) and (6), and we see that

$$(11) \qquad P_r(a+1) - P_r(a) = \frac{(-1)^{r+1}a^r}{r}.$$

Also, if we replace a by zero in (10), this formula must reduce to (8) since $H(z, 0) = J(z)$; hence

$$(12) \qquad P_r(0) = \frac{(-1)^{r+1}B_{r+1}}{r(r+1)}.$$

Now the polynomials $B_r(a)$ of Bernoulli are defined by the equations

$$B_r(a+1)-B_r(a) = r\,a^{r-1}, \quad B_r(0) = B_r,$$

and therefore the equations (11) and (12) are satisfied if we write

$$P_r(a) = \frac{(-1)^{r+1} B_{r+1}(a)}{r(r+1)}. \tag{13}$$

Since the equations (11) and (12) determine completely the polynomials $P_r(a)$, these polynomials necessarily have the values given by (13).

Replacing the polynomials $P_r(a)$ in (10) by these values we obtain the required asymptotic expansion for $\log \Gamma(z+a)$:

$$\log \Gamma(z+a) = \left(z+a-\frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + \frac{B_2(a)}{1\cdot 2z} - \cdots + \frac{(-1)^{n+1} B_{n+1}(a)}{n(n+1)z^n} + O\left(\frac{1}{z^{n+1}}\right),$$

and this expansion has been shown to hold subject to the condition that z should remain in the region D.

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ON THE DIRICHLET-NEUMANN PROBLEM.*

By G. E. Raynor.

1. The purpose of the following paper is to discuss briefly what the writer has chosen to call the Dirichlet-Neumann problem. It will be stated explicitly below but for convenience we shall first state the classic Dirichlet and Neumann problems.

Let R be a connected and bounded region in a plane and B its boundary. Let $U(x, y)$ be a function defined and continuous on B. Then Dirichlet's problem for the plane is to prove the existence of a function continuous on $R+B$, identical with $U(x, y)$ on B, and harmonic in R, that is, satisfying Laplace's differential equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

in R. The problem has been solved for very general types of regions and has been generalized by allowing $U(x, y)$ to be discontinuous. For very complete references see a paper by O. D. Kellogg.[1]

If the boundary B have a normal at each of its points and $V(x, y)$ be a function defined and continuous on B and satisfying the condition $\int_B \frac{\partial v}{\partial n}\, ds = 0$, then Neumann's problem is to prove the existence of a function continuous on $R+B$, harmonic in R, and whose normal derivatives on B are identical with $V(x, y)$.[2]

For the Dirichlet problem we have the following uniqueness theorem:

Theorem I. *If a solution $f(x, y)$ of the Dirichlet problem for the domain $R+B$ and boundary values $U(x, y)$ exist it is unique.*

For the Neumann problem the following theorem is classic.

Theorem II. *If a solution $f(x, y)$ of the Neumann problem, for the region $R+B$ and boundary values $V(x, y)$ of the normal derivative exist, it is unique to within an additive constant.*

Now if we try to find a function $f(x, y)$ harmonic in R and such that both the values of $f(x, y)$ and the values of its normal derivatives on B are specified the uniqueness theorems imply that the problem has in

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[1] Recent Progress with the Dirichlet Problem, Bull. Am. Math. Soc., vol. 32 (1926), pp. 601–625.

[2] For a discussion of the problem and its generalizations see Lovitt, Integral Equations, pp. 110–115 and Evans, The Logarithmic Potential, chap. IV.

general no solution. Or put otherwise, if $f(x, y)$ is to have the properties on B described above it must fail to be harmonic at at least one point P of R, that is it must have a singular point at P. We are thus led to a consideration of the following Dirichlet-Neumann problem: Given two functions $U(x, y)$ and $V(x, y)$ defined and continuous on B and given an arbitrary point P in R to prove the existence of a function continuous in $R+B-P$, identical with $U(x, y)$ on B, having normal derivatives on B identical with $V(x, y)$, and harmonic in $R-P$.

It is not our purpose, in this paper, to enter into a full discussion of the existence of solutions of the Dirichlet-Neumann problem although we shall outline briefly in section three an existence proof for certain fairly general domains. In section two we prove the following uniqueness theorem; where B is assumed to be a rectifiable curve with a normal at each point.

THEOREM III. *If a solution $f(x, y)$ of the Dirichlet-Neumann problem for the domain $R+B$, of any finite order of connectivity, and a point P in R, exist, it is unique.*[3]

2. Let $f_1(x, y)$ and $f_2(x, y)$ be two solutions of the Dirichlet-Neumann problem for the region R and point P, each taking identical boundary values on B and having identical normal derivatives on B. We shall prove that the function

$$F(x, y) \equiv f_1(x, y) - f_2(x, y)$$

is identically zero in $R-P$.

Let C be a circle with center at P and lying entirely in R, and let r be the distance from P to any point of $R+B$. The writer has shown[4] that in the neighborhood of an isolated singular point such as P, the function F may be put in the form

$$F \equiv \Phi + c \log r + W \tag{1}$$

where Φ is identically zero on C, is harmonic in C except at P (unless $\equiv 0$), $c = \dfrac{1}{2\pi}\displaystyle\int_C \frac{\partial F}{\partial n} ds$ and W is harmonic in C including P. In the region bounded by B and C, F is harmonic and hence by a well known theorem of Potential Theory,

$$\int_{B+C} \frac{\partial F}{\partial n} ds = 0. \tag{2}$$

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[3] In the following we shall understand the term normal derivative to be used in the sense of Goursat, Cours d'Analyse Mathématique, vol. 3, p. 179.

[4] G. E. Raynor, Isolated singular points of harmonic functions, Bull. Am. Math. Soc., vol. 32 (1926–27), p. 537.

But on B, $\frac{\partial F}{\partial n}$ is identically zero and hence (2) reduces to

$$\int_C \frac{\partial F}{\partial n} ds = 0. \tag{3}$$

Hence by (3) the function F can have no logarithmic singularity and F has the form

$$F = \Phi + W \tag{4}$$

in and on C.

Now the function $r^m \cos m\theta$, where m is any positive integer and θ is the angle between a fixed line through P and the line along which r is measured, is harmonic everywhere in R. Hence if we apply Green's formula to the functions F and $r^m \cos m\theta$ for the region bounded by B and C we have

$$\int_{B+C} \left(r^m \frac{\partial F}{\partial n} \cos m\theta - F \frac{\partial (r^m \cos m\theta)}{\partial n} \right) ds = 0 \tag{5}$$

where the normal derivatives are taken, say, toward the exterior of our region. On B, $\frac{\partial F}{\partial n}$ and F are identically zero and hence (5) reduces to

$$\int_C \left(r^m \frac{\partial F}{\partial n} \cos m\theta - F \frac{\partial (r^m \cos m\theta)}{\partial n} \right) ds = 0, \tag{6}$$

where since we took the normal derivatives in the direction of the outer normal to the region bounded by B and C the normal derivatives in (6) are taken toward the interior of C. Now $\frac{\partial (r^m \cos m\theta)}{\partial n}$ taken toward the interior of C is

$$- m r^{m-1} \cos m\theta$$

and substituting this in (6) and remembering that r is constant on C, we find that (6) becomes

$$r \int_C \frac{\partial F}{\partial n} \cos m\theta \, ds + m \int_C F \cos m\theta \, ds = 0. \tag{7}$$

Substituting (4) in (7) we have

$$\begin{aligned} r \int_C \frac{\partial \Phi}{\partial n} \cos m\theta \, ds + m \int_C \Phi \cos m\theta \, ds + r \int_C \frac{\partial W}{\partial n} \cos m\theta \, ds \\ + m \int_C W \cos m\theta \, ds = 0. \end{aligned} \tag{8}$$

Now W is harmonic within C and so if we apply Green's formula to the functions $r^m \cos m\theta$ and W for the region bounded by C we obtain a formula analogous to (6) with W in place of F and which will reduce analogously to

$$(9) \qquad r\int_C \frac{\partial W}{\partial n}\cos m\theta\, ds + m\int_C W\cos m\theta\, ds = 0.$$

Hence the sum of the last two terms on the left of (8) vanishes. Furthermore, since as stated above $\Phi \equiv 0$ on C, the second term of (8) vanishes. Hence (8) reduces finally to

$$(10) \qquad \int_C \frac{\partial \Phi}{\partial n}\cos m\theta\, ds = 0.$$

In the same manner, by using the function $r^m \sin m\theta$ in place of $r^m \cos m\theta$ we obtain by a similar discussion

$$(11) \qquad \int_C \frac{\partial \Phi}{\partial n}\sin m\theta\, ds = 0.$$

The writer has shown[5] that Φ may be expanded in C in the form

$$(12) \qquad \Phi = \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{r_1}{r}\right)^m - \left(\frac{r}{r_1}\right)^m\right](\gamma_m \cos m\theta + \delta_m \sin m\theta)$$

where r_1 is the radius of C and γ_m and δ_m are given by the equations

$$(13) \qquad \gamma_m = \frac{1}{2\pi}\int_C \frac{\partial \Phi}{\partial n}\cos m\theta\, ds, \qquad \delta_m = \frac{1}{2\pi}\int_C \frac{\partial \Phi}{\partial n}\sin m\theta\, ds.$$

Hence it follows from (11) that γ_m and δ_m are zero and hence that $\Phi \equiv 0$. Thus we have arrived at the result that F has at most a removable singularity at P. Hence it follows from Theorem 1 on the uniqueness of the solution of Dirichlet's problem that $F \equiv 0$ in the region $B+R-P$ and hence that the solution of the Dirichlet-Neumann problem is unique.

3. O. Hölder[6] has stated necessary and sufficient conditions for the solution of the Dirichlet-Neumann problem for the circle C of radius R with singular point P at the center. His conditions are as follows. Given a number λ which is less than $1/R$ there must exist a positive number M such that the expressions

$$\left.\begin{aligned} &\lambda^m \int_{-\pi}^{\pi}\left[U(\alpha) - \frac{1}{m}V(\alpha)\right]\cos m\alpha\, d\alpha, \\ &\lambda^m \int_{-\pi}^{\pi}\left[U(\alpha) - \frac{1}{m}V(\alpha)\right]\sin m\alpha\, d\alpha \end{aligned}\right\} \qquad (m = 1, 2, 3, \cdots)$$

[5] G. E. Raynor, Note on the expansion of harmonic functions in the neighborhood of isolated singular points; these Annals, vol. 31, 1930.

[6] Leipzig. Berichte, vol. 63 (1911), p. 477.

are less in absolute value than M, where $U(\alpha)$ and $V(\alpha)$ are the boundary value and the value of the outer normal derivative respectively on C and α is the angle made by a radius of C with a polar axis through P.

It may be of interest to point out in closing that Riemann's theorem and its extensions on conformal mapping allow us to solve the Dirichlet-Neumann problem for certain simply connected regions, with arbitrary singular point, whose boundaries are specified below. It is not our purpose, in this paper, to attempt to find the most general regions for which the Dirichlet-Neumann problem has a solution but merely to show that it does have a solution for regions considerably more general than a circle.

For our purposes we state the following theorem which includes that of Riemann.[7]

The interior of any plane simply connected region R whose boundary consists of more than one point can be mapped conformally upon the interior of a circle C in such a way that an inner point P of R is carried into the center of C and a given direction at P is carried into a given direction at the center of C.

The above theorem says nothing about the behavior of the mapping function on the boundary. A sufficient theorem for our purposes is as follows:

If the boundary of R consists of a simple closed Jordan curve, then the conformal map of the interior of C will be one-to-one and continuous on the boundary.[8]

In the second part of this paper we have used various line integrals over our boundary and these line integrals have involved normal derivatives on the boundary. Hence in view of the above theorems and the remarks just made we shall assume that our region is such that its boundary is a simple rectifiable closed Jordan curve at every point of which there is a normal.

The solution of the Dirichlet problem by the method of conformal mapping is of course classic and is briefly as follows. Let B be the boundary of the simply connected region, in the W-plane, for which the solution of Dirichlet's problem is desired and C a circle with center at the origin O in the Z-plane. Let

$$w = u + iv = \varphi(z), \qquad z = x + iy \tag{14}$$

be the transformation which maps the interior of C conformally on the interior of B. Let $\varphi(x, y)$ be the solution of Dirichlet's problem for C.

[7] See Ford, Automorphic Functions, pp. 186, 187.

[8] Osgood und Taylor, Conformal Transformations, Trans. Am. Math. Soc., vol. 14 (1913), p. 294, and Ford l. c. p. 198, Theorem 15.

Now $\varphi(x, y)$ becomes under the transformation (14) a function $F(u, v)$ and it is easily verified that the Laplacian $\nabla^2 F$ takes the form

$$(15)\qquad \nabla^2 F = \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right]\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)$$

and since φ is harmonic it follows that F is a harmonic function of u and v. Furthermore, since $F(u, v) = \varphi(x, y)$ the functions F and φ take the same values at corresponding points of B and C. Hence to solve Dirichlet's problem for the region bounded by B we need merely to determine the transformation (14), then determine by means of the transformation the boundary values for C corresponding to those of B, obtain the solution $\varphi(x, y)$ of Dirichlet's problem for C and then transform $\varphi(x, y)$ by (14) to $F(u, v)$.

Now let P be an arbitrary point of the region R bounded by B. Riemann's theorem allows us to map R on the interior of C so that P corresponds to the center of C. Furthermore it can be shown, by a fairly simple computation, that by means of the transformation (14) the following relation between the normal derivatives of F and φ at corresponding points holds, namely,

$$\frac{\partial F}{\partial n} = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right]^{1/2} \frac{\partial \varphi}{\partial n}.$$

Hence if we desire the solution of the Dirichlet-Neumann problem for the region R with given boundary values $U(u, v)$ and normal derivatives given by the function $V(u, v)$ we need merely find the solution $\varphi(x, y)$ of the Dirichlet-Neumann problem for the interior of C, taking for boundary values of φ the values of U at corresponding points of B and for the normal derivatives of φ the values of the assigned normal derivatives V at corresponding points of B multiplied by

$$\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right]^{-1/2}.$$

Thus we may state

THEOREM IV. *The solution of the Dirichlet-Neumann problem exists for any simply connected plane region whose boundary is a simple rectifiable closed Jordan curve which has a normal at each point, provided the assigned boundary values and those of the normal derivatives are such that the corresponding problem under conformal mapping has a solution for a circle with singular point at the center.*

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EXPANSION OF ANALYTIC FUNCTIONS INTO INFINITE PRODUCTS.*

By S. Borofsky.

1. Introduction. In a paper which is to appear shortly in the Mathematische Zeitschrift,[1] Professor J. F. Ritt has proved that if a function of z is analytic and equal to unity at the origin, the function can be expressed in some neighborhood of $z = 0$ in the form

$$(1 + a_1 z)(1 + a_2 z^2) \cdots (1 + a_n z^n) \cdots,$$

where the a's are constants. The present paper, which is complete in itself, generalizes this theorem in two ways.

On the one hand, we obtain for every function representable in a right half-plane by an absolutely convergent Dirichlet series,

$$(1) \qquad 1 + c_1 e^{-\beta_1 z} + c_2 e^{-\beta_2 z} + \cdots + c_n e^{-\beta_n z} + \cdots,$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots$, $\lim \beta_n = \infty$, an infinite product expansion, in some right half-plane, of the form

$$(1 + a_1 e^{-\lambda_1 z})(1 + a_2 e^{-\lambda_2 z}) \cdots (1 + a_n e^{-\lambda_n z}) \cdots,$$

where the λ's are linear combinations of the β's with positive integral coefficients.

On the other hand, we obtain for functions analytic and equal to unity at the origin expansions of the form

$$[1 + a_1 H_1(z)]\,[1 + a_2 H_2(z)] \cdots [1 + a_n H_n(z)] \cdots,$$

where $H_1(z), H_2(z), \cdots, H_n(z), \cdots$ is any sequence of functions of a general type.

These results are contained in the following general theorem.

Theorem 1. *Let $f(z)$, $z = x + iy$, be representable in a right half-plane by an absolutely convergent series* (1).

Let

$$(2) \qquad \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n, \cdots,$$

be all the distinct quantities, ordered according to increasing magnitude, obtained by forming all expressions of the form

* Received February 4, 1930.

[1] Vol. 32 (1930), pp. 1–3.

$$\delta_1 \beta_{p_1} + \delta_2 \beta_{p_2} + \cdots + \delta_m \beta_{p_m}, \qquad (m = 1, 2, 3, \cdots),$$

where $\beta_{p_1}, \beta_{p_2}, \cdots, \beta_{p_m}$ *are any* m β*'s and* $\delta_1, \delta_2, \cdots, \delta_m$ *is any set of non-negative integers.*

Let

$$(3) \qquad D(z) = d_0 e^{-\lambda_0 z} + d_1 e^{-\lambda_1 z} + \cdots + d_n e^{-\lambda_n z} + \cdots,$$

where $d_0, d_1, \cdots, d_n, \cdots$ *are non-negative real quantities and where* $d_0 \geqq 1$, *be convergent in some right half-plane.*[2]

Let

$$(4) \qquad c_0^{(i)} e^{-\lambda_0 z} + c_1^{(i)} e^{-\lambda_1 z} + \cdots + c_n^{(i)} e^{-\lambda_n z} + \cdots, \qquad (i = 1, 2, 3, \cdots),$$

be a set of series such that

$$(5) \qquad |c_n^{(i)}| \leqq d_n, \qquad c_0^{(i)} = 1, \qquad (i = 1, 2, \cdots; n = 0, 1, 2, \cdots),$$

and let $f_i(z)$ *be the function represented by the series* (4).

Let $m_1, m_2, \cdots, m_n, \cdots$ *be any positive integers.*

Then, in some right half-plane, $f(z)$ *can be expressed as an infinite product,*

$$(6) \qquad [1 + w_1 f_1(z) e^{-\lambda_1 z}]^{m_1} [1 + w_2 f_2(z) e^{-\lambda_2 z}]^{m_2} \cdots [1 + w_n f_n(z) e^{-\lambda_n z}]^{m_n} \cdots,$$

where the w*'s are constants. In this half-plane the product*

$$\prod_{n=1}^{\infty} \Big[1 + m_n |w_n| F_n(z) e^{-\lambda_n z} + \frac{m_n(m_n - 1)}{1 \cdot 2} |w_n^2| F_n^2(z) e^{-2\lambda_n z} + \cdots \\ \cdots + |w_n^{m_n}| F_n^{m_n}(z) e^{-m_n \lambda_n z} \Big],$$

where

$$F_n(z) = |c_0^{(n)}| e^{-\lambda_0 z} + |c_1^{(n)}| e^{-\lambda_1 z} + \cdots + |c_\nu^{(n)}| e^{-\lambda_\nu z} + \cdots, \qquad (n = 1, 2, \cdots)$$

converges uniformly.

For brevity we say that the convergence of (6) is regular.

In investigations on Dirichlet series the assumption is often made that $\limsup (\log n)/\beta_n$ is finite. In paragraph 5 it is shown that when this is the case, then $\limsup (\log n)/\lambda_n$ is also finite.

If we let $m_n = 1$, $u = e^{-z}$, and $f_n(z) = 1$, we obtain Professor Ritt's theorem.

If we let $m_n = 1$, $u = e^{-z}$, and

$$f_n(z) = 1 + u^n + u^{2n} + u^{3n} + \cdots,$$

we obtain for every function representable in a neighborhood of $u = 0$ by a series of the form

[2] Note that $\lambda_0 = 0$.

(7) $$1+c_1u+c_2u^2+\cdots+c_nu^n+\cdots,$$

an infinite product expansion of the form

$$\left(1+w_1\frac{u}{1-u}\right)\left(1+w_2\frac{u^2}{1-u^2}\right)\cdots\left(1+w_n\frac{u^n}{1-u^n}\right)\cdots.$$

This provides an analogue for infinite products of the theorem that an analytic function which vanishes at the origin can be expressed as a Lambert series.[3]

If we let $m_n=1$, $u=e^{-z}$, and $f_n(z)=2^n n!\,J_n(u)/u^n$, where $J_n(u)$ is the Bessel function

$$\sum_{r=0}^{\infty}\left(\frac{u}{2}\right)^{n+2r}\frac{(-1)^r}{r!\,(n+r)!},$$

we obtain for every function represented by a series (7) an expansion of the form

$$[1+w_1J_1(u)]\,[1+w_2J_2(u)]\cdots[1+w_nJ_n(u)]\cdots.$$

Remark: If (6) and

(8) $$\prod_{n=1}^{\infty}[1+w_n'f_n(z)\,e^{-\lambda_n'z}]^{m_n},$$

where

$$0<\lambda_1'<\lambda_2'<\cdots<\lambda_n'<\cdots,\qquad \lim\lambda_n'=\infty,$$

are two regularly convergent products representing $f(z)$ in a half-plane, then $w_n=w_n'$ and $\lambda_n=\lambda_n'$ for every n. For, expanding (6) and (8) into infinite series, we have

$$1+m_1w_1e^{-\lambda_1z}+\cdots=1+m_1w_1'e^{-\lambda_1'z}+\cdots,$$

so that $w_1=w_1'$ and $\lambda_1=\lambda_1'$. Then, dividing out the first factor in (6) and (8), we obtain, in the same way, $w_2=w_2'$ and $\lambda_2=\lambda_2'$. In general, proceeding by induction, we obtain $w_n=w_n'$ and $\lambda_n=\lambda_n'$.

2. Determination of the coefficients in the product. If we assume the existence of a regularly convergent product (6) representing $f(z)$ in a half-plane, the w's can be determined by expanding (6) into an infinite series,

(9) $$1+e_1e^{-\lambda_1z}+e_2e^{-\lambda_2z}+\cdots+e_ne^{-\lambda_nz}+\cdots,$$

and equating the coefficients of $e^{-\lambda_nz}$ in (1) and (9). As each e_n is of the form $m_nw_n+P_n$, where P_n is a polynomial in $w_1, w_2, \cdots, w_{n-1}$,

[3] K. Knopp, Journal für Mathematik, vol. 142 (1913), p. 288.

the w's are uniquely determined. However, as the expressions thus obtained for the w's are not suitable for a convergence proof, we obtain another set of expressions.

Let (3) be convergent for $x \geqq R$ and let M be a positive quantity such that $D(R) \leqq M$.

Let

$$D^k(z) = \left[\frac{D(z)}{2M}\right]^k = d_{k0}\, e^{-\lambda_0 z} + d_{k1}\, e^{-\lambda_1 z} + \cdots + d_{kn}\, e^{-\lambda_n z} + \cdots, \quad (k = 1, 2, \cdots). \tag{10}$$

$$g_i^k(z) = \left[\frac{f_i(z)}{2M}\right]^k = g_{k0}^{(i)}\, e^{-\lambda_0 z} + g_{k1}^{(i)}\, e^{-\lambda_1 z} + \cdots + g_{kn}^{(i)}\, e^{-\lambda_n z} + \cdots, \quad (i, k = 1, 2, \cdots). \tag{11}$$

Then, because of (5), we have

$$|g_{kn}^{(i)}| \leqq d_{kn}, \quad (i, k = 1, 2, \cdots;\ n = 0, 1, 2, \cdots), \tag{12}$$

and,

$$g_{10}^{(i)} = 1/2M, \quad (i = 1, 2, \cdots). \tag{13}$$

From the definition of M and $D^k(z)$, we have

$$d_{k0}\, e^{-\lambda_0 x} + d_{k1}\, e^{-\lambda_1 x} + \cdots + d_{kn}\, e^{-\lambda_n x} + \cdots \leqq 1/2^k, \quad (k = 1, 2, \cdots), \tag{14}$$

for $x \geqq R$.

We write (6) in the form

$$\prod_{n=1}^{\infty} [1 + a_n\, g_n(z)\, e^{-\lambda_n z}]^{m_n}, \tag{15}$$

and we assume that this is regularly convergent to $f(z)$ in some right half-plane.

Since (1) has a half-plane of absolute convergence, the logarithm of $f(z)$ can be expressed in a half-plane as an absolutely convergent series

$$b_1\, e^{-\lambda_1 z} + b_2\, e^{-\lambda_2 z} + \cdots + b_n\, e^{-\lambda_n z} + \cdots, \tag{16}$$

where some of the b's may be zeros.[4]

From (15) we have

$$\log f(z) = \sum_{n=1}^{\infty} m_n \log [1 + a_n\, g_n(z)\, e^{-\lambda_n z}]. \tag{17}$$

The expression on the right in (17) can be written

$$\sum_{n=1}^{\infty} m_n \left[a_n g_n(z)\, e^{-\lambda_n z} - \frac{1}{2} a_n^2\, g_n^2(z)\, e^{-2\lambda_n z} + \frac{1}{3} a_n^3\, g_n^3(z)\, e^{-3\lambda_n z} - \cdots\right]. \tag{18}$$

[4] E. Landau, Handbuch der Primzahlen, p. 734.

We now equate the coefficients of $e^{-\lambda_n z}$ in (16) and (18). Since any linear combination of the λ's with positive integral coefficients is itself one of the λ's, we need not concern ourselves explicitly with the other terms in (18).

Consider any particular λ_n. If it is not the sum of two smaller λ's, distinct or not, we have

$$(19) \qquad b_n = m_n g_{10}^{(n)} a_n.$$

Suppose that λ_n can be expressed as the sum of two smaller λ's, distinct or not. Let $i_1 \neq 0$ be the smallest value of p for which $\lambda_n - \lambda_p$ is equal to a λ. Let q_1 be the greatest integral value of q for which $\lambda_n - q\lambda_{i_1}$ equals a λ. Then $q_1 \geqq 1$.

Since the sum of two λ's is a λ, each of the quantities

$$\lambda_n - q\lambda_{i_1}, \qquad (q = 1, 2, \cdots, q_1),$$

is a λ. Let, then,

$$(20) \qquad \lambda_n = q_1\lambda_{i_1} + \lambda_{11} = (q_1 - 1)\lambda_{i_1} + \lambda_{12} = \cdots = \lambda_{i_1} + \lambda_{1q_1}.$$

In (20) each doubly subscripted λ is, of course, merely one of the λ's of (2). If λ_n is an integral multiple of λ_{i_1}, then λ_{11} will be $\lambda_0 = 0$.

Let

$$\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}, \qquad (0 < i_1 < i_2 < \cdots < i_k < n),$$

be all those λ's which, when subtracted from λ_n, leave a λ. For any t from 1 to k, let q_t be the greatest integral value of q for which

$$\lambda_n - q\lambda_{i_t}$$

is a λ. We will have, as in (20),

$$(21) \quad \lambda_n = q_t\lambda_{i_t} + \lambda_{t1} = (q_t - 1)\lambda_{i_t} + \lambda_{t2} = \cdots = \lambda_{i_t} + \lambda_{tq_t}, \quad (t = 1, 2, \cdots, k).$$

Each λ_{tr} of (21) is, as has already been stated, one of the λ's of (2). Let j_{tr} be used to represent the subscript which λ_{tr} has in (2). Then $\lambda_{tr} \equiv \lambda j_{tr}$. Of course, two different double subscripts tr may give the same j_{tr} in (2).

Then we have, upon equating the coefficients of $e^{-\lambda_n z}$ in (16) und (18),

$$(22) \quad b_n = m_n g_{10}^{(n)} a_n + \sum_{t=1}^{k} m_{i_t}\left[\pm \frac{a_{i_t}^{q_t}}{q_t} g_{q_t, j_{t1}}^{(i_t)} \pm \frac{a_{i_t}^{q_t - 1}}{q_t - 1} g_{q_t - 1, j_{t2}}^{(i_t)} \pm \cdots \pm \frac{a_{i_t}}{1} g_{1, j_{tq_t}}^{(i_t)}\right].$$

3. An inequality for the coefficients. Let $\alpha_i = |m_i a_i|$. We show that there exists a $\sigma > 0$ such that $\alpha_n \leqq e^{\lambda_n \sigma}$ for every n.

Consider any fixed n. If (19) holds, then

$$(23) \qquad g_{10}^{(n)} \alpha_n = |b_n|.$$

If (22) holds, then, throwing away the denominators which appear on the right hand side of (22),

$$(24)\quad g_{10}^{(n)}\alpha_n \leqq |b_n| + \sum_{t=1}^{k} m_{i_t}\Big[|a_{i_t}|^{q_t}|g_{q_t,j_{t1}}^{(i_t)}| + |a_{i_t}|^{q_t-1}|g_{q_t-1,j_{t2}}^{(i_t)}| + \cdots + |a_{i_t}|\cdot|g_{1,j_{tq_t}}^{(i_t)}|\Big].$$

Since $m_{i_t} \geqq 1$, we may replace in (24),

$$m_{i_t}|a_{i_t}|^{q_t},\ m_{i_t}|a_{i_t}|^{q_t-1},\ \cdots,\ m_{i_t}|a_{i_t}|$$

by

$$|m_{i_t}a_{i_t}|^{q_t},\ |m_{i_t}a_{i_t}|^{q_t-1},\ \cdots,\ |m_{i_t}a_{i_t}|,$$

without weakening the inequality. Hence, we have

$$(25)\quad g_{10}^{(n)}\alpha_n \leqq |b_n| + \sum_{t=1}^{k}\Big[\alpha_{i_t}^{q_t}|g_{q_t,j_{t1}}^{(i_t)}| + \alpha_{i_t}^{q_t-1}|g_{q_t-1,j_{t2}}^{(i_t)}| + \cdots + \alpha_{i_t}|g_{1,j_{tq_t}}^{(i_t)}|\Big].$$

Let ω be a quantity so chosen that

$$(26)\qquad \omega \geqq 2\lambda_1/\lambda_2, \qquad \omega \geqq 6M$$

where M is the M of § 2.

Let

$$\sigma_m = \frac{\log(\omega\lambda_m/\lambda_1)}{\log 2}, \qquad (m = 2, 3, \cdots).$$

Let

$$\tau_1 = 1,$$

$$\log\tau_m = (\log 2)\sum_{p=1}^{[\sigma_m]}(p+1/2)2^{-p}, \qquad (m = 2, 3, \cdots),$$

where $[\sigma_m]$ is the greatest integer which does not exceed σ_m.

Let $m > 1$ and m' be two positive integers such that $\lambda_m \geqq 2\lambda_{m'}$. Then

$$\log(\omega\lambda_m/\lambda_1) \geqq \log(\omega\lambda_{m'}/\lambda_1) + \log 2,$$

so that

$$[\sigma_m] \geqq 1 + [\sigma_{m'}].$$

Then,

$$\log\tau_m - \log\tau_{m'} \geqq ([\sigma_m]+1/2)2^{-[\sigma_m]}\log 2 > \sigma_m 2^{-\sigma_m}\log 2,$$

since $(p+1/2)2^{-p}$ is a decreasing function of p for $p \geqq 1$. Thus,

$$\log\tau_m - \log\tau_{m'} > \frac{\log(\omega\lambda_m/\lambda_1)}{\omega\lambda_m/\lambda_1},$$

or,

$$(27)\qquad T(\log\tau_m - \log\tau_{m'}) > \frac{\log(\omega\lambda_m/\lambda_1)}{\lambda_m}, \quad \text{where } T = \omega/\lambda_1.$$

Since (16) has a half-plane of convergence, the sequence $(\log|b_n|)/\lambda_n$ has a finite upper bound.[5] Let H be a positive quantity not less than the least upper bound of this sequence and at least equal to the R of § 2.

Let N be a quantity so chosen that

$$(28) \qquad \log N \geqq (\log 6M)/\lambda_1, \qquad \log N \geqq 0.$$

We show that

$$(29) \qquad (\log \alpha_\nu)/\lambda_\nu \leqq H + T \log \tau_\nu + \log N, \qquad (\nu = 1, 2, \cdots).$$

Since (23) holds for $n = 1$, we have by (13),

$$g_{10}^{(1)} \alpha_1 = |b_1|, \qquad \alpha_1 = 2M|b_1|,$$

whence,

$$\frac{\log \alpha_1}{\lambda_1} = \frac{\log|b_1|}{\lambda_1} + \frac{\log 2M}{\lambda_1} \leqq H + \log N \leqq H + T \log \tau_1 + \log N.$$

Thus (29) is true for $\nu = 1$.

We proceed by induction on ν, supposing that (29) holds for $\nu = 1, 2, \cdots, n-1$ and proving that it then follows for $\nu = n$.

For α_n we have one of the expressions (23), (25). If (23) holds, then

$$\alpha_n = 2M|b_n|,$$

so that

$$\frac{\log \alpha_n}{\lambda_n} \leqq H + \log N < H + T \log \tau_n + \log N.$$

Suppose (25) holds. Let r be any one of the values which q_t assumes in (21). Let

$$t^{(1)}, t^{(2)}, \cdots, t^{(v)},$$

be those values of t, arranged in order of increasing magnitude, for which $q_t = r$. Let

$$u_j = i_{t^{(j)}}, \qquad (j = 1, 2, \cdots, v),$$

where $i_{t^{(j)}}$ is the subscript of the λ in (21) which goes with $q_{t^{(j)}}$.

Consider those terms under the summation sign in (25) for which $q_t = r$. The totality of these terms is

$$(30) \qquad \sum_{i=1}^{v} \left[\alpha_{u_i}^{r} |g_{rj_{t^{(i)}1}}^{(u_i)}| + \alpha_{u_i}^{r-1} |g_{r-1, j_{t^{(i)}2}}^{(u_i)}| + \cdots + \alpha_{u_i} |g_{1, j_{t^{(i)}r}}^{(u_i)}| \right].$$

As $u_i < n$ and as (29) holds for every $\nu < n$, the expression (30) is equal to or less than

[5] O. Szász, Mathematische Annalen, Bd. 85 (1922), p. 108.

$$\sum_{i=1}^{v}\Big[e^{r\lambda_{u_i}H}\,\tau_{u_i}^{r\lambda_{u_i}T}\,N^{r\lambda_{u_i}}\,|\,g_{rj_{t^{(i)}_1}}^{(u_i)}\,| + e^{(r-1)\lambda_{u_i}H}\,\tau_{u_i}^{(r-1)\lambda_{u_i}T}\,N^{(r-1)\lambda_{u_i}}\,|\,g_{r-1,j_{t^{(i)}_2}}^{(u_i)}\,| + \cdots$$
$$\cdots + e^{\lambda_{u_i}H}\,\tau_{u_i}^{\lambda_{u_i}T}\,N^{\lambda_{u_i}}\,|\,g_{1,j_{t^{(i)}_r}}^{(u_i)}\,|\Big].$$

Since $\tau_{u_i} \geqq 1$, $N \geqq 1$ and $\lambda_{u_1} < \lambda_{u_2} < \cdots < \lambda_{u_v}$, the last expression cannot exceed

$$(31)\quad \tau_{u_v}^{r\lambda_{u_v}}\,N^{r\lambda_{u_v}}\sum_{i=1}^{v}\Big[e^{r\lambda_{u_i}H}\,|\,g_{rj_{t^{(i)}_1}}^{(u_i)}\,| + e^{(r-1)\lambda_{u_i}H}\,|\,g_{r-1,j_{t^{(i)}_2}}^{(u_i)}\,| + \cdots + e^{\lambda_{u_i}H}\,|\,g_{1,j_{t^{(i)}_r}}^{(u_i)}\,|\Big].$$

From (21) we have

$$\lambda_n = r\,\lambda_{u_i} + \lambda_{t^{(i)}1} = (r-1)\,\lambda_{u_i} + \lambda_{t^{(i)}2} = \cdots = \lambda_{u_i} + \lambda_{t^{(i)}r}, \quad (i = 1, 2, \cdots, v).$$

Hence, (31) can be written in the form

$$(32)\qquad e^{\lambda_n H}\,\tau_{u_v}^{r\lambda_{u_v}T}\,N^{r\lambda_{u_v}}\sum_{i=1}^{v}\Big[\,|\,g_{rj_{t^{(i)}_1}}^{(u_i)}\,|\,e^{-\lambda_{t^{(i)}_1}H} + |\,g_{r-1,j_{t^{(i)}_2}}^{(u_i)}\,|\,e^{-\lambda_{t^{(i)}_2}H} + \cdots$$
$$\cdots + |\,g_{1,j_{t^{(i)}_r}}^{(u_i)}\,|\,e^{-\lambda_{t^{(i)}_r}H}\Big].$$

Consider the expression

$$(33)\qquad \sum_{i=1}^{v}|\,g_{r-q+1,j_{t^{(i)}_q}}^{(u_i)}\,|\,e^{-\lambda_{t^{(i)}_q}H},$$

for a fixed q such that $1 \leqq q \leqq r$.

As $\lambda_n = (r-q+1)\,\lambda_{u_i} + \lambda_{t^{(i)}q}$, and $\lambda_{u_1}, \lambda_{u_2}, \cdots, \lambda_{u_v}$ are all distinct, the quantities $\lambda_{t^{(i)}q}$, $(i = 1, 2, \cdots, v)$, are all distinct. Hence, by (12), the expression (33) cannot exceed

$$\sum_{i=1}^{v} d_{r-q+1,j_{t^{(i)}_q}}\,e^{-\lambda_{t^{(i)}_q}H},$$

which is less than

$$\sum_{n=0}^{\infty} d_{r-q+1,\,n}\,e^{-\lambda_n H}.$$

As $H \geqq R$, the last expression, by (14), does not exceed $1/2^{r-q+1}$. Therefore, the expression (32) is less than

$$e^{\lambda_n H}\,\tau_{u_v}^{r\lambda_{u_v}T}\,N^{r\lambda_{u_v}}\,(1/2 + 1/2^2 + \cdots + 1/2^r),$$

which, in turn, is less than

$$(34)\qquad e^{\lambda_n H}\,\tau_{u_v}^{r\lambda_{u_v}T}\,N^{r\lambda_{u_v}}.$$

If r takes on the value unity, let

$$(35)\qquad e^{\lambda_n H}\,\tau_s^{\lambda_s T}\,N^{\lambda_s}$$

be the expression which corresponds to (34) for $r=1$. We have

$$\lambda_n \geqq \lambda_s + \lambda_1 .$$

If r takes on values greater then unity, let

$$e^{\lambda_n H} \tau_q^{p\lambda_q T} N^{p\lambda_q} \tag{36}$$

be one of the greatest of the expressions (34), for all the values which r assumes greater than unity. We have

$$\lambda_n \geqq p\lambda_q, \qquad p \geqq 2.$$

By (21) we see that r can in no case exceed λ_n/λ_1.

By (25), (35), (36), we have, if the only value assumed by r is unity,

$$g_{10}^{(n)} \alpha_n < |b_n| + e^{\lambda_n H} \tau_s^{\lambda_s T} N^{\lambda_s}. \tag{37}$$

If the values of r all exceed unity, we have

$$g_{10}^{(n)} \alpha^n < |b_n| + e^{\lambda_n H} \tau_q^{p\lambda_q T} N^{p\lambda_q} \lambda_n/\lambda_1. \tag{38}$$

If r takes on the value unity as well as greater values, we have

$$g_{10}^{(n)} \alpha_n < |b_n| + e^{\lambda_n H} \tau_s^{\lambda_s T} N^{\lambda_s} + e^{\lambda_n H} \tau_q^{p\lambda_q T} N^{p\lambda_q} \lambda_n/\lambda_1. \tag{39}$$

We suppose that (39) holds. Obviously, if (37) or (38) holds, we can treat them in the same way.

We have, now, at least one of the following three possibilities, depending upon which of the three terms on the right in (39) is one of the greatest:

Case (a) $$g_{10}^{(n)} \alpha_n < 3|b_n|.$$

Then

$$\frac{\log \alpha_n}{\lambda_n} < \frac{\log 6M}{\lambda_n} + \frac{\log|b_n|}{\lambda_n} < \log N + H < H + T\log \tau_n + \log N.$$

Case (b) $$g_{10}^{(n)} \alpha_n < 3e^{\lambda_n H} \tau_s^{\lambda_s T} N^{\lambda_s}.$$

Then

$$\frac{\log \alpha_n}{\lambda_n} < H + \frac{T\lambda_s}{\lambda_n} \log \tau_s + \frac{\log 6MN^{\lambda_s}}{\lambda_n}.$$

Since $s<n$, therefore,

$$\frac{T\lambda_s}{\lambda_n} \log \tau_s < T\log \tau_n .$$

Since $\lambda_n \geqq \lambda_s + \lambda_1$, therefore,

$$\frac{\log 6MN^{\lambda_s}}{\lambda_n} \leqq \frac{\log 6M}{\lambda_n} + \frac{(\lambda_n - \lambda_1)\log N}{\lambda_n}.$$

By (28)

$$\lambda_1 \log N \geqq \log 6M,$$

so that,

$$\log 6M - \lambda_1 \log N \leqq 0.$$

Therefore,

$$\frac{\log 6MN^{\lambda_s}}{\lambda_n} \leqq \log N.$$

Hence (29) holds for $\nu = n$.

Case (c) $$g_{10}^{(n)}\, \alpha_n < 3e^{\lambda_n H}\, \tau_q^{p\lambda_q T}\, N^{p\lambda_q}\, \lambda_n/\lambda_1.$$

Then,

$$\frac{\log \alpha_n}{\lambda_n} < H + \frac{p\lambda_q T}{\lambda_n}\log \tau_q + \frac{1}{\lambda_n}\log\frac{6M\lambda_n}{\lambda_1} + \frac{p\lambda_q}{\lambda_n}\log N.$$

Since $\lambda_n \geqq p\lambda_q \geqq 2\lambda_q$, we have, because of (26) and (27),

$$\begin{aligned}\frac{p\lambda_q T}{\lambda_n}\log\tau_q + \frac{1}{\lambda_n}\log\frac{6M\lambda_n}{\lambda_1} &\leqq T\log\tau_q + \frac{1}{\lambda_n}\log\frac{\omega\lambda_n}{\lambda_1} \\ &< T\log\tau_q + T(\log\tau_n - \log\tau_q) \\ &= T\log\tau_n.\end{aligned}$$

Obviously, $(p\lambda_q/\lambda_n)\log N \leqq \log N$.

Thus (29) again holds for $\nu = n$.

We have thus seen that in every case (29) is true for $\nu = n$.

It is easy to see that $\log\tau_n < (5/2)\log 2$ for every n. We have shown, consequently, that

(40) $$\alpha_n \leqq e^{\lambda_n \sigma}, \qquad (n = 1, 2, \cdots),$$

where $\sigma = H + (5\omega/2\lambda_1)\log 2 + \log N$.

4. Convergence proof. Let ϵ be any fixed positive quantity and let $c = \sigma + \epsilon$. Let

$$A_n = \alpha_n e^{-\lambda_n c}, \quad B_n = |b_n|\, e^{-\lambda_n c}, \quad \sigma_{kn}^{(i)} = |g_{kn}^{(i)}|\, e^{-\lambda_n c},$$
$$(i, k = 1, 2, \cdots;\ n = 0, 1, 2, \cdots).$$

We show that there is a constant $L > 0$ such that

(41) $$A_1 + A_2 + \cdots + A_n \leqq e^{\lambda_n L}, \qquad (n = 1, 2, \cdots).$$

Consider a fixed n. If (23) holds, we have, upon multiplying through by $e^{-\lambda_n c}$

(42) $$g_{10}^{(n)} A_n = B_n.$$

If (25) holds, we have, in the same way, because of (21),

$$(43)\quad g_{10}^{(n)} A_n \leqq B_n + \sum_{t=1}^{k} [A_{i_t}^{q_t} \sigma_{q_t j_{t1}}^{(i_t)} + A_{i_t}^{q_t-1} \sigma_{q_t-1, j_{t2}}^{(i_t)} + \cdots + A_{i_t} \sigma_{1, j_{tq_t}}^{(i_t)}].$$

For every n we have an expression corresponding to one or the other of (42), (43). Let ν be any fixed positive integer and consider the expressions obtained for $n = 1, 2, \cdots, \nu$. We add up the left-hand sides and the corresponding right-hand sides of these expressions.

If all the expressions for $n = 1, 2, \cdots, \nu$ are of the form (42), we have

$$g_{10}^{(1)} A_1 + g_{10}^{(2)} A_2 + \cdots + g_{10}^{(\nu)} A_\nu = B_1 + B_2 + \cdots + B_\nu,$$

or, from (13),

$$(44)\qquad A_1 + A_2 + \cdots + A_\nu = 2M(B_1 + B_2 + \cdots + B_\nu).$$

In the contrary case, let A_{i_t} be an A which appears in the sum of the right-hand sides, and let q be one of the powers to which it appears.

Suppose $A_{i_t}^q$ appears in the expressions for which $n = v_1, v_2, \cdots, v_u$. Since each of the differences $\lambda_{v_j} - q\lambda_{i_t}$ is equal to a λ, suppose

$$\lambda_{v_j} = q\lambda_{i_t} + \lambda_{s_j}, \qquad (j = 1, 2, \cdots, u).$$

Then, in the sum of the right-hand sides, the coefficient of $A_{i_t}^q$ is

$$(45)\quad \sigma_{qs_1}^{(i_t)} + \sigma_{qs_2}^{(i_t)} + \cdots + \sigma_{qs_u}^{(i_t)} = |g_{qs_1}^{(i_t)}| e^{-\lambda_{s_1} c} + |g_{qs_2}^{(i_t)}| e^{-\lambda_{s_2} c} + \cdots + |g_{qs_u}^{(i_t)}| e^{-\lambda_{s_u} c}.$$

By (12), the right-hand side of (45) cannot exceed

$$d_{qs_1} e^{-\lambda_{s_1} c} + d_{qs_2} e^{-\lambda_{s_2} c} + \cdots + d_{qs_u} e^{-\lambda_{s_u} c},$$

which is less than

$$d_{q0} + d_{q1} e^{-\lambda_1 c} + d_{q2} e^{-\lambda_2 c} + \cdots.$$

By (14), the preceeding expression is less than unity, since

$$c = \sigma + \varepsilon > H \geqq R.$$

We have, therefore,

$$(46)\quad g_{10}^{(1)} A_1 + g_{10}^{(2)} A_2 + \cdots + g_{10}^{(\nu)} A_\nu < B_1 + B_2 + \cdots + B_\nu + \sum_{i=1}^{m} (A_i + A_i^2 + \cdots),$$

where m is some integer for which we have $\lambda_\nu \geqq \lambda_m + \lambda_1$.

Let

$$P = 1/(1 - e^{-\lambda_1 \varepsilon}).$$

Since

$$A_i = \alpha_i e^{-\lambda_i c} \leqq e^{-\lambda_i (c-\sigma)} = e^{-\lambda_i \varepsilon} \leqq e^{-\lambda_1 \varepsilon} < 1,$$

therefore,

$$A_i + A_i^2 + \cdots \leqq PA_i.$$

Hence, we have from (46), because of (13),

$$(47) \quad A_1 + A_2 + \cdots + A_\nu < 2M(B_1 + B_2 + \cdots + B_\nu) + 2MP(A_1 + A_2 + \cdots + A_m).$$

Since (16) has a half-plane of absolute convergence, the sequence $[\log(B_1 + B_2 + \cdots + B_n)]/\lambda_n$ has a finite upper bound.[6] Let Q be a non-negative quantity at least equal to the least upper bound of this sequence.

Let V be a quantity so chosen that

$$\log V \geqq (\log 4MP)/\lambda_1, \qquad \log V \geqq 0,$$

where M is the quantity defined in § 2.

We proceed by induction on n to show that

$$(48) \qquad \frac{\log(A_1 + A_2 + \cdots + A_n)}{\lambda_n} \leqq Q + \log V$$

for every n.

Since (48) obviously holds for $n = 1$, we suppose it is true for $n = 1, 2, \cdots, \nu - 1$.

From (44) and (47), we have either

$$\text{(a)} \qquad A_1 + A_2 + \cdots + A_\nu < 4M(B_1 + B_2 + \cdots + B_\nu)$$

$$\frac{\log(A_1 + A_2 + \cdots + A_\nu)}{\lambda_\nu} < \frac{\log 4M}{\lambda_\nu} + \frac{\log(B_1 + B_2 + \cdots + B_\nu)}{\lambda_\nu}$$

$$< Q + \log V,$$

or,

$$\text{(b)} \qquad \frac{\log(A_1 + A_2 + \cdots + A_\nu)}{\lambda_\nu} < \frac{\log 4MP}{\lambda_\nu} + \frac{\log(A_1 + A_2 + \cdots + A_m)}{\lambda_\nu}$$

$$< \frac{\log 4MP}{\lambda_\nu} + \frac{\lambda_m}{\lambda_\nu}(Q + \log V).$$

As

$$\frac{\log 4MP}{\lambda_\nu} + \frac{\lambda_m}{\lambda_\nu}\log V \leqq \frac{\log 4MP + (\lambda_\nu - \lambda_1)\log V}{\lambda_\nu} \leqq \log V,$$

we see that (48) holds for $n = \nu$.

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[6] From the proof of Theorem 5, E. Landau, loc. cit., p. 732–4, it can be seen that when $L = \limsup (\log |a_1 + a_2 + \cdots + a_n|)/\lambda_n$, then,

(a) if x_0 is any positive value of x for which the series $\sum a_n e^{-\lambda_n x}$ converges, we have $L \leqq x_0$,

(b) the series converges for any $x_0 > 0$ for which $x_0 > L$.

The series $\sum_{n=1}^{\infty} A_n e^{-\lambda_n z}$, therefore, converges for $x > Q + \log V$, so that the series $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n z}$ converges for $x > Q + \log V + \sigma$.

We can now complete the convergence proof. The product

$$\prod_{n=1}^{\infty} [1 + a_n g_n(z) e^{-\lambda_n z}]^{m_n}$$

$$= \prod_{n=1}^{\infty} \left[1 + m_n a_n g_n(z) e^{-\lambda_n z} + \frac{m_n(m_n - 1)}{1 \cdot 2} a_n^2 g_n^2(z) e^{-2\lambda_n z} + \cdots + a_n^{m_n} g_n^{m_n}(z) e^{-m_n \lambda_n z}\right]$$

will converge regularly if the product

$$\prod_{n=1}^{\infty} e^{|m_n a_n G_n(z) e^{-\lambda_n z}|},$$

where

$$G_n(z) = |g_{10}^{(n)}| e^{-\lambda_0 z} + |g_{11}^{(n)}| e^{-\lambda_1 z} + \cdots,$$

converges uniformly. Since $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n z}$ converges uniformly and

$$|G_n(z)| \leqq G_n(x) \leqq D'(x) \leqq 1/2$$

for $x > Q + \log V + \sigma$, our theorem is proved.

5. **The exponents in the product.** If lim sup $(\log n)/\lambda_n$ is finite, then the fact that $\alpha_n \leqq e^{\lambda_n \sigma}$ for every n is sufficient to insure the existence of a half-plane of convergence for the series $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n z}$. For, since there is then a constant $E > 0$ such that $\lambda_n > E \log(n+1)$, we have, for $x > \sigma$,

$$\sum \alpha_n e^{-\lambda_n x} \leqq \sum e^{-\lambda_n (x-\sigma)} < \sum \frac{1}{(n+1)^{(x-\sigma)E}}.$$

We show that if lim sup $(\log n)/\beta_n$ is finite, then lim sup $(\log n)/\lambda_n$ is finite.

Consider any $\lambda_n > 0$. Since it is a linear combination of the β's with positive integral coefficients, it can be expressed in the form

$$\delta_1 \beta_{p_1} + \delta_2 \beta_{p_2} + \cdots + \delta_m \beta_{p_m}, \tag{49}$$

where $\delta_1, \delta_2, \cdots, \delta_m$ are positive integers. If there are several expressions of this form for λ_n, we fix our attention upon a particular one.

Since lim sup $(\log n)/\beta_n$ is finite, there is a constant $D > 0$ such that $\beta_n > D \log(n+1)$ for every n.

If to the λ_n with the expression (49) we make correspond the integer

$$p = (p_1 + 1)^{\delta_1} (p_2 + 1)^{\delta_2} \cdots (p_m + 1)^{\delta_m},$$

we have

$$\lambda_n > D\delta_1 \log(p_1+1) + D\delta_2 \log(p_2+1) + \cdots + D\delta_m \log(p_m+1) = D \log p.$$

Given any fixed p there are at most as many λ's to which this p corresponds as there are ways, $N(p)$, of writing p in the form $\gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\cdots\gamma_s^{\varepsilon_s}$, where the γ's and ε's are positive integers and $\gamma_i \geqq 2$. We show that $N(p) \leqq p^2$.

Certainly $N(p) \leqq p^2$ for $p = 2$. Hence, we proceed by induction on p, supposing $N(i) \leqq i^2$ for $i = 2, 3, \cdots, p-1$.

If $\gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\cdots\gamma_s^{\varepsilon_s}$ is a representation of p in the desired form, where $\gamma_i \nmid p$, then $\gamma_1^{\varepsilon_1-1}\gamma_2^{\varepsilon_2}\cdots\gamma_s^{\varepsilon_s}$ is a representation of a divisor of p in this same form. Also, two distinct representations of p induce either representations of different divisors of p or different representations of the same divisor. Thus, if $d_1, d_2, \cdots, d_r$ are all the distinct divisors of p different from unity,

$$N(p) \leqq 1 + N(p/d_1) + N(p/d_2) + \cdots + N(p/d_r) < 1 + p^2(1/2^2 + 1/3^2 + \cdots)$$
$$= 1 + p^2(\pi^2/6 - 1) < p^2$$

for $p > 2$.

We put all the λ's which are in correspondance with a given p into one group and order the groups according to increasing p's, while ordering the particular members of each group in any way at all. We let the λ's so ordered be called $\xi_1, \xi_2, \cdots, \xi_n, \cdots$.

If the integer corresponding to ξ_n is p, the number of ξ's preceeding ξ_n is at most $2^2 + 3^2 + \cdots + p^2 < p^3$. Hence, $(\log n)/\xi_n \leqq \log p^3/D \log p = 3/D$. Thus, $\xi_n \geqq E \log n$ for every n, where $E = D/3$.

The sequence $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ is simply a rearrangement of the sequence $\xi_1, \xi_2, \cdots, \xi_n, \cdots$. Hence, for every n there is a $\varkappa$ such that $\lambda_n = \xi_\varkappa$.

If $\varkappa \geqq n$, then $\lambda_n = \xi_\varkappa \geqq E \log \varkappa \geqq E \log n$.

If $\varkappa < n$, then among $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ there is a $\lambda_j = \xi_i$ with $i \geqq n$. Then $\lambda_n > \lambda_j = \xi_i \geqq E \log i \geqq E \log n$.

Thus, lim sup $(\log n)/\lambda_n \leqq 3/D$.

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NOTE ON AN INFINITE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS.[1]

By William T. Reid.[2]

In a recent paper[3] the author has treated an infinite system of ordinary linear differential equations of the first order for which the solution is a vector in Hilbert space. It is the object of this note to show that the results there obtained may be extended to the more general system

$$\frac{dy_i}{dx} = \sum_{j=1}^{\infty} A_{ij}(x) y_j \qquad (i = 1, 2, \cdots), \tag{1}$$

where $A_{ij}(x)$ $(i, j = 1, 2, \cdots)$ is a Lebesgue summable function on the interval $X: 0 \leqq x \leqq 1$ and furthermore there is a non-negative summable function $\varphi(x)$ on X such that for every integer n and all real quantities $\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_n$ we have,

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, \xi_i\, \eta_j \right| \leqq \varphi(x) \left(\sum_{i=1}^{n} |\xi_i|^q \right)^{1/q} \left(\sum_{j=1}^{n} |\eta_j|^p \right)^{1/p},$$

where $p > 1$ and $q = p/(p-1)$. A solution of (1) is defined as a set of absolutely continuous functions $y_1(x), y_2(x), \cdots$ which satisfy (1) "almost everywhere" on X and such that $\sum_{i=1}^{\infty} |y_i(x)|^p$ is bounded by a finite quantity uniformly on X.

1. Matrix Properties.

If $\eta_1, \eta_2, \cdots$ is a countable set of real numbers such that $\sum_{i=1}^{\infty} |\eta_i|^p$ $(p > 1)$ is a finite quantity, then $\eta \equiv (\eta_i)$ is said to be a point or a vector in the space R_p and the positive real quantity $\left(\sum_{i=1}^{\infty} |\eta_i|^p \right)^{1/p}$ is the length or modulus of η and will be denoted by $M_p[\eta]$. For $p = 2$ we have Hilbert real space of an infinite number of dimensions. The space R_q, where

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[2] National Research Fellow in Mathematics.

[3] W. T. Reid, Transactions of the American Mathematical Society, vol. 32 (1930), pp. 284–318. This paper will be referred to as A.

$q = p/(p-1)$, will be called the space complementary to R_p. If $\eta \equiv (\eta_i)$ and $\xi \equiv (\xi_i)$ are vectors in R_p and R_q respectively, we have the Cauchy-Hölder inequality[4]

$$(1.1) \qquad \left| \sum_{i=1}^{\infty} \xi_i \eta_i \right| \leqq M_p[\eta]\, M_q[\xi].$$

For $\eta^{(1)} \equiv (\eta_i^{(1)})$ and $\eta^{(2)} \equiv (\eta_i^{(2)})$ two vectors in R_p the vector $\eta^{(1)} + \eta^{(2)} \equiv (\eta_i^{(1)} + \eta_i^{(2)})$ is defined and we have the inequality[5]

$$(1.2) \qquad M_p[\eta^{(1)} + \eta^{(2)}] \leqq M_p[\eta^{(1)}] + M_p[\eta^{(2)}].$$

All the matrices considered in this paper are supposed to be real and to have an infinite number of rows and columns. Matrices are denoted by capital letters and the element in the i-th row and the j-th column is denoted by the same letter with the subscript ij. The matrix O is the matrix each of whose elements is zero and E is used to denote the unit matrix $E \equiv \|E_{ij}\|$, where $E_{ij} = 0$ if $i \neq j$, $E_{ii} = 1$.

Definition. If for the matrix $A \equiv \|A_{ij}\|$ there is a constant k such that for every integer n and for all real quantities $\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_n$ we have

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \xi_i \eta_j \right| \leqq k \left(\sum_{i=1}^{n} |\xi_i|^q \right)^{1/q} \left(\sum_{j=1}^{n} |\eta_j|^p \right)^{1/p},$$

then A is a matrix of class L_p and is said to be bounded by the constant k.

For $p = 2$ we have the class of limited matrices as defined by Hilbert.[6] Some of the elementary properties of matrices of class L_p may be established in the same manner that has been used by Hellinger and Toeplitz[7] to prove the corresponding properties for limited matrices. Clearly the sum of two matrices of class L_p is also a matrix of class L_p. If A is a matrix of class L_p and bounded by k, then every row of A is a vector in R_q

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[4] O. Hölder, Göttinger Nachrichten, 1889, pp. 38–47. See also F. Riesz, *Les systèmes d'équations linéaires*, Paris, 1913, p. 45.

[5] See Riesz, loc. cit., p. 45. For a finite number of terms this inequality was established by Minkowski. See Minkowski, *Diophantische Approximationen*, Leipzig, 1907, p. 95.

[6] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Berlin, 1912, p. 147.

[7] E. Hellinger und O. Toeplitz, Mathematische Annalen, vol. 69 (1910), pp. 289–330. See also F. Riesz, loc. cit., pp. 78–121. The referee has kindly pointed out that the theory of bilinear forms in the space R_p is but a very special case of the general results of Radon. See J. Radon, *Theorie und Anwendung der absolut additiven Mengenfunktionen*, Wiener Sitzungsberichte, vol. 122 (1913), pp. 1295–1438, in particular pp. 1351–1412.

whose modulus is at most k, and every column of A is a vector in R_p of modulus at most k. Furthermore, if η is a vector in R_p, then $v \equiv (v_i)$, where $v_i = A_{i\alpha}\,\eta_\alpha$[8] $(i = 1, 2, \cdots)$, is a vector in R_p whose modulus does not exceed $k M_p[\eta]$ and is denoted by $A\eta$. Similarly, if ξ is a vector in R_q, then $u \equiv (u_i)$, where $u_i = A_{\alpha i}\,\xi_\alpha$ $(i = 1, 2, \cdots)$, is a vector in R_q whose modulus is not greater than $k M_q[\xi]$ and is denoted by ξA. If A and B are two matrices of class L_p and bounded by k_1 and k_2 respectively, then the product matrix $AB \equiv \|A_{i\alpha}\,B_{\alpha j}\|$ is of class L_p and is bounded by $k_1 k_2$; also, for matrices of class L_p multiplication is associative.

Definition. The matrix B is a right-hand [left-hand] reciprocal of the matrix A of class L_p if B is of class L_p and $AB = E[BA = E]$.

As for limited matrices[9] we have that if A is a matrix of class L_p and has a left-hand reciprocal and also a right-hand reciprocal, then it has a unique left-hand and right-hand reciprocal and they are equal, in which case A is said to possess a unique reciprocal; also, if A has a unique left-hand reciprocal or a unique right-hand reciprocal, then A has a unique reciprocal.

We may prove the following theorem:[10]

THEOREM 1.1. *If A is a matrix of class L_p, then a necessary and sufficient condition that A have a unique reciprocal is that there exist a positive constant k such that for all vectors η of R_p we have $M_p[A\eta] \geqq k M_p[\eta]$ and for all vectors ξ* of *R_q, $M_q[\xi A] \geqq k M_q[\xi]$.*

Definition. A matrix $A \equiv \|A_{ij}\|$ is of class C_p if for every positive ϵ there exists a positive $N(\epsilon)$ such that if ξ and η are any pair of vectors in R_q and R_p respectively whose moduli are not greater than unity, then for all integers $m, n \geqq N(\epsilon)$ we have,

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\,\xi_i\,\eta_j - \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}\,\xi_i\,\eta_j \right| < \epsilon.$$

Clearly every matrix of class C_p is also a matrix of class L_p. For $p = 2$ we have the class of "vollstetig" or completely continuous matrices as defined by Hilbert. If A is a matrix of class C_p and B is any matrix of

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[8] The repetition of a subscript is used to denote summation with respect to that subscript over all positive integral values, i. e., $A_{i\alpha}\,\eta_\alpha = A_{i1}\,\eta_1 + A_{i2}\,\eta_2 + \cdots$.

[9] Hellinger and Toeplitz, loc. cit., p. 311. Radon does not introduce the analogue of the right-hand and left-hand reciprocals separately, but only the idea of the unique reciprocal.

[10] See Radon, loc. cit., p. 1397. For $p = 2$ the theorem follows from a more general theorem established by Hyslop. See J. Hyslop, Proceedings of the London Mathematical Society, vol. 24 (1925), pp. 264–304. See also Riesz, loc. cit., p. 86; also p. 61. The method used by Riesz may be extended to the general case $p > 1$.

class L_p, it follows that AB and BA are also matrices of class C_p[11]. We then have the following theorem:[12]

THEOREM 1.2. *If A is a matrix of class C_p, then there are at most a finite number of linearly independent solutions of the vector equation $(E+A)u=0$, where u is a vector in R_p, and the number of linearly independent solutions of the systems $(E+A)u=0$ and $v(E+A)=0$, where v is a vector in R_q, is the same; furthermore, a necessary and sufficient condition that $(E+A)u=0$ have only the identically vanishing solution is that the matrix $(E+A)$ have a unique reciprocal.*

If $A \equiv \|A_{ij}\|$ is an infinite matrix and r is a positive integer, we say that the matrix $B \equiv \|B_{ij}\|$ is formed by omitting the rth column of A if $B_{ij} = A_{ij} (j<r;\ i=1,2,\cdots)$ and $B_{ij} = A_{i,j+1} (j \geqq r;\ i=1,2,\cdots)$. In a corresponding way we may define a matrix formed by omitting a row of A. The idea may be extended readily to defining a matrix formed by omitting a finite or an infinite number of rows and columns of A. We may prove, as in the case $p=2$[13], the following

THEOREM 1.3. *If A is a matrix of class C_p and $(E+A)u=0$ has exactly n linearly independent solutions, then no infinite matrix formed by omitting s columns, $s<n$, and by omitting a countable number of rows has a unique reciprocal; furthermore, there exists a matrix formed by omitting n rows and n columns which possesses a unique reciprocal.*

2. THE DIFFERENTIAL SYSTEM.

a. **Existence theorem and properties of solutions.** We now consider the infinite set of ordinary linear differential equations of the first order written in vector form as

$$y'(x) = A(x)\,y(x) + b(x), \tag{2.1}$$

where $A(x)$ is a matrix bounded in R_p by a non-negative summable function $\varphi(x)$ on $X: 0 \leqq x \leqq 1$ and $b(x)$ is a vector in R_p which is, together with $M_p[b(x)]$, summable on X. For $b(x) \equiv 0$ we have,

$$y'(x) = A(x)\,y(x), \tag{2.2}$$

which is the homogeneous differential system (1). The equation (2.1) may be written in the equivalent integral form

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[11] See Radon, loc. cit., p. 1411. For the case $p=2$, see Hilbert, loc. cit., p. 152 and Riesz, loc. cit., p. 97.

[12] See Radon, loc. cit., pp. 1401–1412. For $p=2$, see Hilbert, loc. cit., pp. 165–169 and also Riesz, loc. cit., pp. 98–102. The method of proof used by Riesz may be extended quite readily to the general case $p>1$.

[13] Reid, loc. cit., pp. 307–308.

$$y(x) = \int_0^x [A(t)\, y(t) + b(t)]\, dt + y(0). \tag{2.1'}$$

The homogeneous differential equation

$$z'(x) = -z(x)\, A(x), \tag{2.3}$$

where $z(x)$ is a vector in R_q for x on X, is called the equation adjoint to the homogeneous equation (2.2).

Definitions. If $Y(x) \equiv \| Y_{ij}(x) \|$ is a matrix each column of which is a solution of (2.2), then $Y(x)$ is a *matrix* of (2.2). Similarly, if $Z(x)$ is a matrix each row of which is a solution of (2.3), then $Z(x)$ is a *matrix* of the adjoint equation. If for each value of x on X the matrices $Y(x)$ and $Z(x)$ are of class L_p, then $Y(x)$ and $Z(x)$ are *bounded matrices* of (2.2) and (2.3) respectively. In particular, $Y(x)$ is the *principal matrix* of (2.2) at a point x_0 of X if $Y(x_0) = E$. Likewise, $Z(x)$ is the *principal matrix* of (2.3) at a point x_0 if $Z(x_0) = E$.

We prove the following general existence theorem:

THEOREM 2.1. *If $B(x)$ is a matrix each element of which is summable on X and the matrix $\int_0^x B(t)\, dt$ is for each x on X of class L_p and is bounded by a finite constant uniformly with respect to x on X, and C is a constant matrix of class L_p, then there exists a unique "matrix $Y(x)$ of absolutely continuous functions" such that*

$$Y(x) = \int_0^x [A(t)\, Y(t) + B(t)]\, dt + C, \tag{2.4}$$

and $Y(x)$ is of class L_p and bounded uniformly on X by a finite constant.

For let

$$\begin{aligned} Y^{(0)}(x) &= \int_0^x B(t)\, dt + C, \\ Y^{(n)}(x) &= \int_0^x [A(t)\, Y^{(n-1)}(t) + B(t)]\, dt + C \qquad (n = 1, 2, \cdots), \\ H^{(0)}(x) &= Y^{(0)}(x), \\ H^{(n)}(x) &= Y^{(n)}(x) - Y^{(n-1)}(x) \qquad (n = 1, 2, \cdots). \end{aligned} \tag{2.5}$$

Then

$$Y^{(n)}(x) = \sum_{\alpha=0}^{n} H^{(\alpha)}(x),$$

and

$$H^{(n)}(x) = \int_0^x A(t)\, H^{(n-1)}(t)\, dt \qquad (n = 1, 2, \cdots). \tag{2.6}$$

In the proof of Theorem 2.1 we make repeated use of the following lemma which we state without proof:

LEMMA. *If $M(x)$ is a matrix each element of which is summable on X and $M(x)$ is bounded in R_p by the summable function $\Phi(x)$, then for every x on X the matrix $\int_0^x M(t)\,dt$ is of class L_p and is bounded by the quantity $\int_0^x \Phi(t)\,dt$.*[14]

Since the matrix $\int_0^x B(t)\,dt$ is bounded by a finite constant independent of the value of x on X, there exists a finite constant k which bounds in R_p the matrix $\int_0^x B(t)\,dt + C$ independent of x on X. Now by (2.6),

$$H^{(1)}(x) = \int_0^x A(t)\,H^{(0)}(t)\,dt,$$

and since $A(x)$ is bounded by $\varphi(x)$ and $H^{(0)}(x)$ is bounded by k, we have by the above lemma that $H^{(1)}(x)$ is bounded by $k\int_0^x \varphi(t)\,dt$. By mathematical induction we may show that $H^{(n)}(x)$ is bounded in R_p by

$$k\left[\int_0^x \varphi(t)\,dt\right]^n \Big/ n! \qquad (n = 1, 2, \cdots).$$

Therefore $\sum_{\alpha=0}^{\infty} H^{(\alpha)}(x) = \lim_{n\to\infty} Y^{(n)}(x)$ exists and is bounded in R_p by $k \exp\left[\int_0^1 \varphi(t)\,dt\right]$ uniformly on X. Let $Y(x) = \lim_{n\to\infty} Y^{(n)}(x)$. Then since $Y(x) - Y^{(n)}(x) \equiv \sum_{\alpha=1}^{\infty} H^{(n+\alpha)}(x)$ is bounded by $k\sum_{\alpha=1}^{\infty}\left[\int_0^1 \varphi(t)\,dt\right]^{n+\alpha}/(n+a)!$, it follows that the matrix $\int_0^x A(t)\,[Y(t) - Y^{(n)}(t)]\,dt$ is bounded for every x by the constant $k\sum_{\alpha=1}^{\infty}\left[\int_0^1 \varphi(t)\,dt\right]^{n+1+\alpha}/(n+\alpha)!$ and therefore

$$\lim_{n\to\infty}\int_0^x A(t)\,Y^{(n)}(t)\,dt = \int_0^x A(t)\,Y(t)\,dt. \tag{2.7}$$

Hence $Y(x)$ is a solution of (2.4).

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[14] In particular, let $M_{i1} = f_i(x)$, $M_{ij} = 0$ if $j \neq 1$, where $f(x) \equiv (f_i(x))$ is a vector in R_p which is, together with $M_p[f(x)]$, summable on X. Then for every x on X, $M(x)$ is of class L_p and bounded by $M_p[f(x)]$. Then by the above lemma the matrix $\int_0^x M(t)\,dt$ is bounded by $\int_0^x M_p[f(t)]\,dt$ and therefore $\left(\sum_{\alpha=1}^{\infty}\left[\int_0^x f_\alpha(t)\,dt\right]^p\right)^{1/p} \equiv M_p\left[\int_0^x f(t)\,dt\right] \leqq \int_0^x M_p[f(t)]\,dt$. This inequality is of use in proving some of the theorems stated in this paper.

Let $B_r(x)$ and C_r denote the vectors $(B_{\alpha r}(x))$ and $(C_{\alpha r})$ respectively and $Y_r^*(x) \equiv (Y_{\alpha r}^*(x))$ be any solution of the vector differential equation

$$y(x) = \int_0^x [A(t)\, y(t) + B_r(t)]\, dt + C_r.$$

Then $Y^{**}(x) \equiv \| Y_{ij}^{**}(x) \|$, where $Y_{ir}^{**} = Y_{ir}^*$ $(i = 1, 2, \cdots)$ and $Y_{ij}^{**} = Y_{ij}$ $(i = 1, 2, \cdots)$ if $j \neq r$ and where $Y_{ij}(x)$ is determined as above by the method of successive approximations, is a matrix which is a solution of (2.4) and is bounded uniformly on X by a finite constant k^{**}. Furthermore, $Y^{(0)}(x) - Y^{**}(x)$ is bounded uniformly on X by the constant $k + k^{**}$, and since

$$Y^{(n)}(x) - Y^{**}(x) = \int_0^x A(t)\, [Y^{(n-1)}(t) - Y^{**}(t)]\, dt \qquad (n = 1, 2, \cdots),$$

it follows that for every n the matrix $Y^{(n)}(x) - Y^{**}(x)$ is bounded by $[k + k^{**}] \left[\int_0^x \varphi(t)\, dt\right]^n \Big/ n!$, and therefore $Y^{**}(x) \equiv Y(x)$ and the solution of (2.4) is unique.

COROLLARY 1. *There exists a unique solution of the equation* (2.1) *satisfying the condition* $y(0) = a$, *where* a *is an arbitrary constant vector in* R_p.

COROLLARY 2. *If* $y(x)$ *is a solution of* (2.1), *then* $y(x)$ *is a vector which is strongly continuous*[15] *on* X.

COROLLARY 3. *If* $y(x)$ *is a solution of* (2.1), *then* $M_p[y(x)]$ *is an absolutely continuous function on* X.

COROLLARY 4. *If* $Y(x)$ *is a matrix of the equation* (2.2) *and for some point* x_0 *of* X *the matrix* $Y(x_0)$ *is bounded by the constant* l, *then* $Y(x)$ *is a bounded matrix of* (2.2) *and is bounded uniformly on* X *by* $l \exp \left[\int_0^1 \varphi(t)\, dt\right]$.

In a manner analogous to that used[16] to prove the corresponding theorems for the differential system in Hilbert space, we may prove:

THEOREM 2.2. *If* $Y(x)$ *and* $Z(x)$ *are bounded matrices of* (2.2) *and* (2.3) *respectively, then the matrix* $Z(x)\, Y(x)$ *is a constant matrix on* X.

THEOREM 2.3. *If* $Y(x)$ *is a bounded matrix of* (2.2) *which for some point of* X *has a unique reciprocal, then there exists a matrix* $Z(x)$ *which is the unique reciprocal of* $Y(x)$ *on* X *and* $Z(x)$ *is a bounded matrix of* (2.3).

Definition. If $Y(x)$ is a bounded matrix of (2.2) and has a unique reciprocal for each value of x on X, then $Y(x)$ is a *matrix solution* of (2.2). A *matrix solution* of the adjoint system (2.3) is defined in a corresponding manner.

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[15] A vector $y(x)$ is said to be strongly continuous on an interval X if for every point x_0 of X we have $\lim_{x \to x_0} M_p[y(x) - y(x_0)] = 0$.

[16] See paper A, pp. 292–294.

THEOREM 2.4. *If $Y(x)$ is a bounded matrix of* (2.2) *and C is a constant matrix of class L_p, then $Y(x)C$ is a bounded matrix of* (2.2).

THEOREM 2.5. *If $Y(x)$ is a matrix solution of* (2.2), *the most general bounded matrix of* (2.2) *is of the form $Y(x)C$, where C is a constant matrix of class L_p.*

If the matrix $A(x)$ of (2.2) is also a function of a parameter μ, then properties of the solution $y(x;\mu)$ as a function of μ may be studied as in paper A for the corresponding differential equation in Hilbert space. Also, if with the vector differential equation we associate two-point boundary conditions of the form

$$My(0)+Ny(1) = 0,$$

where M and N are constant matrices of class L_p such that there exists a matrix formed by omitting an infinite number of columns of $\|M; N\|$[17] which has a unique reciprocal, adjoint boundary conditions may be determined. We state here, however, only a few properties of a more specialized system which may be proved with the aid of the theorems given in § 1.

b. **A boundary value problem.** *Definition.* A matrix $A(x) \equiv \|A_{ij}(x)\|$ is said to be a matrix of class C_p^* if: (a) $A_{ij}(x)$ $(i, j = 1, 2, \cdots)$ is a Lebesgue summable function on $X: 0 \leqq x \leqq 1$ and $A(x)$ is bounded in R_p by a non-negative summable function $\varphi(x)$ on X; (b) for values of x "almost everywhere" on X the matrix $A(x)$ is a matrix of class C_p.

We now consider the homogeneous vector differential equation

$$y'(x) = A(x)\,y(x), \tag{2.8}$$

in which the matrix $A(x)$ is of class C_p^* on X. This equation is a special case of (2.2). As in the case $p = 2$,[18] we may prove:

THEOREM 2.6. *If $Y(x)$ is the principal matrix of* (2.8) *at $x = 0$, then $Y(x) = E + R(x)$, where $R(x)$ is a matrix of class C_p for each x on X.*

With (2.8) we associate the boundary conditions

$$(E+G_0)y(0)+\delta(E+G_1)y(1) = 0, \tag{2.9}$$

where G_0 and G_1 are constant matrices of class C_p such that $E+G_0$ and $E+G_1$ have unique reciprocals, and δ is a real constant such that $0 \neq \delta \neq -1$. It follows that the reciprocals of $E+G_0$ and $E+G_1$ are of the form $E+T_0$ and $E+T_1$ respectively, where T_0 and T_1 are of class C_p.

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[17] If M and N are matrices of class L_p, then we denote by $\|M; N\|$ the matrix $U \equiv \|U_{ij}\|$, where $U_{i,2j-1} = M_{ij}$ and $U_{i,2j} = N_{ij}$ $(i, j = 1, 2, \cdots)$.

[18] See paper A, pp. 308-311.

Since from Theorem 2.5 the general solution of (2.8) is of the form $Y(x)c$, where $Y(x)$ is the principal matrix of (2.8) at $x = 0$ and c is a constant vector in R_p, if the system (2.8), (2.9) is compatible it is necessary and sufficient that there exist a non-null vector c in R_p such that $[(E+G_0)Y(0)+\delta(E+G_1)Y(1)]c=0$. Since $[G_0+\delta(G_1+R(1)+G_1R(1))]$ is a matrix of class C_p, we have from Theorem 1.2,

THEOREM 2.7. *A necessary and sufficient condition that* (2.8), (2.9) *be incompatible is that the matrix* $E+Q$, *where* $Q=[G_0+\delta(G_1+R(1)+G_1R(1))]/(1+\delta)$, *have a unique reciprocal.*[19]

In view of Theorem 1.2 we have

THEOREM 2.8. *The system* (2.8), (2.9) *has at most a finite number of linearly independent solutions.*

The non-homogeneous system corresponding to (2.8), (2.9) is

$$y'(x) = A(x)y(x)+b(x), \tag{2.10}$$

$$(E+G_0)y(0)+\delta(E+G_1)y(1) = h, \tag{2.11}$$

where $b(x)$ is "almost everywhere" on X a vector in R_p and is, together with $M_p[b(x)]$, summable on X, and $h \equiv (h_\alpha)$ is a constant vector in R_p. We have in view of Theorem 2.7,

THEOREM 2.9. *A necessary and sufficient condition that the non-homogeneous system* (2.10), (2.11) *have a unique solution is that the homogeneous system* (2.8), (2.9) *be incompatible.*

With the adjoint differential equation

$$z'(x) = -z(x)A(x) \tag{2.12}$$

we associate the adjoint boundary conditions

$$z(0)(E+T_0)\delta+z(1)(E+T_1) = 0. \tag{2.13}$$

By Theorem 2.6 the principal matrix $Z(x)$ of 2.12 at $x=0$ is of the form $E+R^*(x)$, where $R^*(x)$ is a matrix which is of class C_p for each x on X. If the system (2.12), (2.13) is compatible there exists a non-null vector c in R_q such that $c[Z(0)(E+T_0)\delta+Z(1)(E+T_1)]=0$. Also we have that $[Z(0)(E+T_0)\delta+Z(1)(E+T_1)] = (1+\delta)[E+Q^*]$, where $Q^*=[\delta T_0+T_1+R^*(1)+R^*(1)T_1]/(1+\delta)$ and is of class C_p. It is easy to show that if $[(E+G_0)Y(0)+\delta(E+G_1)Y(1)]$ has a unique reciprocal V,

[19] In paper A the system (2.8), (2.9) is considered for $p=2$, only in the case $\delta=1$; however, it is readily seen that the method of proof there used is valid for all real values of δ distinct from 0 and -1.

then the matrix $[Z(0)(E+T_0)\delta+Z(1)(E+T_1)]$ has a unique reciprocal given by $(E+G_0)Y(0)V(E+G_1)Y(1)$. Hence we have

THEOREM 2.10. *A necessary and sufficient condition that the adjoint system* (2.12), (2.13) *be compatible is that the system* (2.8), (2.9) *be compatible.*

By a method similar to that used in the finite case,[20] and making use of Theorems 1.2 and 1.3 we may prove

THEOREM 2.11. *The number of linearly independent solutions of the system* (2.12), (2.13) *is the same as the number of linearly independent solutions of the system* (2.8), (2.9).

Whenever the system (2.8), (2.9) is incompatible it may be shown, as in paper A for the differential system in Hilbert space, that a Green's matrix may be defined for the system.

[20] See G. A. Bliss, Transactions of the American Mathematical Society, vol. 28 (1926), p. 566.

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ON THE WAVE EQUATION OF THE HYDROGEN ATOM.*

By T. H. Gronwall.

This note gives a quite elementary derivation of the (non-relativistic) set of energy levels and characteristic functions of the hydrogen atom, and it is also proved that the set obtained is complete.

1. **Introduction.** The non-relativistic wave equation of the hydrogen atom is, in the customary notation,

$$\nabla^2 \psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{e^2}{r}\right)\psi = 0, \tag{1}$$

and those values of the constant E (energy levels or characteristic constants) are to be determined for which there exist solutions of (1) (characteristic functions) *which are single-valued and finite everywhere.*

Schrödinger[1] solves this problem by introducing polar coördinates r, θ, φ, whereby the equation becomes separable, and writing $\psi = \chi(r)\, u(\theta, \varphi)$, where $u(\theta, \varphi)$ is any one of the surface harmonics of the unit sphere

$$\begin{aligned} u &= P_n^{\mu}(\cos\theta)\cos\mu\varphi, \quad 0 \leqq \mu \leqq n, \\ u &= P_n^{\mu}(\cos\theta)\sin\mu\varphi, \quad 1 \leqq \mu \leqq n, \end{aligned} \qquad n = 0, 1, 2, \cdots, \tag{2}$$

(1) yields the following differential equation for $\chi(r)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\chi}{dr}\right) + \left(\frac{8\pi^2 m E}{h^2} + \frac{8\pi^2 m e^2}{h^2 r} - \frac{n(n+1)}{r^2}\right)\chi = 0. \tag{3}$$

The transformation of Laplace is now used to determine the characteristic constants and functions of (3), and the characteristic functions of (1) are then obtained by multiplying those of (3) by the $2n+1$ spherical harmonics (2), and making $n = 0, 1, 2, \cdots$ successively. As Schrödinger[2] points out, this method labors under the disadvantage of not showing that the set of energy levels and characteristic functions thus obtained is complete.

The present note avoids the somewhat complicated use of the Laplace transformation, and derives the set of energy levels and characteristic functions of (1) by a short and quite elementary process, which is so arranged as to prove at the same time the completenes of the set.

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[1] E. Schrödinger, Ann. d. Phys. *79*, 361, 1926.

[2] E. Schrödinger, l. c. 1), middle of page 370.

2. Proof that the complete set of energy levels and characteristic functions of (1) is obtainable from the corresponding set of (3). The foundation of this proof is the well known fact that the harmonics (2) form a closed set of orthogonal functions on the unit sphere, that is, when a continuous function $f(\theta, \varphi)$ of the position on the unit sphere is such that, $dS = \sin\theta \, d\theta \, d\varphi$ being the surface element of the unit sphere, the coefficients

$$\iint f(\theta, \varphi) \, u \, dS \tag{4}$$

of the expansion of $f(\theta, \varphi)$ in a series of spherical harmonics vanish for *all* the u's defined by (2), then $f(\theta, \varphi)$ is identically zero.[3] Applying this to the left hand member of (1), it is seen that the wave equation is equivalent to the statement that the equation

$$\iint \left[\nabla^2 \psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{e^2}{r}\right)\psi\right] u \, dS = 0 \tag{5}$$

holds for every u defined by (2) (and of course, ψ being a function of r also, for every positive value of r). The expansion of any characteristic function ψ of (1) in spherical harmonics is

$$\psi = \sum \chi(r) \cdot u(\theta, \varphi) \Big/ \iint u^2 \, dS, \tag{6}$$

the sum extending over all the u's defined by (2) and the $\chi(r)$ corresponding to each u being, by (4),

$$\chi(r) = \iint \psi \, u \, dS. \tag{7}$$

The series (6) being convergent,[4] it is evident that a complete set of characteristic functions of (1) is given by the expressions

$$\chi(r) \cdot u(\theta, \varphi), \tag{8}$$

where $u(\theta, \varphi)$ takes all the values (2), the corresponding $\chi(r)$ being given by (7), and in the latter equation, ψ runs through all the single valued

[3] This closure property of the spherical harmonics may be proved in many ways. A proof by integral equations is given in Courant and Hilbert, Methoden der mathematischen Physik, vol. 1, p. 421. The simplest proof, however, follows from the summability of Laplace's series by Cesàro means of the second order; see Fejér, Math. Ann. 67, 76, 1909 and Math. Zs. 24, 267, 1925.

[4] Since, by (1), ψ must have partial derivatives of the second order. For the proof compare Courant and Hilbert, l. c. [3]), p. 422, or C. Jordan, Cours d'Analyse, vol. II, chapter 5.

and everywhere finite solutions of (5) (which we have just shown to be equivalent to (1)).

After these somewhat tedious preliminaries, we come to the gist of the completeness proof, namely that (5) is equivalent to (3), provided that χ is defined by (7). This may be shown by expressing $\nabla^2\psi$ in polar coördinates in (5), noting that $dS = \sin\theta\, d\theta\, d\varphi$ and performing a few integrations by parts. A neater method is however the following. Applying Green's formula to the spherical shell $r_1 \leqq r \leqq r_2$, we obtain the identity in ψ and u

$$\iiint (u\nabla^2\psi - \psi\nabla^2 u)\, r^2\, dr\, dS$$
$$= \iint_{r=r_2}\left(u\frac{\partial\psi}{\partial r} - \psi\frac{\partial u}{\partial r}\right) r^2\, dS - \iint_{r=r_1}\left(u\frac{\partial\psi}{\partial r} - \psi\frac{\partial u}{\partial r}\right) r^2\, dS.$$

Now let u be any one of the functions (2); then $\partial u/\partial r = 0$ and $\nabla^2 u = -n(n+1)u/r^2$ (since $\nabla^2 r^n u = 0$), moreover, (7) shows that

$$\chi'(r) = \iint \frac{\partial\psi}{\partial r} u\, dS,$$

and the preceding identity becomes

$$r_2^2\chi'(r_2) - r_1^2\chi'(r_1) - \iiint\left(\nabla^2\psi + \frac{n(n+1)}{r^2}\psi\right) u r^2\, dr\, dS = 0.$$

Divide by $r_2 - r_1$ and let r_1 and r_2 approach the limit r:

$$\frac{d}{dr}\left(r^2\frac{d\chi(r)}{dr}\right) - r^2\iint\left(\nabla^2\psi + \frac{n(n+1)}{r^2}\psi\right) u\, dS = 0.$$

Dividing this identity by r^2, adding to (5) and using (7), we obtain (3), which equation is therefore equivalent to (5).

Consequently, we obtain a complete set of characteristic functions of (1) by substituting for $\chi(r)$ in (8) a complete set of characteristic functions of (3).

The singular points of (3) are $r=0$ and $r=\infty$, and the exponent ϱ to which a solution of (3) belongs at $r=0$ is determined by $\varrho(\varrho+1) - n(n+1) = 0$, so that $\varrho = n$ or $\varrho = -n-1$. Any solution of (3) which is finite at $r=0$ therefore belongs to the exponent n and is uniquely determined except for an arbitrary constant factor. Consequently we have only to determine the values of E for which this solution of (3) remains finite as r increases indefinitely, and here we must distinguish the two cases $E \geqq 0$ and $E < 0$.

3. The continuous spectrum, $E \geqq 0$. Writing

$$f(r) = \frac{8\pi^2 mE}{h^2} + \frac{8\pi^2 me^2}{h^2 r} - \frac{n(n+1)}{r^2}, \tag{9}$$

and

(10) $$r\chi(r) = v,$$

equation (3) becomes

$$\frac{d^2 v}{dr^2} + f(r)\, v = 0;$$

multiplying by $2\,dv/dr$ and integrating from r_0 to r, we find

$$\left(\frac{dv}{dr}\right)^2 - \left(\frac{dv}{dr}\right)^2_{r=r_0} + \int_{r_0}^{r} 2v\frac{dv}{dr} f(r)\, dr = 0,$$

or integrating by parts

(11) $$\left(\frac{dv}{dr}\right)^2 + f(r)\, v^2 = \left[\left(\frac{dv}{dr}\right)^2 + f(r)\, v^2\right]_{r=r_0} + \int_{r_0}^{r} f'(r)\, v^2\, dr.$$

When r_0 is sufficiently large, it follows from (9) that $f(r) > 0$ and $f'(r) < 0$ for $r > r_0$, so that the integral in (11) is negative, and hence $f(r)\, v^2$ is less than the bracket to the right in (11) which is a constant. Since $f(r) > 0$ for $r > r_0$, it now follows at once from (9) and (10) that as r increases indefinitely, $\chi(r)$ approaches zero at least as rapidly as r^{-1} for $E > 0$ and $r^{-\frac{1}{2}}$ for $E = 0$. Consequently, to every $E \geqq 0$, there corresponds one characteristic function of (3).

In the case $E < 0$, $f(r) < 0$ for r sufficiently large, and the argument based on (11) breaks down, so that this case must be treated differently.

4. **The line spectrum,** $E < 0$. Write

(12) $$\xi = r \cdot \frac{4\pi}{h}\sqrt{-2mE}$$

and

(13) $$\chi(r) = \xi^n \exp\left(-\frac{1}{2}\xi\right) v(\xi);$$

introducing ξ as the independent and v as the dependent variable in (3), it is readily seen that the introduction of the factors ξ^n and $\exp\left(-\frac{1}{2}\xi\right)$ in (13) has the effect of removing the last and the first term, respectively, in the coefficient of χ in (3), and this equation becomes

(14) $$\xi\frac{d^2 v}{d\xi^2} + (2n+2-\xi)\frac{dv}{d\xi} + (l-n-1)\, v = 0,$$

where

(15) $$l = \frac{\pi e^2}{h}\sqrt{-\frac{2m}{E}}.$$

Since the solution of (3) which is finite at $r = 0$ belongs to the exponent n, it is evident from (13) that v belongs to the exponent 0 at $\xi = 0$, and the only other singular point of (14) being $\xi = \infty$, v is expansible in an everywhere convergent power series

$$v = \sum_{\nu=0}^{\infty} a_\nu \xi^\nu. \tag{16}$$

Substitution in (14) gives the recurrent formula

$$a_{\nu+1} = \frac{\nu - l + n + 1}{(\nu+1)(\nu+2n+2)} a_\nu. \tag{17}$$

First assume that $l - n - 1$ is neither zero nor a positive integer. Then (17) shows that none of the a_ν vanish (except in the trivial case where v vanishes identically), and that all a_ν with $\nu > l - n - 1$ have the same sign, which we may assume positive. To any positive $\varepsilon < \frac{1}{2}$ there belongs a $\nu_0 > l - n - 1$ such that

$$\frac{\nu - l + n + 1}{\nu + 2n + 2} > 1 - \varepsilon \quad \text{for} \quad \nu > \nu_0,$$

whence by (17)

$$a_{\nu+1} > \frac{1-\varepsilon}{\nu+1} a_\nu,$$

so that $\nu!\,(1-\varepsilon)^{-\nu} a_\nu$ increases with ν for $\nu > \nu_0$, and consequently there exists a positive constant c such that

$$a_\nu > c \cdot \frac{(1-\varepsilon)^\nu}{\nu!} \quad \text{for} \quad \nu > \nu_0. \tag{18}$$

From (16) and (18) it follows that when ξ increases indefinitely, v increases at least as rapidly as $\exp((1-\varepsilon)\xi)$, and since $\varepsilon < \frac{1}{2}$, (13) shows that $\chi(r)$ becomes infinite with r.

Consequently, there exists no solution of (3) which is finite everywhere unless $l - n - 1$ is a positive integer or zero. In this case we have, since n is a positive integer or zero,

$$l = \text{positive integer}, \; n < l. \tag{19}$$

and by (15) we obtain the Bohr energy levels

$$E = -\frac{2\pi^2 m e^4}{h^2 l^2}, \qquad l = 1, 2, 3, \cdots \tag{20}$$

From (19) and (17) it follows that $a_\nu = 0$ for $\nu > l - n - 1$, so that $v(\xi)$ is a polynomial of degree $l - n - 1$, and (19) shows that $\chi(r)$ tends toward zero as r increases indefinitely. Hence for $E < 0$, there corresponds one characteristic function of (3) to each value of E given by (20), but none to any other negative value of E. Moreover, the comparison of (14) and the differential equation for the derivatives of the Laguerre polynomials[5] shows that, except for a constant factor, v equals $L_{n+l}^{(2n+1)}(\xi)$, the $(2n+1)$th derivative of the $(n+l)$th Laguerre polynomial.

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[5] E. Schrödinger, Ann. d. Phys. *80*, 437 (1926), eq. (102) on p. 483 or Riemann-Weber, Differentialgleichungen der mathematischen Physik, 7. ed., vol. I (1925), p. 341.

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June 30, 1928.

ON THE CESÀRO SUMS OF FOURIER'S AND LAPLACE'S SERIES.[1]

By T. H. Gronwall.

1. **Introduction.** Let $f(x)$ be a continuous function of x with the period 2π, and $s_n^{(k)}\{f(x)\}$ the n : th Cesàro sum of order k of its Fourier series. Similarly, let $s_n^{(k)}\{f(\theta,\varphi)\}$ be the corresponding sum of the Laplace series of a function continuous on the unit sphere. Fejér has shown that

$$(1) \qquad s_n^{(1)}\{f(x)\} \to f(x) \text{ and } s_n^{(2)}\{f(\theta,\varphi)\} \to f(\theta,\varphi) \text{ as } n \to \infty,$$

and also that

$$(2) \qquad m \leqq s_n^{(1)}\{f(x)\} \leqq M, \quad m \leqq s_n^{(2)}\{f(\theta,\varphi)\} \leqq M, \qquad n = 0, 1, 2, \cdots$$

where m and M denote the minimum and maximum of the function concerned.[2] The summability property (1) has been extended to any order $k > 0$ for Fourier's series,[3] and $k > \frac{1}{2}$ for Laplace's series,[4] these lower bounds for k being the best obtainable.

On the contrary, the mean value property (2) cannot be extended to smaller values of k; it is the purpose of the present paper to prove the following two theorems:

Theorem I. *To every k, where $0 \leqq k < 1$, and every $n = 1, 2, 3, \cdots$ there corresponds a continuous function $f(x)$ of period 2π such that*

$$s_n^{(k)}\{f(0)\} > M.$$

Theorem II. *To every k, where $0 \leqq k < 2$, and every odd $n = 1, 3, 5, \cdots$ as well as every even n satisfying the inequality*

$$(3) \qquad n^{2-k} > c,$$

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[1] Received September 22, 1930. — Presented to the American Mathematical Society, Oct. 25, 1924.

[2] L. Fejér, Untersuchungen über Fouriersche Reihen, Math. Annalen, vol 58 (1904), p. 51–69, and Über die Laplacesche Reihe, ibid. vol. 67 (1909), p. 76–109.

[3] M. Riesz, Sur les séries de Dirichlet et les séries entières, Comptes Rendus, vol. 149 (1909), p. 909–912. S. Chapman, Non-integral orders of summability of series and integrals, Proc. London Math. Soc., ser. 2, vol. 9 (1911), p. 369–409.

[4] T. H. Gronwall, Über die Laplacesche Reihe, Math. Annalen, vol 74 (1913), p. 213–270, and Über die Summierbarkeit der Reihen von Laplace and Legendre, ibid. vol. 75 (1914), p. 321–375.

where $c>1$ *is a constant independent of* k, *there exists a function* $f(\theta, \varphi)$ *continuous on the unit sphere such that*

$$s_n^{(k)}\{f(0,0)\} > M.^5$$

2. **Proof of Theorem I.** For convenient reference, we recall some familiar formulas for the Cesàro sums. Writing

$$(4) \quad A_0^{(k)} = 1, \; A_n^{(k)} = \frac{k(k+1)\cdots(k+n-1)}{n!} = \frac{\Gamma(n+k)}{\Gamma(k)\,\Gamma(n+1)}, \; n>0,$$

the n: th Cesàro sum $s_n^{(k)}$ of order k of the series

$$u_0 + u_1 + \cdots + u_n + \cdots$$

is defined by

$$(5) \qquad A_n^{(k+1)} s_n^{(k)} = S_n^{(k)} = \sum_{\nu=0}^{n} A_\nu^{(k+1)} u_{n-\nu},$$

and the generating function of $S_n^{(k)}$ is

$$(6) \qquad \frac{1}{(1-z)^{k+1}} \sum_0^\infty u_n z^n = \sum_0^\infty S_n^{(k)} z^n,$$

whence the identity

$$(7) \qquad S_n^{(k)} = \sum_{\nu=0}^{n} A_\nu^{(k-l)} S_{n-\nu}^{(l)}.$$

In case of the Fourier series, we have

$$(8) \qquad A_n^{(k+1)} s_n^{(k)}\{f(0)\} = \frac{1}{\pi}\int_0^{2\pi} f(x)\, S_n^{(k)}(x)\, dx,$$

where $S_n^{(k)}(x)$ is the sum (5) formed with $u_0 = \frac{1}{2}$, $u_n = \cos nx$, $(n>0)$, so that in particular

$$(9) \qquad S_n^{(1)}(x) = \frac{1-\cos(n+1)x}{1-\cos x}.$$

When $S_n^{(k)}(x) \geqq 0$ for $0 \leqq x \leqq 2\pi$, (8) gives at once for $k \geqq 0$

$$A_n^{(k+1)} s_n^{(k)}\{f(0)\} \leqq \frac{1}{\pi} M \int_0^{2\pi} S_n^{(k)}(x)\, dx = M A_n^{(k+1)},$$

or $s_n^{(k)}\{f(0)\} \leqq M$. When, on the other hand, $S_n^{(k)}(x)$ changes its sign between 0 and 2π, the discontinuous function

[5] The corresponding statements for the minimum are obtained by the trivial substitution of $-f$ for f.

$$f(x) = \operatorname{sgn} S_n^{(k)}(x) \tag{10}$$

gives

$$A_n^{(k+1)} s_n^{(k)}\{f(0)\} = \frac{1}{\pi}\int_0^{2\pi} |S_n^{(k)}(x)|\, dx > \frac{1}{\pi}\int_0^{2\pi} S_n^{(k)}(x)\, dx,$$

so that, since $M = 1$ for (10),

$$s_n^{(k)}\{f(0)\} > M, \tag{11}$$

and approximating (10) closely enough by a continuous function for which $M = 1$, (11) will still hold for this function. Since $S_n^{(k)}(x)$ is positive for $x = 0$, $0 \leqq k < 1$, all terms in (5) being positive, the proof of Theorem I is thus reduced to showing that $S_n^{(k)}(x)$ takes negative values. Making $x = 2\pi/(n+1)$, (9) shows that $S_n^{(1)} = 0$, $S_{n-\nu}^{(1)} > 0$ for $\nu > 0$, and by (4), $A_\nu^{(k-1)} < 0$ for $0 \leqq k < 1$ and $\nu > 0$. Hence, making $l = 1$ in (7), we find

$$S_n^{(k)}\left(\frac{2\pi}{n+1}\right) < 0, \qquad 0 \leqq k < 1, \qquad n \geqq 1.$$

3. Proof of Theorem II for n odd. As in the preceding paragraph, it is seen that the necessary and sufficient condition for the existence of a continuous function $f(\theta, \varphi)$ for which $s_n^{(k)}\{f(0, 0)\} > M$ is that the sum $S_n^{(k)}(\theta)$ formed by making $u_n = (2n+1) P_n(\cos\theta)$ in (5), shall change its sign in the interval $0 \leqq \theta \leqq \pi$. According to (6), the generating function is now

$$\frac{1}{(1-z)^{k+1}} \cdot \frac{1-z^2}{(1-2z\cos\theta + z^2)^{3/2}} = \sum_0^\infty S_n^{(k)}(\theta) z^n, \tag{12}$$

whence it follows at once that for $\theta = 0$

$$S_n^{(k)}(0) > 0 \quad \text{for} \quad k \geqq 0, \tag{13}$$

and also, making $k = 1$ and $\theta = \pi$,

$$S_n^{(1)}(\pi) = \begin{cases} (n+2)/2\,, & n \text{ even}, \\ -(n+1)/2, & n \text{ odd}. \end{cases} \tag{14}$$

Making $l = 1$, in (7), we obtain

$$(-1)^n S_n^{(k)}(\pi) = \sum_{\nu=0}^{n} (-1)^\nu v_\nu,$$

where

$$v_\nu = \begin{cases} A_\nu^{(k-1)}(n-\nu+2)/2, & n-\nu \text{ even}, \\ A_\nu^{(k-1)}(n-\nu+1)/2, & n-\nu \text{ odd}. \end{cases}$$

By (4), $v_\nu > 0$ for $\nu \geqq 0$ and $1 < k < 2$, but $v_0 > 0$ and $v_\nu < 0$ for $\nu \geqq 1$ and $0 < k < 1$; moreover

$$\frac{v_{\nu+1}}{v_\nu} = \begin{cases} \dfrac{k-1+\nu}{\nu+1} \cdot \dfrac{n-\nu}{n-\nu+2}, & n-\nu \text{ even,} \\ \dfrac{k-1+\nu}{\nu+1}, & n-\nu \text{ odd,} \end{cases}$$

so that $v_{\nu+1}/v_\nu < 1$ for $\nu \geqq 0$, $1 < k < 2$ and for $\nu \geqq 1$, $0 < k < 1$. The familiar argument on alternating series with numerically decreasing terms now shows that $\operatorname{sgn}(-1)^n S_n^{(k)}(\pi) = 1$; (the cases $k = 0$ and $k = 1$ left out in the discussion are taken care of by (12) and (14)). Thus Theorem II is proved for n odd.

4. **Proof of Theorem II for n even.** Making $n = 2$, we obtain by direct calculation from (5)

$$S_2^{(k)}(\theta) = (k+1)(k+2)/2 + 3(k+1)\cos\theta + 5(3\cos^2\theta - 1)/2,$$

whence

$$10 S_2^{(k)}(\theta) = 3(5\cos\theta + k + 1)^2 + (k+6)(2k-3),$$

the necessary and sufficient condition for a change of sign being consequently $2k < 3$. This algebraic process of determining the range of values of k for which a change of sign in $S_n^{(k)}(\theta)$ is possible becomes unmanageable for larger even values of n, and we must apply an asymptotic method. By Christoffel's formula, we have

$$S_n^{(0)}(\theta) = \sum_0^n (2\nu+1) P_\nu(\cos\theta) = \frac{(n+1)(P_n(\cos\theta) - P_{n+1}(\cos\theta))}{1-\cos\theta},$$

and consequently

$$(1-\cos\theta) S_n^{(1)}(\theta) = \sum_0^n P_\nu(\cos\theta) - (n+1) P_{n+1}(\cos\theta).$$

Using Mehler's integral, this becomes

$$(15)\quad (1-\cos\theta) S_n^{(1)}(\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sum_0^n \sin(\nu+\frac{1}{2})\varphi - (n+1)\sin(n+\frac{3}{2})\varphi}{\sqrt{2(\cos\theta - \cos\varphi)}}\, d\varphi,$$

and since

$$\sigma_n(\varphi) = \sum_0^n \sin\left(\nu + \frac{1}{2}\right)\varphi = \frac{\sin^2((n+1)\varphi/2)}{\sin(\varphi/2)},$$

(15) and (7) give

$$(16)\quad \begin{aligned} (1-\cos\theta) S_n^{(k)}(\theta) &= \frac{2}{\pi}\int_\theta^\pi \frac{\sum_0^n A_\nu^{(k-1)} \sigma_{n-\nu}(\varphi)\, d\varphi}{\sqrt{2(\cos\theta-\cos\varphi)}} \\ &\quad - \frac{2}{\pi}\int_\theta^\pi \frac{\sum_0^n A_\nu^{(k-1)} (n-\nu+1)\sin(n-\nu+\frac{3}{2})\varphi\, d\varphi}{\sqrt{2(\cos\theta-\cos\varphi)}} \\ &= I_1 + I_2. \end{aligned}$$

For $\pi/2 \leqq \theta \leqq \pi$, we have $0 \leqq \sigma_n(\varphi)/\sin(\varphi/2) \leqq 2$, and consequently, using Mehler's integral for $P_0(\cos\theta) = 1$,

$$I_1 < 2\sum_0^n |A_\nu^{(k-1)}|.$$

Now $A_\nu^{(k-1)}$ is positive when $k>1$, but negative for $\nu>1$ when $0<k<1$ and consequently

$$\sum_0^n |A_\nu^{(k-1)}| = \sum_0^n A_\nu^{(k-1)} = A_n^{(k)}, \qquad k \geqq 1,$$

$$\sum_0^n |A_\nu^{(k-1)}| = 2 - \sum_0^n A_\nu^{(k-1)} = 2 - A_n^{(k)}, \qquad 0 \leqq k \leqq 1.$$

Using Stirling's formula in (4), we therefore have

$$I_1 < c_1 + c_2 n^{k-1}, \qquad 0 \leqq k \leqq 2, \qquad \pi/2 \leqq \varphi \leqq \pi, \tag{17}$$

where the constants c_1 and c_2 are independent of k.

To obtain an asymptotic expression for I_2 when θ is in the neighborhood of π, we observe that the generating function of

$$\tau_n(\varphi) = \sum_0^n A_\nu^{(k-1)}(n-\nu+1)\, e^{\left(n-\nu+\frac{3}{2}\right)\varphi i} \tag{18}$$

is evidently

$$\frac{1}{(1-z)^{k-1}} \cdot \frac{e^{\frac{3}{2}\varphi i}}{(1-z\,e^{\varphi i})^2},$$

so that, replacing z by $1/z$ and using Cauchy's integral formula,

$$\tau_n(\varphi) = \frac{1}{2\pi i}\int_C \frac{e^{\frac{3}{2}\varphi i} z^{n+k}\,dz}{(z-1)^{k-1}(z-e^{\varphi i})^2} \tag{19}$$

where C is the circle $|z| = r$, where $r>1$, and those branches of z^{n+k} and $(z-1)^{k-1}$ are used which are real and positive for z real and >1.[6] By Cauchy's theorem, we have

$$\tau_n(\varphi) = \frac{1}{2\pi i}\int_C = \frac{1}{2\pi i}\int_{C_1} + \frac{1}{2\pi i}\int_{C_2}, \tag{20}$$

where C_1 is a loop enclosing $z=0$ and $z=1$, but not $z=e^{\varphi i}$, and C_2 a contour enclosing $z = e^{\varphi i}$ and exterior to C_1. Since $k-1<1$, C_1 may

[6] For this method, originally used by Stieltjes to obtain the asymptotic expansions of Legendre's polynomials, compare T. H. Gronwall, On the summability of Fourier's series, Bulletin Am. Math. Soc., vol. 20 (1914), p. 139–146.

be contracted into the straight line segment from 1 to 0, followed by the segment from 0 to 1, and on the former $\arg z = 0$, $\arg(z-1) = \pi$ while on the latter, $\arg z = 2\pi$, $\arg(z-1) = \pi$.

Consequently

$$\frac{1}{2\pi i}\int_{C_1} = \frac{1}{2\pi i}\left(e^{\frac{3}{2}\varphi i+(1-k)\pi i}\int_1^0 \frac{z^{n+k}(1-z)^{1-k}\,dz}{(z-e^{\varphi i})^2}\right.$$

$$\left.+\, e^{\frac{3}{2}\varphi i+(1-k)\pi i+2k\pi i}\int_0^1 \frac{z^{n+k}(1-z)^{1-k}\,dz}{(z-e^{\varphi i})^2}\right)$$

$$= -e^{\frac{3}{2}\varphi i}\,\frac{\sin k\pi}{\pi}\int_0^1 \frac{z^{n+k}(1-z)^{1-k}\,dz}{(z-e^{\varphi i})^2},$$

and since, for $\pi/2 \leqq \varphi \leqq \pi$, we have $|z-e^{\varphi i}|>1$ on the integration interval,

$$\left|\frac{1}{2\pi i}\int_{C_1}\right| < \frac{|\sin k\pi|}{\pi}\int_0^1 z^{n+k}(1-z)^{1-k}\,dz$$

$$= \frac{|\sin k\pi|}{\pi}\,\frac{\Gamma(n+k+1)\,\Gamma(2-k)}{\Gamma(n+3)} = \frac{|1-k|\,\Gamma(n+k+1)}{\Gamma(k)\,\Gamma(n+3)},$$

or since $k<2$,

$$\left|\frac{1}{2\pi i}\int_{C_1}\right| < \frac{|1-k|}{\Gamma(k)} < c_3, \tag{21}$$

where c_3 is independent of k. The second integral in (20) is simply the residue of the integrand at $z = e^{\varphi i}$; at this point the argument of $z-1$ is $(\pi+\varphi)/2$ (exterior angle at 1 of the triangle with vertices $0, 1, e^{\varphi i}$), so that for $z = e^{\varphi i}+h$ and h sufficiently small

$$e^{\frac{3}{2}\varphi i}\, z^{n+k} = e^{\left(n+\frac{3}{2}+k\right)\varphi i}\,(1+he^{-\varphi i})^{n+k},$$

$$\frac{1}{(z-1)^{k-1}} = e^{\frac{1-k}{2}(\pi+\varphi)i}\left(1+\frac{h}{e^{\varphi i}-1}\right)^{1-k}.$$

Expanding by the binomial theorem, multiplying and taking the coefficient of h in the result, we obtain the residue, so that, after some slight algebraic transformations,

$$\frac{1}{2\pi i}\int_{C_2} = e^{\left(n+\frac{3}{2}\right)\varphi i+\frac{1-k}{2}(\pi-\varphi)i}\left(n+\frac{e^{\varphi i}-k}{e^{\varphi i}-1}\right). \tag{22}$$

Now we restrict φ to the interval

$$\pi - c_4/n \leqq \varphi \leqq \pi, \tag{23}$$

where c_4 (independent of k) is to be determined later. Then

$$\left| e^{\frac{1-k}{2}(\pi-\varphi)i} - 1 \right| < c_5/n,$$

and (22) gives

$$\left| \frac{1}{2\pi i} \int_{C_2} -n\, e^{\left(n+\frac{3}{2}\right)\varphi i} \right| < c_6, \tag{24}$$

where c_6 is independent of k. From (16) and (18), we have

$$I_2 = -\frac{2}{\pi} \int_\theta^\pi \frac{\mathfrak{J}\, \tau_n(\varphi)\, d\varphi}{\sqrt{2(\cos\theta - \cos\varphi)}},$$

where $\mathfrak{J}$ denotes the imaginary part of the expression following it, and (20), (21) and (24) give, for $0 \leqq k < 2$ and θ in the interval (23),

$$I_2 < -\frac{2n}{\pi} \int_\theta^\pi \frac{\sin\left(n+\frac{3}{2}\right)\varphi\, d\varphi}{\sqrt{2(\cos\theta - \cos\varphi)}} + c_7 = -n P_{n+1}(\theta) + c_7$$

by Mehler's formula. Consequently, using (16) and (17)

$$(1-\cos\theta)\, S_n^{(k)}(\theta) < c_8 + c_2 n^{k-1} - n P_{n+1}(\theta), \tag{25}$$

for θ in (23) and $0 \leqq k < 2$, the constants being independent of k. We now give to θ the value nearest to π which makes $P_{n+1}(\theta)$ an extreme. It has been shown by Blumenthal[7] that this θ lies in (23) when $c_4 = 5\pi/4$, and that the absolute value of the extreme lies between the bounds 0.406 ± 0.069. When $n+1$ is odd, this extreme is positive, and consequently, the right hand member of (25) for n even does not exceed

$$c_8 + c_2 n^{k-1} - 0.3\, n.$$

It is now evident that this expression is negative when n satisfies (3), provided that c is taken large enough, and consequently $S_n^{(k)}(\theta)$ is negative for the value of θ considered, which proves Theorem II in the case of n even.

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[7] O. Blumenthal, Über asymptotische Integration linearer Differentialgleichungen, mit Anwendung auf eine asymptotische Theorie der Kugelfunktionen, Archiv d. Math. u. Phys., ser. 3, vol. 19 (1912), p. 136–174. See p. 173–174.

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THE DISCRIMINANT MATRICES OF A LINEAR ASSOCIATIVE ALGEBRA.[1]

By C. C. MacDuffee.

1. **Introduction.** One of the most important theorems in the modern theory of algebraic equations is named for Borchardt and Jacobi, although it is the culmination of a long series of researches by Sturm, Sylvester, Hermite and others.[2] As finally evolved, the theorem is as follows:[3]

Let $f(x) = 0$ be an algebraic equation of degree n with real coefficients. Let s_k denote the sum of the kth powers of the roots. Let

$$M = \left\| \begin{matrix} n & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{matrix} \right\|.$$

Then the rank of M is the number of distinct roots, and the signature of M is the number of distinct real roots of $f(x) = 0$. The determinant of M is the discriminant of the equation.

In the theory of linear algebras, the matrix

$$\left\| \sum_{i,j} c_{rsi}\, c_{ijj} \right\|, \qquad (r = \text{row},\ s = \text{column}),$$

where the c's are the constants of multiplication is continually appearing.[4] Thus the order of the semi-simple component of an algebra is the rank of this matrix.[5]

It is the purpose of this paper (mainly expository) to show the close relationship of these two matrices, and to discuss some of their simpler properties.

2. **Definitions.** Let $\mathfrak{A}$ be a linear associative algebra over a field $\mathfrak{F}$, the basis numbers being $e_1, e_2, \cdots, e_n$ and the constants of multiplication c_{ijk}. Let

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[1] Received April 13 and June 2, 1930. Presented to the American Mathematical Society Nov. 30, 1928.

[2] For a history of this theorem, see *Abhandlung über die Auflösung der numerischen Gleichungen* by C. Sturm, ed. by A. Loewy, Leipzig, 1904. (Ostwald's Klassiker, No. 143.)

[3] Bieberbach-Bauer, *Vorlesungen über Algebra,* Teubner, 1928, p. 168.

[4] One example of an early appearance is Frobenius, *Sitzungsberichte der preußischen Akademie zu Berlin*, 1896, p. 601.

[5] Dickson, *Algebren und ihre Zahlentheorie,* Zürich, 1927, p. 110.

$$R_i = (c_{isr}), \qquad S_i = (c_{ris})$$

be the first and second matrices[6] respectively of e_i. Let $x = \sum x_i e_i$ be a number of $\mathfrak{A}$, and call

$$R(x) = \sum x_i R_i, \qquad S(x) = \sum x_i S_i$$

the first and second matrices, respectively, of x. Then

$$C_1(\omega) = |\omega I - R(x)| = 0, \qquad C_2(\omega) = |\omega I - S(x)| = 0$$

are respectively the first and second characteristic equations of x. The traces $t_1(x)$ and $t_2(x)$ of $R(x)$ and $S(x)$ are called the first and second traces, respectively, of x. They are evidently the coefficients of $-\omega^{n-1}$ in $C_1(\omega) = 0$ and $C_2(\omega) = 0$.

We shall define the *first discriminant matrix* of n numbers $x_1, x_2, \cdots, x_n$ of $\mathfrak{A}$ as follows:

$$T_1(x_1, x_2, \cdots, x_n) = \left\| \begin{matrix} t_1(x_1^2) & t_1(x_1 x_2) & \cdots & t_1(x_1 x_n) \\ t_1(x_2 x_1) & t_1(x_2^2) & \cdots & t_1(x_2 x_n) \\ \cdot & \cdot & \cdot & \cdot \\ t_1(x_n x_1) & t_1(x_n x_2) & \cdots & t_1(x_n^2) \end{matrix} \right\|.$$

Similarly the *second discriminant matrix* is

$$T_2(x_1, x_2, \cdots, x_n) = \| t_2(x_r x_s) \|.$$

3. The discriminant matrices in terms of the constants of multiplication. Denote by

$$x^{(r)} = x_1^{(r)} e_1 + x_2^{(r)} e_2 + \cdots + x_n^{(r)} e_n$$

the numbers of $\mathfrak{A}$ where the $x_i^{(r)}$ range over the numbers of $\mathfrak{F}$. We have defined $t_1(x^{(r)})$ and $t_2(x^{(r)})$ to be the traces of $R(x^{(r)})$ and $S(x^{(r)})$ respectively. Hence

$$t_1(x^{(r)}) = \sum_i x_i^{(r)} t_1(e_i).$$

Since the matrices $R(x^{(r)})$ and $R(x^{(s)})$ are isomorphic under multiplication with $x^{(r)}$ and $x^{(s)}$, we have

$$\begin{aligned} R(x^{(r)} x^{(s)}) = R(x^{(r)}) R(x^{(s)}) &= \sum_{i,j} x_i^{(r)} x_j^{(s)} R(e_i) R(e_j) \\ &= \sum_{i,j} x_i^{(r)} x_j^{(s)} R(e_i e_j), \end{aligned}$$

[6] MacDuffee, *Bulletin of the American Mathematical Society*, 35 (1929), p. 344.

so that

$$t_1(x^{(r)} x^{(s)}) = \sum_{i,j} x_i^{(r)} x_j^{(s)} t_1(e_i e_j).$$

It follows then upon forming the matrix in which the above expression lies in row r and column s that

$$(1) \qquad T_1(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = X\, T_1(e_1, e_2, \cdots, e_n)\, \bar{X}$$

where X is the matrix $(x_s^{(r)})$ and $\bar{X}$ is its transpose.

A similar result holds for T_2.

By definition

$$R(e_i)\, R(e_j) = (\sum_k c_{ikr} c_{jsk}),$$

so that

$$t_1(e_i e_j) = \sum_{h,k} c_{ikh} c_{jhk}.$$

Hence, writing T_1 for $T_1(e_1, e_2, \cdots, e_n)$, we have

$$T_1 = (\sum_{h,k} c_{rkh} c_{shk}),$$

and similarly

$$T_2 = (\sum_{h,k} c_{hrk} c_{ksh}).$$

Evidently both these matrices are symmetric.

If in the associativity conditions

$$\sum_k c_{rkh} c_{slk} = \sum_k c_{rsk} c_{klh}$$

we set $l = h$ and sum for h, we have

$$(2) \qquad \sum_{h,k} c_{rkh} c_{shk} = \sum_{h,k} c_{rsk} c_{khh}.$$

Thus T_1 is identical with the matrix to which reference was made in the introduction.

If $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ are linearly independent with respect to $\mathfrak{F}$, i. e., if X is non-singular, they can be taken for a new basis of $\mathfrak{A}$, and conversely every basis of $\mathfrak{A}$ is obtained from one basis $e_1, e_2, \cdots, e_n$ by such a transformation. We have then from (1),

THEOREM 1. *Under a linear transformation of basis, the matrices T_1 and T_2 are transformed like the matrices of quadratic forms.*

COROLLARY 1. *The ranks of T_1 and T_2 are invariant. If $\mathfrak{F}$ is a real field, the signatures of T_1 and T_2 are likewise invariant.*

4. **Another approach to the discriminant matrices.** Since the matrices T_1 and T_2 are of such fundamental importance, it may be of interest to derive them by an entirely different method of approach.

The first matrices $R_i = (c_{isr})$ of the basis numbers e_i will be linearly independent if and only if

$$\sum_i a_i c_{isr} = 0 \qquad (r, s = 1, 2, \cdots, n)$$

implies $a_i = 0$ for every i, that is, if the matrix

$$C = \left\| \begin{matrix} c_{111} & c_{112} & \cdots & c_{1nn} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n11} & c_{n12} & \cdots & c_{nnn} \end{matrix} \right\|$$

of n rows and n^2 columns is of rank n.

Denote by C' the matrix obtained from C by changing rows to columns and then permuting row $c_{1ij} \cdots c_{nij}$ with row $c_{1ji} \cdots c_{nji}$. Now

$$C C' = \left(\sum_{i,j} c_{rij} c_{sji}\right) = T_1.$$

Since the rank of the product cannot exceed the rank of either factor, it follows that if $\mathfrak{A}$ is semi-simple so that T_1 is of rank n, then C is of rank n and $R_1, R_2, \cdots, R_n$ are linearly independent. The converse does not follow.

An analogous treatment of the second matrices S_i leads to T_2.

5. **A method for expressing an algebra as a sum of a nilpotent and a semi-simple algebra.** If $\mathfrak{A}$ is neither semi-simple nor nilpotent, it contains a maximal nilpotent invariant subalgebra $\mathfrak{N}$, and $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ where $\mathfrak{S}$ is semi-simple.[7] If the rank of T_1 is r, a new basis for $\mathfrak{A}$ may be chosen so that $e_1, e_2, \cdots, e_r$ is a basis for $\mathfrak{S}$ and $e_{r+1}, \cdots, e_n$ is a basis for $\mathfrak{N}$.

The actual determination of this basis may be made very simply by the theory of matrices. If T_1 is of rank r, there exists a non-singular matrix X with elements in $\mathfrak{F}$ such that

$$X T_1 \bar{X} = \left\| \begin{matrix} T_r & 0 \\ 0 & 0 \end{matrix} \right\|$$

where T_r is non-singular of order r. A transformation of basis with matrix X puts $\mathfrak{A}$ into the desired form. The proof is as given by Dickson,[8] the only change being the use of matric notation.

6. **Relation to algebraic fields.** Suppose that $\mathfrak{A}$ is the algebraic field $\mathfrak{F}(\theta)$ where θ satisfies the irreducible equation

(3) $$\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_1 \omega + a_0 = 0,$$

[7] Loc. cit. 5, p. 136.
[8] Loc. cit. 5, p. 109.

the a's being rational. Every number x of $\mathfrak{F}(\theta)$ can be written in the form

$$x = x_0 + x_1\theta + x_2\theta^2 + \cdots + x_{n-1}\theta^{n-1},$$

where the x's are rational. Let $\theta_1 (= \theta)$, $\theta_2, \cdots, \theta_n$ be the conjugates of θ, i. e., the n roots of (3). Then

$$x^{(i)} = x_0^{(i)} + x_1^{(i)}\theta_i + x_2^{(i)}\theta_i^2 + \cdots + x_{n-1}^{(i)}\theta_i^{n-1}$$

are the conjugates of $x = x^{(1)}$. The equation

$$R(\omega) = (\omega - x^{(1)})(\omega - x^{(2)}) \cdots (\omega - x^{(n)}) = 0 \tag{4}$$

has rational coefficients, the leading coefficient being 1. This equation is irreducible in the rational field, for in particular θ satisfies no equation of degree less than n. Then (4) is the rank equation of $\mathfrak{A}$.

Let $C_1(\omega) = 0$ and $C_2(\omega) = 0$ be the first and second characteristic equations of x as defined in § 1. It is known[9] that the distinct irreducible factors of $C_1(\omega)$, $C_2(\omega)$ and $R(\omega)$ are identical. Since each of these functions is of degree n, and the leading coefficient of each is 1, and $R(\omega)$ is irreducible, they are identical.

From (3) it now follows that

$$t_1(x) = t_2(x) = x^{(1)} + x^{(2)} + \cdots + x^{(n)}$$

and therefore

$$t_1(x^k) = t_2(x^k) = x^{(1)^k} + x^{(2)^k} + \cdots + x^{(n)^k}.$$

The discriminant of (3) is customarily defined as

$$D = \begin{vmatrix} 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^{n-1} \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \theta_n & \theta_n^2 & \cdots & \theta_n^{n-1} \end{vmatrix}^2 = \prod_{i<j} (\theta_i - \theta_j)^2.$$

Now

$$\left\Vert \begin{matrix} 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \cdots & \theta_n \\ \cdot & \cdot & \cdot & \cdot \\ \theta_1^{n-1} & \theta_2^{n-2} & \cdots & \theta_n^{n-1} \end{matrix} \right\Vert \left\Vert \begin{matrix} 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^{n-1} \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \theta_n & \theta_n^2 & \cdots & \theta_n^{n-1} \end{matrix} \right\Vert \tag{5}$$

$$= \left\Vert \begin{matrix} n & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{matrix} \right\Vert$$

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[9] Loc. cit. 5, p. 261.

where

$$s_k = \theta_1^k + \theta_2^k + \cdots + \theta_n^k = t_1(\theta^k) = t_2(\theta^k).$$

Thus a matrix having D as determinant can be written

$$\| t_1(\theta^{r-1}\theta^{s-1}) \| = T_1(1, \theta, \theta^2, \cdots, \theta^{n-1}) = T_2(1, \theta, \theta^2, \cdots, \theta^{n-1})$$

where r denotes the row and s the column in which an element stands.

We have therefore proved

THEOREM 2. *When $\mathfrak{A}$ is the algebraic field $\mathfrak{F}(\theta)$, each of the discriminant matrices $T_1(1, \theta, \theta^2, \cdots, \theta^{n-1})$ and $T_2(1, \theta, \theta^2, \cdots, \theta^{n-1})$ is equal to the matrix of Borchardt's Theorem.*

We are thus led to a method for calculating the discriminant matrix of an equation which is about as rapid as the usual method using Newton's identities.

Consider the reduced cubic

$$x^3 + px + q = 0.$$

Take $(1, x, x^2) = (e_1, e_2, e_3)$ as a basis for an algebra over the field of the coefficients. Then $e_1 e_i = e_i e_1 = e_i$, $e_2^2 = e_3$, $e_2 e_3 = e_3 e_2 = -q e_1 - p e_2$, $e_3^2 = -q e_2 - p e_3$. Thus $c_{1ij} = c_{i1j} = \delta_{ij}$, $c_{221} = 0$, $c_{222} = 0$, $c_{223} = 1$, $c_{231} = c_{321} = -q$, $c_{232} = c_{322} = -p$, $c_{233} = c_{323} = 0$, $c_{331} = 0$, $c_{332} = -q$, $c_{323} = -p$.

If we let

$$T_1(1, x, x^2) = \| \tau_{rs} \|,$$

then from (2)

$$\tau_{rs} = \sum_k c_{rsk} d_k, \qquad d_k = \sum_h c_{khh}.$$

We readily obtain

$$d_1 = 3, \qquad d_2 = 0, \qquad d_3 = -2p,$$

$$T_1(1, x, x^2) = \left\| \begin{matrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{matrix} \right\|.$$

Thus the Sturm functions are equivalent to

$$3, \qquad -6p, \qquad -4p^3 - 27q^2,$$

the latter being the discriminant.

7. Interpretation of the signature when $\mathfrak{A}$ is a field. It is known that the totality of units in an algebraic field $\mathfrak{F}(\theta)$ is given by

$$\varepsilon\,\eta_1^{e_1}\,\eta_2^{e_2}\,\cdots\,\eta_r^{e_r}$$

where ε ranges over the roots of unity in the field, and $e_1, e_2, \cdots, e_r$ range over all positive and negative integers and zero. The units $\eta_1, \eta_2, \cdots, \eta_r$ are said to constitute a *fundamental system* of units, and their number is given by

$$r = r_1 + r_2 - 1$$

where r_1 is the number of real roots and $2r_2$ the number of non-real roots of (3).[10]

Since (3) is irreducible, $r_1 + 2r_2 = n$, and from Borchardt's Theorem $r_1 = s$ where s is the signature of the discriminant matrix. Hence

$$r = \frac{n+s}{2} - 1.$$

Thus the quadratic field $\mathfrak{F}(\sqrt{m})$ has a basis $1, \theta$ where $\theta^2 = m$ in Case I, i. e., when $m = 2$ or $3 \pmod{4}$, and $\theta^2 = \theta + m'$, $4m' = m - 1$ in Case II, i. e., when $m = 1 \pmod{4}$. In the respective cases

$$T_1(1, \theta) = \left\| \begin{matrix} 1 & 0 \\ 0 & m \end{matrix} \right\|, \qquad \left\| \begin{matrix} 1 & 0 \\ 0 & m' \end{matrix} \right\|.$$

When $m < 0$ in both cases $s = 0$ so that $r = 0$, and when $m > 0$, $s = 2$ so that $r = 1$.[11]

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[10] Landau, *Vorlesungen über Zahlentheorie*, Leipzig, 1927, v. 3, p. 165.

[11] L. W. Reid, *Theory of algebraic numbers*, Macmillan, 1910, p. 404 and p. 417.

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ON MULTIPLE FACTORIAL SERIES.[1]

By C. Raymond Adams.

For some time the writer has felt that multiple factorial series might well be made to play a useful and significant rôle in the study of linear partial difference equations. If such is to be the case, it is necessary first to develop a considerable number of the properties, including those of convergence, of such series. To accomplish this purpose is the object of the present paper. It is hoped that the results obtained will also be of some intrinsic interest.

To gain simplicity of statement we confine ourselves to the *double* series[2]

$$(1) \qquad \sum_{i,j=0}^{\infty} \frac{(i-1)!\,(j-1)!\,c_{ij}}{x(x+1)\cdots(x+i-1)\,y(y+1)\cdots(y+j-1)},$$

in which the c_{ij} are constants, x and y are complex variables, and the quantities

$$(-1)!,\ 0!,\ x(x+1)\cdots(x+i-1) \text{ for } i=0,$$

and

$$y(y+1)\cdots(y+j-1) \text{ for } j=0$$

are all to be interpreted as unity. The treatment of series of multiplicity greater than 2 involves only formal modifications.

We begin (§§ 1—5) with a study of convergence properties following the lines of Landau's fundamental paper on simple factorial series,[3] with which the reader is assumed to be familiar and to the details of which frequent reference is made. For this approach to the problem one must employ an adaptation to multiple series of Abel's device of partial summation, so valuable in many problems of simple series. Such an adaptation has recently been made by C. N. Moore;[4] a particular case of one of his theorems is basic in this part of our work. We next direct our efforts (§§ 6, 7) toward obtaining generalizations of some of Nörlund's results

[1] Received March 14, 1930. — Presented to the American Mathematical Society, December 27, 1929.

[2] The author has already drawn attention to a factorial series consisting essentially of the main diagonal of (1): "Factorial series in two variables", *Bulletin of the American Mathematical Society*, vol. 34 (1928), pp. 473–475.

[3] Landau, "Über die Grundlagen der Theorie der Fakultätenreihen", *Sitzungsberichte der Münchener Akademie* (math.-phys.), vol. 36 (1906), pp. 151–218.

[4] C. N. Moore, "On convergence factors in multiple series", *Transactions of the American Mathematical Society*, vol. 29 (1927), pp. 227–238.

concerning simple factorial series.[5] The concluding portion of the paper (§§ 8, 9) is devoted to the proof of an expansion theorem. The proofs which have been given for expansion theorems concerning simple factorial series do not seem to be capable of ready extension to multiple series, and it seems necessary to content ourselves with a rather restricted theorem. This theorem, however, can be proved in a simple and direct way which appears not to have been used previously in the study of expansion theorems for the simple factorial series.

The Pringsheim definition of convergence[6] is used throughout the paper.

1. If in Moore's first theorem we set $r = 1$, take (α, β) as a single pair of values, and extend the summations from 0 to ∞ rather than from 1 to ∞, we may express the sufficiency of his conditions in the form of[7]

Theorem I. *Sufficient conditions that the double series* $\sum_{i,j=0}^{\infty} a_{ij} b_{ij}$ *converge with* S_{ij} *bounded*[8] *whenever the series* $\sum_{i,j=0}^{\infty} a_{ij}$ *is convergent with* S_{ij} *bounded are that the convergence factors* b_{ij} *satisfy the following conditions*:

(A) $\sum_{i,j=0}^{\infty} | b_{i+1,j+1} - b_{i+1,j} - b_{i,j+1} + b_{ij} |$ *converges*;

(B₁) $\lim_{j \to \infty} | b_{i+1,j} - b_{ij} | = 0 \qquad (i = 0, 1, 2, \cdots)$;

(B₂) $\lim_{i \to \infty} | b_{i,j+1} - b_{ij} | = 0 \qquad (j = 0, 1, 2, \cdots)$;

(C) b_{ij} *is bounded for* $i, j = 0, 1, 2, \cdots$.

As a consequence we have

Theorem 1. *If a double factorial series* (1) *converges with* S_{ij} *bounded for the place*[9] (x_0, y_0), *it converges with* S_{ij} *bounded for every place* (x_1, y_1) *satisfying the conditions*[10] $R(x_1) > R(x_0)$, $R(y_1) > R(y_0)$; $x_1, y_1 \neq 0, -1, -2, \cdots$.

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[5] Nörlund, "Sur les séries de facultés", *Acta Mathematica*, vol. 37 (1914), pp. 327-387; especially pp. 342-346.

[6] See Pringsheim, *Encyklopädie der mathematischen Wissenschaften*, vol. I, A3, pp. 97-98.

[7] Roman numerals will be used for theorems due to Moore and their extensions, arabic numerals for our own theorems.

[8] S_{ij} is used, in a generic sense, to designate the sum of the terms in the first i rows and the first j columns of any double series to which the phrase "convergent with S_{ij} bounded" is applied. At this point the phrase "with S_{ij} bounded" has been added by us to Moore's statement; this addition is justified immediately by an examination of his proof.

[9] A pair of values x, y will be spoken of as the *place* (x, y).

[10] $R(x)$ denotes the *real part* of x.

Proof. Let us set

$$a_{ij} = \frac{(i-1)!\,(j-1)!\,c_{ij}}{x_0(x_0+1)\cdots(x_0+i-1)\,y_0(y_0+1)\cdots(y_0+j-1)},$$

$$b_{ij} = \frac{x_0(x_0+1)\cdots(x_0+i-1)\,y_0(y_0+1)\cdots(y_0+j-1)}{x_1(x_1+1)\cdots(x_1+i-1)\,y_1(y_1+1)\cdots(y_1+j-1)}.$$

Then by hypothesis $\sum\limits_{i,j=0}^{\infty} a_{ij}$ is convergent with S_{ij} bounded, and we shall show that for $R(x_1)>R(x_0)$, $R(y_1)>R(y_0)$ the b_{ij} satisfy the conditions (A), (B_1), (B_2), and (C).

First it should be observed[11] that for any particular i, $\lim\limits_{j\to\infty} b_{ij}=0$ and for any particular j, $\lim\limits_{i\to\infty} b_{ij}=0$. Therefore, since b_{ij} is the product of a function of i and a function of j, we have

$$\lim_{\substack{i\to\infty\\ j\to\infty}} b_{ij} = \lim_{i\to\infty}\,[\lim_{j\to\infty} b_{ij}] = \lim_{j\to\infty}\,[\lim_{i\to\infty} b_{ij}] = 0$$

and (C) is fulfilled. And from the equalities

$$b_{i+1,j}-b_{i,j} = \frac{x_0-x_1}{x_1+i}\,b_{ij}, \qquad b_{i,j+1}-b_{ij} = \frac{y_0-y_1}{y_1+j}\,b_{ij},$$

it follows that conditions (B_1) and (B_2) are satisfied.

For brevity let the quantity within absolute value signs in (A) be denoted by d_{ij}; then we have

$$d_{ij} = \frac{x_0-x_1}{x_1+i}\cdot\frac{y_0-y_1}{y_1+j}\,b_{ij},$$

and thus, since d_{ij} is the product of a function of i and a function of j,[11]

$$\lim_{\substack{i\to\infty\\ j\to\infty}} |d_{ij}|\,(i-1)^{1+R(x_1-x_0)}(j-1)^{1+R(y_1-y_0)}$$
$$= \left|\frac{\Gamma(x_1)}{\Gamma(x_0)}\right|\,|x_1-x_0|\,\left|\frac{\Gamma(y_1)}{\Gamma(y_0)}\right|\,|y_1-y_0|.$$

Accordingly for $i\geqq I$ and $j\geqq J$, where I and J are sufficiently large, the inequality

$$|d_{ij}| < \left|\frac{\Gamma(x_1)\,\Gamma(y_1)}{\Gamma(x_0)\,\Gamma(y_0)}(x_1-x_0)\,(y_1-y_0)\right|\,\frac{2}{(i-1)^{1+R(x_1-x_0)}(j-1)^{1+R(y_1-y_0)}}$$

[11] Cf. Landau, loc. cit., pp. 158-159.

is valid. Let r denote the lesser of the two numbers $R(x_1-x_0)$, $R(y_1-y_0)$. Then, since the first $I-1$ rows and the first $J-1$ columns are simple series, known to be convergent,[12] the series $\sum_{i,j=0}^{\infty}|d_{ij}|$ will converge and condition (A) be satisfied if the series

$$\sum_{i,j=1}^{\infty}(ij)^{-1-r}$$

is convergent; but this is certainly true, for the double series may be regarded as the product $\sum_{i=1}^{\infty} i^{-1-r} \sum_{j=1}^{\infty} j^{-1-r}$ and r is positive.

Thus the hypotheses of Theorem I are all satisfied and our Theorem 1 follows.

From this theorem we conclude that *in general there exist two related real numbers* λ_1 *and* λ_2 *such that the series* (1) *converges with* S_{ij} *bounded for* $R(x)>\lambda_1$, $R(y)>\lambda_2$ *and fails to converge with* S_{ij} *bounded for* $R(x)<\lambda_1$, $R(y)<\lambda_2$; *in particular either one or both of these numbers may become positively or negatively infinite.*

By Landau's proof[12] of his Theorem I we have at once the important fact that *within the related half-planes of convergence with* S_{ij} *bounded, each row and each column of* (1) *is convergent.* Hence the double series may be summed either by rows or by columns in this region.[13]

2. The sufficiency proof of Moore's first theorem requires only obvious modifications to yield

THEOREM II. *Sufficient conditions that the double series* $\sum_{i,j=0}^{\infty} a_{ij}\, b_{ij}(x, y)$ *converge uniformly with* S_{ij} *uniformly bounded for* (x, y) *in* G *when the series* $\sum_{i,j=0}^{\infty} a_{ij}$ *is convergent with* S_{ij} *bounded are that the convergence factors* $b_{ij}(x, y)$ *satisfy the conditions* (A), (B_1), (B_2), *and* (C) *uniformly in* G.

From this follows

THEOREM 2. *The double factorial series* (1) *converges uniformly with* S_{ij} *uniformly bounded in a certain neighborhood of every place* (x, y) *satisfying the conditions* $R(x)>\lambda_1$; $R(y)>\lambda_2$; $x, y \neq 0, -1, -2, \cdots$.

Proof. It is sufficient to show that if (1) converges with S_{ij} bounded for (x_0, y_0), it converges uniformly for $(x = u+vi,\ y = w+zi)$ in the region defined by the inequalities[14]

[12] Cf. Landau, loc. cit., pp. 158–159.

[13] Cf. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre*, Leipzig, I. 2, 1926, p. 457.

[14] The modification of Landau's work needed for this proof we give in some detail, since his paper contains two slight errors at this point.

$$u_0+\gamma_1 \leqq u \leqq u_0+\gamma_2, \qquad v_0-\gamma_3 \leqq v \leqq v_0+\gamma_3,$$
$$w_0+\gamma_4 \leqq w \leqq w_0+\gamma_5, \qquad z_0-\gamma_6 \leqq z \leqq z_0+\gamma_6,$$

where $\gamma_1, \gamma_2, \cdots, \gamma_6$ are six positive quantities ($\gamma_1 < \gamma_2$, $\gamma_4 < \gamma_5$) so chosen that this pair of associated rectangles includes none of the points $x, y = 0, -1, -2, \cdots$ in their interiors or on their boundaries. For manifestly we may inclose in such a pair of rectangles, which we designate as the region G, every place (x, y) satisfying the conditions stated in the theorem.

We first define a_{ij} as in the proof of Theorem 1 and set

$$b_{ij}(x, y) = \frac{x_0(x_0+1)\cdots(x_0+i-1)\,y_0(y_0+1)\cdots(y_0+j-1)}{x(x+1)\cdots(x+i-1)\,y(y+1)\cdots(y+j-1)}.$$

Next let integers γ and γ' be so chosen that γ exceeds the three numbers $|u_0|$, $|u_0+\gamma_i|+|v_0|+\gamma_3$ $(i=1, 2)$ and γ' exceeds the three numbers $|w_0|$, $|w_0+\gamma_i|+|z_0|+\gamma_6$ $(i = 4, 5)$. Then for all (x, y) in G we have

$$|x| = |u+vi| \leqq |u|+|v| < \gamma, \qquad |x_0| < \gamma,$$
$$|y| = |w+zi| \leqq |w|+|z| < \gamma', \qquad |y_0| < \gamma'.$$

Moreover let β and β' be chosen so that in G

$$\left|\prod_{\nu=0}^{2\gamma} \frac{x_0+\nu}{x+\nu}\right| < \beta, \qquad \left|\prod_{\nu'=0}^{2\gamma'} \frac{y_0+\nu'}{y+\nu'}\right| < \beta'. \tag{2}$$

We then find[15] that for $i-1 > 2\gamma$, $j-1 > 2\gamma'$ and all (x, y) in G,

$$\begin{aligned} &\left|\prod_{\nu=2\gamma+1}^{i-1} \frac{x_0+\nu}{x+\nu} \prod_{\nu'=2\gamma'+1}^{j-1} \frac{y_0+\nu'}{y+\nu'}\right| \\ &< \exp\left(-\gamma_1 \sum_{\nu=2\gamma+1}^{i-1} \frac{1}{\nu} + 2\gamma^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} - \gamma_4 \sum_{\nu'=2\gamma'+1}^{j-1} \frac{1}{\nu'} + 2\gamma'^2 \sum_{\nu'=1}^{\infty} \frac{1}{\nu'^2}\right). \end{aligned} \tag{3}$$

From these inequalities (2) and (3) it follows that for all $i \geqq I$, $j \geqq J$ (where I and J are sufficiently large) and for all (x, y) in G we have

$$\begin{aligned} |b_{ij}(x, y)| &= \left|\prod_{\nu=0}^{i-1} \frac{x_0+\nu}{x+\nu} \prod_{\nu'=0}^{j-1} \frac{y_0+\nu'}{y+\nu'}\right| \\ &< \alpha\beta\beta' \exp\left[-\frac{\gamma_1}{2}\log(i-1) - \frac{\gamma_4}{2}\log(j-1)\right] \\ &= \alpha\beta\beta'(i-1)^{-\gamma_1/2}(j-1)^{-\gamma_4/2}, \end{aligned} \tag{4}$$

[15] Cf. Landau, loc. cit., p. 162.

where

$$\alpha = \exp\left(2\gamma^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} + 2\gamma'^2 \sum_{\nu'=1}^{\infty} \frac{1}{\nu'^2}\right).$$

This shows that

$$\lim_{\substack{i\to\infty \\ j\to\infty}} b_{ij}(x, y) = 0$$

uniformly in G. From this and the facts that for any particular i, $\lim_{j\to\infty} b_{ij}(x, y) = 0$ uniformly in G and that for any particular j, $\lim_{i\to\infty} b_{ij}(x, y) = 0$ uniformly in G, we see that condition (C) is satisfied.

As for conditions (B_1) and (B_2), we have

$$b_{i+1,j}(x, y) - b_{ij}(x, y) = \frac{x_0 - x}{x+i} b_{ij}(x, y),$$

$$b_{i,j+1}(x, y) - b_{i,j}(x, y) = \frac{y_0 - y}{y+i} b_{ij}(x, y);$$

since with either subscript fixed, $b_{ij}(x, y)$ approaches zero uniformly in G as the other subscript becomes infinite, these conditions also are satisfied.

That the condition (A) is satisfied may be shown as in the proof of Theorem 1, making use now of the inequality

$$|d_{ij}| < 16\alpha\beta\beta'\gamma\gamma'(i-1)^{-1-\gamma_1/2}(j-1)^{-1-\gamma_4/2},$$

which holds uniformly in G by virtue of (4).

Our theorem follows.

Now the well-known Weierstrass "series theorem"[16] is capable of immediate extension to a double series of functions of two independent complex variables, convergence being defined according to Pringsheim and regarded as uniform when the inequality for simple convergence is satisfied uniformly over a region. Although a formal proof of this generalization may not be in the literature, it is hardly necessary here to do more than make the two following remarks: first, that the ordinary theorem to the effect that a uniformly convergent series of continuous functions represents a continuous function is readily extended to double series in two variables; secondly, that the familiar proof of the Weierstrass theorem depending upon the use of Cauchy's integral formula[17] can then be adapted to proving the corresponding theorem for double series in two variables, using the Cauchy formula for functions of two variables.[18] From this generalization and Theorem 2 follows

[16] Cf. Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, 5th ed. (1928), p. 319.

[17] Cf., e. g., Osgood, loc. cit., pp. 319–320, especially the footnote on p. 320.

[18] Cf., e. g., Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part 1, 1916, pp. 225–226.

THEOREM 3. *Within the related half-planes in which the series* (1) *converges with S_{ij} bounded* $(x, y \neq 0, -1, -2, \cdots)$, *it represents an analytic function and in this region may be differentiated partially with respect to either x or y as many times as may be desired.*

As in the case of simple factorial series[19] it may be proved that such of the points $x, y = 0, -1, -2, \cdots$ as lie within the related half-planes of convergence with S_{ij} bounded are either regular points or simple poles of the function represented by (1).

3. We turn now to a consideration of absolute convergence and establish two theorems.

THEOREM 4. *In general the region of absolute convergence*[20] *of* (1) *consists of two related half-planes bounded on the left by two lines $R(x) = \mu_1$, $R(y) = \mu_2$; the lines themselves may or may not be included in the region; in particular either or both of the numbers μ_1, μ_2 may become positively or negatively infinite.*

Proof. We need only show that if (1) converges absolutely for (x_0, y_0), then it converges absolutely for (x_1, y_1) whenever we have $R(x_1) \geqq R(x_0)$, $R(y_1) \geqq R(y_0)$. This follows at once from the fact that if a double series $\sum_{i,j=0}^{\infty} a_{ij}$ converges absolutely and the constants b_{ij} $(i, j = 0, 1, 2, \cdots)$ are bounded, then the series $\sum_{i,j=0}^{\infty} a_{ij} b_{ij}$ converges absolutely. We define a_{ij} and b_{ij} as in the proof of Theorem 1; then by hypothesis $\sum_{i,j=0}^{\infty} a_{ij}$ converges absolutely, $\lim\limits_{\substack{i\to\infty \\ j\to\infty}} |b_{ij}|$ is[21] zero if $R(x_1) \geqq R(x_0)$, $R(y_1) \geqq R(y_0)$ (both equality signs not holding simultaneously) or is $\left|\dfrac{\Gamma(x_1)\,\Gamma(y_1)}{\Gamma(x_0)\,\Gamma(y_0)}\right|$ if $R(x_1) = R(x_0)$, $R(y_1) = R(y_0)$, and $\lim |b_{ij}|$ exists if either subscript is fixed and the other allowed to become infinite.

With every factorial series (1) there are therefore associated two pairs of related characteristic numbers λ_1, λ_2 and μ_1, μ_2 such that $\lambda_1 \leqq \mu_1$, $\lambda_2 \leqq \mu_2$ and (provided $x, y \neq 0, -1, -2, \cdots$) the series diverges, or converges without S_{ij} being bounded, for either $R(x) < \lambda_1$ or $R(y) < \lambda_2$ or both; converges conditionally with S_{ij} bounded for $\lambda_1 < R(x)$ and $\lambda_2 < R(y)$ and either $R(x) < \mu_1$ or $R(y) < \mu_2$ or both; and converges absolutely for $R(x) > \mu_1$, $R(y) > \mu_2$. Naturally we may have $\lambda_1 = \mu_1$ or $\lambda_2 = \mu_2$ or both, and it is understood that some or all of the numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$ may become positively or negatively infinite.

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[19] Cf. Landau, loc. cit., p. 164.

[20] Of course a series cannot converge absolutely without S_{ij} being bounded.

[21] Cf. Landau, loc. cit., equation (7), p. 159.

THEOREM 5. *If the factorial series* (1) *converges with* S_{ij} *bounded for* (x_0, y_0) *and if* $R(x_1)$ *is* $> R(x_0)+1$ *and* $R(y_1)$ *is* $> R(y_0)+1$, *the series is absolutely convergent for* (x_1, y_1).

This theorem is valid even if the series (1) does not converge for (x_0, y_0), or if S_{ij} is not bounded, provided only the terms of the series are then bounded. For, let the terms of the series for $x = x_0$, $y = y_0$ be denoted by a_{ij} as defined in the proof of Theorem 1 and let $|a_{ij}|$ be $< A\ (i, j = 0, 1, 2, \cdots)$. Then the theorem follows at once, since each row and column is absolutely convergent for $x = x_1$, $y = y_1$[22] and for $i \geqq I$, $j \geqq J$ (where I and J are sufficiently large),

$$\frac{(i-1)!\,(j-1)!\,|c_{ij}|}{|x_1(x_1+1)\cdots(x_1+i-1)\,y_1(y_1+1)\cdots(y_1+j-1)|}$$

is less than[23]

$$2A\left|\frac{\Gamma(x_1)\,\Gamma(y_1)}{\Gamma(x_0)\,\Gamma(y_0)}\right|\frac{1}{(i-1)^{R(x_1-x_0)}\,(j-1)^{R(y_1-y_0)}},$$

which is the term in the ith row and jth column of a convergent double series.

4. We now propose to establish a connection between the factorial series

$$(5)\qquad \sum_{i,j=1}^{\infty}\frac{(i-1)!\,(j-1)!\,c_{ij}}{x(x+1)\cdots(x+i-1)\,y(y+1)\cdots(y+j-1)}$$

and the double Dirichlet series

$$(6)\qquad \sum_{i,j=1}^{\infty}\frac{c_{ij}}{(i-1)^x\,(j-1)^y},$$

in which 0^x and 0^y are interpreted as unity and the coefficients c_{ij} are identical with those in (5). The natural analogue of Landau's sixth theorem — which would be to the effect that the series (5) and (6) have the same region of convergence with S_{ij} bounded — seems to be untrue in general.[24] We may, however, readily prove the following parallel to his seventh theorem.

THEOREM 6. *The places of absolute convergence of the series* (5) *and* (6) *are the same.*

The reasoning here is the same as in the proof of Theorem 4; it is only necessary to observe that, defining b_{ij} as

[22] Cf. Landau, loc. cit., Theorem V, p. 166.

[23] Cf. Landau, loc. cit., pp. 166–167.

[24] That it may sometimes be true is shown by Adams, loc. cit., Theorem 6, p. 474.

$$\frac{i!\,j!\,i^x j^y}{x(x+1)\cdots(x+i)\,y(y+1)\cdots(y+j)},$$

$\lim b_{ij}$ exists when either subscript is held fast and the other allowed to become infinite and we have

$$\lim_{\substack{i\to\infty\\ j\to\infty}} b_{ij} = \lim_{i\to\infty}[\lim_{j\to\infty} b_{ij}] = \lim_{j\to\infty}[\lim_{i\to\infty} b_{ij}] = \Gamma(x)\,\Gamma(y),$$

whence $|b_{ij}|$ is bounded.

5. The fact that when c_{ij} is zero for $i \neq j$ the series (1) has the same region of convergence[25] as the simple factorial series in $x+y$,

$$\sum_{i=0}^{\infty} \frac{i!\,c_{i+1,\,i+1}}{(x+y)(x+y+1)\cdots(x+y+i)},$$

enables one to exhibit examples illustrating the possibilities of positively and negatively infinite values for λ_1, λ_2, μ_1, and μ_2, of equal and unequal values for λ_1 and μ_1 and for λ_2 and μ_2, etc. We shall not give such examples here, for their construction is obvious in the light of Landau's examples.

With the foregoing sections in mind one may very naturally inquire concerning the convergence, *with S_{ij} unbounded,* of the factorial series (1). In particular one might well raise *the question of whether a theorem similar to Theorem* 1 *is valid for convergence with S_{ij} unbounded.* This question *is easily answered in the negative* by means of the following example. Consider the factorial series (1) with

$$\begin{array}{lll} c_{i0} = c_{j0} = 0, & & (i, j = 0, 1, 2, \cdots); \\ c_{1j} = 1, & c_{2j} = k, & (j = 1, 2, 3, \cdots); \\ c_{ii} = 1/(i-1)!, & c_{ij} = 0\,(i \neq j), & (i = 3, 4, 5, \cdots;\ j = 1, 2, 3, \cdots). \end{array}$$

The second and third rows of this double series are convergent for $R(y) > 1$ and divergent for $R(y) < 1$, whatever finite value $x(\neq 0, -1)$, may have.[26] The double series remaining after the deletion of the first three rows converges with S_{ij} bounded for all finite values of x and y $(\neq 0, -1, -2, \cdots)$.[27] Thus the double series under discussion converges with S_{ij} bounded for any finite $x(\neq 0, -1, -2, \cdots)$ and any y whose real part exceeds 1. It also converges, with S_{ij} unbounded, for $x = -k-1$ $(\neq 0, -1, -2, \cdots)$ and any finite $y(\neq 0, -1, -2, \cdots)$ whose real part is < 1. As an illu-

[25] Cf. Adams, loc. cit., pp. 474–475.
[26] Cf. Landau, loc. cit., pp. 171–174.
[27] Cf. Adams, loc. cit., pp. 474–475.

stration take $k=1/2$ (whence $x=-3/2$) and $y=\sqrt{-1}$; for this pair of values the double series is convergent with S_{ij} unbounded. Yet for any $x(\neq 0,-1)$ whose real part is $>-3/2$ and any y whose real part is positive and less than 1 the series is divergent.

6. We now proceed to generalize some of Nörlund's results[28] concerning simple factorial series. Our first object is to establish the formula

$$
\begin{aligned}
\Omega(x,y) &= \sum_{i,j=0}^{\infty} \frac{i!\,j!\,c_{i+1,j+1}}{x(x+1)\cdots(x+i)\,y(y+1)\cdots(y+j)} \\
&= \sum_{i,j=0}^{\infty} \frac{i!\,j!\sum_{\mu=0}^{i}\sum_{\nu=0}^{j}\binom{\beta+\mu-1}{\mu}\binom{\gamma+\nu-1}{\nu}c_{i+1-\mu,j+1-\nu}}{(x+\beta)(x+\beta+1)\cdots(x+\beta+i)(y+\gamma)(y+\gamma+1)\cdots(y+\gamma+j)},
\end{aligned} \tag{7}
$$

where β and γ are any two numbers each of which has a positive real part and where we have used the abbreviation

$$\binom{A+l-1}{l} = \frac{A(A+1)\cdots(A+l-1)}{l!}.$$

It should be observed that the first series in (7) is identical with (5).

In a fashion entirely analogous to Nörlund's it can be shown that the relation (7) is valid in the domain of absolute convergence of (5) if further $R(x)$ and $R(y)$ are both positive. Briefly the details are as follows. An expansion for $1/y$ involving γ can clearly be obtained just as Nörlund finds one for $1/x$ involving β (his equation (11)). The product of the two yields the relation

$$\frac{1}{xy} = \sum_{r,s=0}^{\infty} \frac{\beta(\beta+1)\cdots(\beta+r-1)\gamma(\gamma+1)\cdots(\gamma+s-1)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+r)(y+\gamma)(y+\gamma+1)\cdots(y+\gamma+s)}, \tag{8}$$

the double series being absolutely convergent for $R(x)>0$, $R(y)>0$ and $x\neq-\beta,-\beta-1,\cdots;\ y\neq-\gamma,-\gamma-1,\cdots$. Forming the difference Δ_{ij} of both members of (8) we find

$$
\begin{aligned}
&\frac{i!\,j!}{x(x+1)\cdots(x+i)y(y+1)\cdots(y+j)} \\
&= \sum_{r,s=0}^{\infty} \frac{\beta(\beta+1)\cdots(\beta+r-1)\gamma(\gamma+1)\cdots(\gamma+s-1)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+i+r)(y+\gamma)(y+\gamma+1)\cdots(y+\gamma+j+s)} \\
&\qquad\times\frac{(i+r)!}{r!}\,\frac{(j+s)!}{s!}.
\end{aligned}
$$

[28] Cf. Nörlund, loc. cit.

If we define

$$u_{i,\,i+r,\,j,\,j+s} = \binom{\beta+r-1}{r}\binom{\gamma+s-1}{s}$$
$$\times\frac{(i+r)!\;(j+s)!\;c_{i+1,\,j+1}}{(x+\beta)(x+\beta+1)\cdots(x+\beta+i+r)(y+\gamma)(y+\gamma+1)\cdots(y+\gamma+j+s)},$$

with $u_{i,r,j,s}=0$ for $r<i$ or $s<j$ or both, the absolute convergence of the multiple series

$$\sum_{i,r,j,s=0}^{\infty} u_{i,r,j,s},$$

for $R(x)>\mu_1$ and positive and $R(y)>\mu_2$ and positive, follows from Nörlund's inequalities. Since this multiple series converges absolutely, it may be summed in any desired manner; one method of summation gives the result stated in equation (7).

From the uniform convergence of both series in (7) we conclude that the relation is true not only in the region of absolute convergence of (5) but everywhere that the two series converge.

7. For $\beta=\gamma=1$ the relation (7) becomes

$$(9)\qquad \begin{aligned}\Omega(x,y) &= \sum_{i,j=0}^{\infty}\frac{i!\;j!\;c_{i+1,\,j+1}}{x(x+1)\cdots(x+i)\,y(y+1)\cdots(y+j)}\\ &= \sum_{i,j=0}^{\infty}\frac{i!\;j!\;\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{i+1-\mu,\,j+1-\nu}}{(x+1)(x+2)\cdots(x+i+1)(y+1)(y+2)\cdots(y+j+1)}.\end{aligned}$$

This formula is of special interest, for by its aid we may materially extend the results already found in § 2 concerning the uniform convergence of (1). To this end we first prove the following

LEMMA. *If S_{ij} for the series* (5) *is bounded when x and y have a particular pair of positive real values x_0, y_0, and if ε_1, and ε_2 are any two positive numbers* (*arbitrarily small*), *there exist positive integers S and T and a positive number K such that for all $s \geqq S$ and all $t \geqq T$ we have*

$$(10)\qquad |A_{st}| < K s^{x_0+\varepsilon_1}\, t^{y_0+\varepsilon_2},$$

where

$$|A_{st}| = \sum_{i=1}^{s}\sum_{j=1}^{t} c_{ij}.$$

Proof.[29] Let us define a_{ij} as in the proof of Theorem 1 and set

$$b_{ij} = \frac{x_0(x_0+1)\cdots(x_0+i-1)\,y_0(y_0+1)\cdots(y_0+j-1)}{(i-1)!\,(j-1)!}.$$

[29] Cf. the proof given by Landau, loc. cit., pp. 175–176, of the formula for the abscissa of convergence of a Dirichlet series, the formula and the essentials of the proof being due to Cahen, "Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues", *Annales Scientifiques de l'École Normale Supérieure,* ser. 3, vol. 11 (1894), pp. 89–91.

Then by Moore's identity[30] (5) we have

$$A_{st}=\sum_{i=1}^{s}\sum_{j=1}^{t}a_{ij}b_{ij}=\sum_{i=1}^{s-1}\sum_{j=1}^{t-1}S_{ij}\Delta_{11}b_{ij}+\sum_{i=1}^{s-1}S_{it}\Delta_{10}b_{it}+\sum_{j=1}^{t-1}S_{sj}\Delta_{01}b_{sj}+S_{st}b_{st}.$$

Now it is clear that the quantities

$$\Delta_{10}b_{ij}=b_{ij}\frac{x}{i+1},\qquad \Delta_{01}b_{ij}=b_{ij}\frac{y}{j+1},\qquad \Delta_{11}b_{ij}=b_{ij}\frac{x}{i+1}\,\frac{y}{j+1}$$

are all positive under the hypotheses made above. Hence, if B represents an upper bound of S_{ij}, we have

$$|A_{st}|<B\left[\sum_{i=1}^{s-1}\sum_{j=1}^{t-1}\Delta_{11}b_{ij}+\sum_{i=1}^{s-1}\Delta_{10}b_{it}+\sum_{j=1}^{t-1}\Delta_{01}b_{sj}+b_{st}\right].$$

But the sums within the brackets reduce respectively to the values

$$b_{st}-b_{s1}-b_{1t}+b_{11},\quad b_{st}-b_{1t},\quad \text{and}\quad b_{st}-b_{s1}.$$

Since b_{ij} is positive, we thus obtain

$$|A_{st}|<B[4b_{st}+b_{11}].$$

From the Gauss formula for the Gamma function follows the equality

$$\lim_{\substack{i\to\infty\\ j\to\infty}}\frac{b_{ij}}{(i-1)^{x_0}(j-1)^{y_0}}=\frac{1}{\Gamma(x_0)\,\Gamma(y_0)},$$

whence for $i\geqq I$, $j\geqq J$ (where I and J are sufficiently large) we have

$$b_{ij}<\frac{2}{\Gamma(x_0)\,\Gamma(y_0)}\,i^{x_0}\,j^{y_0}.$$

Thus if ϵ_1 and ϵ_2 are any two positive numbers, arbitrarily small, there exist integers S and T and a positive constant K such that the inequality (10) is satisfied for all $s\geqq S$ and all $t\geqq T$. This completes the proof of the Lemma.

Let it now be supposed that χ and ψ are any two positive numbers respectively exceeding λ_1 and λ_2 *for the series* (5), while x and y are any two numbers whose real parts are not less than χ and ψ respectively. From (9) we then have

$$|\Omega(x,y)|<\frac{(\chi+1)(\psi+1)}{|(x+1)(y+1)|}$$

$$\times\sum_{i,j=0}^{\infty}\frac{i!\,j!\left|\sum\limits_{\mu=0}^{i}\sum\limits_{\nu=0}^{j}c_{i+1-\mu,j+1-\nu}\right|}{(\chi+1)(\chi+2)\cdots(\chi+i+1)(\psi+1)(\psi+2)\cdots(\psi+j+1)}.$$

[30] Moore, loc. cit., p. 230.

By (10) and Stirling's formula there exists a positive number M independent of i and j and satisfying the inequality

$$\left|\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{i+1-\mu,\, j+1-\nu}\right|$$
$$< M\frac{(x_0+\epsilon+1)(x_0+\epsilon+2)\cdots(x_0+\epsilon+i)(y_0+\epsilon+1)(y_0+\epsilon+2)\cdots(y_0+\epsilon+j)}{i!\,j!},$$
$$(i, j = 0, 1, 2, \cdots, \text{ but not simultaneously } = 0),$$

where x_0 and y_0 are any two positive numbers respectively larger than λ_1 and λ_2 (for (5)) and ϵ is an arbitrarily small positive number. Taking care to choose $M > |c_{11}|$ we thus have

$$|\Omega(x, y)| < M\frac{(\chi+1)(\psi+1)}{|(x+1)(y+1)|}$$
$$\times\sum_{i,j=0}^{\infty}\frac{(x_0+\epsilon+1)(x_0+\epsilon+2)\cdots(x_0+\epsilon+i)(y_0+\epsilon+1)(y_0+\epsilon+2)\cdots(y_0+\epsilon+j)}{(\chi+1)(\chi+2)\cdots(\chi+i+1)(\psi+1)(\psi+2)\cdots(\psi+j+1)}.$$

It is possible, however, to choose x_0, y_0, and ϵ to satisfy the inequalities

$$\chi > x_0+\epsilon, \qquad \psi > y_0+\epsilon;$$

the series on the right is then convergent and (8) gives us

$$|\Omega(x, y)| < M\frac{(\chi+1)(\psi+1)}{|(x+1)(y+1)|}\,\frac{1}{(\chi-x_0-\epsilon)(\psi-y_0-\epsilon)}.$$

From this inequality it follows that $|xy\,\Omega(x, y)|$ is bounded in the region $R(x) \geqq \chi$, $R(y) \geqq \psi$.

Let us next set

$$\Omega(x, y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{i!\,j!\,c_{i+1,\,j+1}}{x(x+1)\cdots(x+i)\,y(y+1)\cdots(y+j)} + R_{mn}(x, y).$$

Since in the region of convergence of (5) each row and each column is convergent, $R_{mn}(x, y)$ is the sum of a double series, consisting of the terms of (5) for which i is $\geqq m$ and j is $\geqq n$, and of $m+n$ simple series. To the double series we may apply the same analysis as above, and to each of the simple series Nörlund's inequalities are directly applicable; thus we find

$$R_{mn}(x, y) = R_{mn1}(x, y) + R_{mn2}(x, y) + R_{mn3}(x, y),$$

where

$$R_{mn1}(x, y) = \sum_{i=0}^{m-1}\frac{i!}{x(x+1)\cdots(x+i)}\sum_{j=n}^{\infty}\frac{j!\sum_{\nu=n}^{j} c_{i+1,\,\nu+1}}{(y+1)(y+2)\cdots(y+j+1)},$$

$$R_{mn2}(x, y) = \sum_{j=0}^{n-1} \frac{j!}{y(y+1)\cdots(y+j)} \sum_{i=m}^{\infty} \frac{i! \sum_{\mu=m}^{i} c_{\mu+1, j+1}}{(x+1)(x+2)\cdots(x+i+1)},$$

$$R_{mn3}(x, y) = \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \frac{i!\, j! \sum_{\mu=m}^{i} \sum_{\nu=n}^{j} c_{i+1-\mu, j+1-\nu}}{(x+1)(x+2)\cdots(x+i+1)(y+1)(y+2)\cdots(y+j+1)}.$$

As in the preceding paragraph we may deduce the inequality

$$|R_{mn3}(x,y)| < M \frac{(\chi+1)(\chi+2)\cdots(\chi+m+1)(\psi+1)(\psi+2)\cdots(\psi+n+1)}{|(x+1)(x+2)\cdots(x+m+1)(y+1)(y+2)\cdots(y+n+1)|} \times$$

$$\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \frac{(x_0+\varepsilon+1)(x_0+\varepsilon+2)\cdots(x_0+\varepsilon+i)(y_0+\varepsilon+1)(y_0+\varepsilon+2)\cdots(y_0+\varepsilon+j)}{(\chi+1)(\chi+2)\cdots(\chi+i+1)(\psi+1)(\psi+2)\cdots(\psi+j+1)}.$$

The series (8) being absolutely convergent for $R(x) > 0$, $R(y) > 0$, the double series on the right is a portion of the remainder term of an absolutely convergent double series whose terms are independent of x and y. Furthermore, for $R(x) > \chi$, $R(y) > \psi$ the quotient preceding the double series is $\leqq 1$. It therefore follows that, given an arbitrarily small positive number ε, there exist an m_1 and an n_1 such that for $m \geqq m_1$ and $n \geqq n_1$ the quantity $|R_{mn3}(x, y)|$ in the region $R(x) > \chi$, $R(y) > \psi$ is uniformly less than $\varepsilon/(m+n+1)$.

By Nörlund's discussion we know that the ith series ($i = 0, 1, \cdots, m-1$) in $R_{mn1}(x, y)$ represents a function whose modulus for $n \geqq N_i$ (where N_i is sufficiently large) is uniformly less than $\varepsilon/(m+n+1)$; similarly the jth series ($j = 0, 1, \cdots, n-1$) in $R_{mn2}(x, y)$ represents a function whose modulus for $m \geqq M_j$ (where M_j is sufficiently large) is uniformly less than $\varepsilon/(m+n+1)$ in the region in question. If the largest of the quantities N_i ($i = 0, 1, \cdots, m-1$) and n_1 be denoted by $\mathfrak{N}$ and the largest of the numbers M_j ($j = 0, 1, \cdots, n-1$) and m_1 by $\mathfrak{M}$, it is clear that for $m \geqq \mathfrak{M}$, $n \geqq \mathfrak{N}$ we have

$$|R_{mn}(x, y)| < \varepsilon$$

uniformly in the region $R(x) \geqq \chi$, $R(y) \geqq \psi$. Thus the series (5) is uniformly convergent in this region and we obtain

THEOREM 7. *The series* (5) *is uniformly convergent in the domain* $R(x) \geqq \chi$, $R(y) \geqq \psi$, *where* χ *and* ψ *are any two positive numbers respectively greater than* λ_1 *and* λ_2, *the related abscissae of convergence of* (5).

That Theorem 7 *holds also for the series* (1) *is now seen immediately.*

8. In this and the following section we concern ourselves with the possibility of expanding an arbitrary function in a series of the type (1). Fundamental to our proof is the fact that $1/x^p$ ($p = 2, 3, \cdots$) can be

expanded in a factorial series in x whose coefficients are all real and positive, valid for $R(x) > 0$. Nielsen[31] remarks the existence of a factorial series expansion for $1/x^p$ but does not state, nor is it apparent from his work, that the coefficients are of the desired character. We therefore pause to derive the expansion by a different method.

The relation

$$\frac{1}{x^p} = \frac{(-1)^{p-1}}{(p-1)!}\int_0^1 (\log t)^{p-1}\, t^{x-1}\, dt \qquad (R(x) > 0;\ p = 2, 3, \cdots)$$

is cited by Nielsen and may easily be verified. By expanding[32] $\log t$ in powers of $(1-t)$ we clearly obtain

$$(\log t)^{p-1} = (-1)^{p-1}[k_{p-1}(1-t)^{p-1} + k_p(1-t)^p + \cdots] \qquad (k_{p-1} = 1),$$

the series being uniformly convergent in the interval $\epsilon \leqq t \leqq 1$ (ϵ positive and arbitrarily small) and all the coefficients k_i ($i = p-1,\ p, \cdots$) being positive reals. After multiplying by t^{x-1} the series, even though divergent for $t = 0$, may be integrated termwise over the interval $(0, 1)$.[33] From the well known integral formula for the Beta function we have

$$\int_0^1 (1-t)^n\, t^{x-1}\, dt = B(x, n+1) = \frac{\Gamma(x)\,\Gamma(n+1)}{\Gamma(x+n+1)} = \frac{n!}{x(x+1)\cdots(x+n)}.$$

Thus termwise integration leads to the expansion

$$(11) \qquad \frac{1}{x^p} = \sum_{s=p}^{\infty} \frac{C_s^{s-p}}{x(x+1)\cdots(x+s-1)} \qquad (R(x) > 0;\ p = 2, 3, \cdots),$$

in which all the coefficients are positive.

9. Let us now assume $f(x, y)$ to be any function analytic at the place (∞, ∞) and so expressible in the form

$$(12) \qquad f(x, y) = \sum_{i,j=0}^{\infty} a_{ij}\, x^{-i}\, y^{-j} \qquad (|x| > R_1,\ |y| > R_2),$$

R_1 and R_2 being associated radii of convergence. For $i \geqq 2$, $j \geqq 2$ the powers of $1/x$ and $1/y$ occurring in (12) may be replaced formally by expansions in factorial series as given by (11). In this manner we obtain the multiply infinite series

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[31] Nielsen, *Handbuch der Theorie der Gammafunktion*, Leipzig, Teubner, 1906, p. 247.

[32] For the very brief proof given here we are indebted to a suggestion by R. E. Gilman. The expansion (11) can also be obtained from the integral expression for $1/x^p$ by repeated integration by parts, but this requires a rather lengthy convergence proof.

[33] Cf., for example, Tannery, *Introduction à la Théorie des Fonctions d'une Variable*, Paris, Hermann, 1910, pp. 109–110.

$$\sum_{i,j,r,s=0}^{\infty} a_{ij}\, u_{ijrs}, \tag{13}$$

where

$$u_{ijrs} = C_r^{r-i}\, C_s^{s-j}/[x(x+1)\cdots(x+r-1)y(y+1)\cdots(y+s-1)],$$

the factorials in the denominator being interpreted as in the series (1) for $r = 0$ or $s = 0$ and the C's not already defined by (11) being assigned the following values:

$$C_t^{t-k}\begin{cases} = 1 & \text{if} \quad k = 0, 1;\ t = k, \\ = 0 & \text{if} \quad k = 0, 1;\ t \neq k \quad \text{or} \quad k \geqq 2;\ t < k. \end{cases}$$

If x, y be given any fixed real values ξ, η respectively greater than R_1, R_2 and u_{ijrs} be summed with respect to r and s, we obtain

$$\sum_{r,s=0}^{\infty} u_{ijrs} = \xi^{-i}\,\eta^{-j} \qquad (i, j = 0, 1, \cdots).$$

Moreover each term in this series is a real number $\geqq 0$, since ξ, η, and the C's ($\neq 0$) are all positive reals; i. e., the series converges absolutely and the series of absolute values represents the absolute value of $1/\xi^i \eta^j$. Let this process be followed by summing $a_{ij}\, u_{ijrs}$ on i and j; the double series in question converges absolutely by (12). Hence for $(x, y) = (\xi, \eta)$ the multiple series (13) is absolutely convergent; by comparison it is therefore absolutely convergent for all x, y whose real parts are not less than ξ, η respectively. In other words if ε is a positive quantity, arbitrarily small, the series (13) converges absolutely for $R(x) \geqq R_1 + \varepsilon$, $R(y) \geqq R_2 + \varepsilon$.

In this region we may therefore sum (13) in any manner we please; in particular we may sum it in such a way as to obtain for $f(x, y)$ the factorial series expansion

$$f(x, y) = \sum_{i,j=0}^{\infty} \frac{b_{ij}}{x(x+1)\cdots(x+i-1)y(y+1)\cdots(y+j-1)}.$$

Thus we have proved

THEOREM 8. *Any function $f(x, y)$ analytic at the place (∞, ∞) can be represented in a suitable pair of related right half-planes by a convergent factorial series of the type* (1).

BROWN UNIVERSITY, PROVIDENCE, R. I.
March 12, 1930.

ON THE RELATION BETWEEN CERTAIN METHODS OF SUMMABILITY.[1]

By Henry L. Garabedian.

1. **Introduction.** A fundamental problem in the theory of summable series is the comparison of the various definitions of summability with regard to their effectiveness, consistency, and other properties. The present paper is a contribution to this problem.

A definition, A, is said to be more effective than or to *include*[2] another, B, in case every sequence summable B is summable A to the same value. Two definitions are said to be *equivalent* if each includes the other. Two definitions are said to be *mutually consistent* if, whenever each of them evaluates a sequence, the two values are the same. We shall be concerned exclusively in this paper with the question of relative inclusion. The main object of the paper is to prove that certain definitions due to Lindelöf (vide infra) include that of Cesàro.

All of the definitions with which this account is occupied may easily be proved to be *regular*; that is, they belong to a class of definitions which sum a convergent series to the sum which it has in the ordinary sense. In particular, these definitions are to be studied in connection with the infinite series

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots, \tag{1.1}$$

where we define

$$s_n = u_0 + u_1 + \cdots + u_n. \tag{1.2}$$

We now state the definitions of summability occurring in the present paper.

A. *Cesàro's definition.*[3] We write

$$c_n^{(r)} = \frac{S_n^{(r)}}{\binom{n+r}{r}},$$

[1] Received May 29, 1930.—Presented to the American Mathematical Society, April 19, 1930.

[2] The terminology used here may be found in a paper by W. A. Hurwitz, Bull. Amer. Math. Soc., 28 (1922), pp. 17–36, p. 17.

[3] Bull. Sci. Math., (2) 14 (1890), pp. 114–120.

where

$$S_n^{(r)} = \binom{n+r-1}{r-1} s_0 + \binom{n+r-2}{r-1} s_1 + \cdots + \binom{r-1}{r-1} s_n$$
$$= \binom{n+r}{r} u_0 + \binom{n+r-1}{r} u_1 + \cdots + \binom{r}{r} u_n.$$

If, for some integral value of r, $\lim_{n\to\infty} c_n^{(r)} = s$, we say that the series (1.1) is summable (Cr) to the sum s.

The definitions that remain to be listed are in the main special cases of a very general definition due to E. Lindelöf.

B. *Lindelöf's definition.*[4] Let $\varphi(z, \alpha)$ be an analytic function of the complex variable $z = \varrho e^{i\psi}$, where α is a real and positive parameter, which satisfies the conditions:

(i) $\varphi(z, \alpha)$ is holomorphic in $-\psi_0' \leqq \psi \leqq \psi_0$ and in $|z| < 1$, where $0 \leqq \psi_0, \psi_0' \leqq \pi$;

(ii) $|\varphi(z, \alpha)| < e^{K(\alpha)|z|}$ for $-\psi_0' \leqq \psi \leqq \psi_0$, where $K(\alpha)$ tends to zero with α;

(iii) $\varphi(z, \alpha) \to 1$ uniformly as $\alpha \to 0$ in any finite portion of the domain for which condition (i) is fulfilled;

(iv) $\lim_{n\to\infty} |\varphi(n, \alpha)|^{1/n} = 0$.

Then

$$\lim_{\alpha\to 0} \sum_{n=0}^{\infty} \varphi(n, \alpha)\, u_n = s$$

is the generalized value of the series (1.1) when this limit exists.

B 1. *Le Roy's definition.*[5] For this definition we have

$$\varphi(z, \alpha) = \frac{\Gamma[(1-\alpha)z+1]}{\Gamma(1+z)}.$$

Enlisting the aid of Stirling's formula we may readily show that the four conditions of Lindelöf's definition are fulfilled by this function.

B 2. *Mittag-Leffler's definition.*[6] Here we have

$$\varphi(z, \alpha) = \frac{1}{\Gamma(1+z\alpha)}.$$

[4] Journal de Math., (5) 9 (1903), pp. 213–221, p. 213. Lindelöf deals only with the problem of representing an analytic function defined by a power series outside of the circle of convergence of the series.

[5] Annales Fac. Sci. Toulouse, (2) 2 (1900), pp. 317–430, p. 327.

[6] Atti del IV Congresso Internazionale dei Matematici, 1 (1908), pp. 67–85, p. 82.

Again, we need Stirling's formula to show that the conditions of Lindelöf's definition are fulfilled by this function.

B 3. *The Dirichlet's series definitions.* For this definition we have

$$\varphi(z, \alpha) = e^{-\lambda(z)\alpha},$$

where $\Re\,\lambda(z) > 0$ and where $\lambda(z)$ tends to infinity with z. Moreover, $\lambda(z)$ must be such that the conditions of Lindelöf's definition are fulfilled. In particular, in order that condition (ii) be satisfied, we must have $\Re\,\lambda(z) > C|z|$,[7] and in order to fulfill condition (iv), $\lambda(n)$ must tend to infinity faster than n. The particular choice $\lambda(n) \equiv \lambda_n = n \log n$ has been utilized extensively in the theory of analytic continuation by Lindelöf who shows that it gives a representation of an analytic function in its principal star. The same property holds for the Le Roy and Mittag-Leffler definitions. It was shown by D. S. Morse[8] that this special definition of Lindelöf includes (Cr) for every r. Morse tried to prove the same thing for Le Roy's definition but failed to complete the argument. In a certain sense, then, the present paper is a continuation of Morse's article.

It is to be observed that a much more general definition can be stated here, but one which on the other hand does not necessarily come under the head of the Lindelöf definition. In point of fact, a series (1.1) may be said to be summable by the Dirichlet's series method provided that

$$\lim_{s\to 0} \sum_{n=0}^{\infty} u_n e^{-\lambda_n s} \tag{1.3}$$

exists, where λ_n is a sequence of positive increasing real numbers whose limit is infinite, and where the Dirichlet's series converges when $\Re(s) > 0$, s being restricted to this half-plane.[9] This definition can be shown to be effective if suitable limitations are imposed on the rate of growth of the sequence λ_n.

In § 2 we prove that *Le Roy summability includes* $(C\,1)$ *summability.* From this result it follows in particular that *the Fourier series of an integrable function is summable Le Roy for almost every value of the variable.* This particular result was the first one found by the author and served as the point of departure for the present investigation.

[7] We shall always use the letter C to denote an unspecified constant, not always the same, or a bounded function of r where r also is bounded.

[8] Amer. Journal of Math., 45 (1923), pp. 259–285. Morse, by the way, attributes this definition to G. H. Hardy, Quar. Journal of Math., 42 (1911), pp. 181–215, p. 193.

[9] Compare the corresponding definition given by D. S. Morse, loc. cit., p. 261. Morse's quantity t_0 must be zero in order that the definition be regular.

In § 3 we extend the methods of § 2 to show that *Le Roy summability includes* (Cr) *summability for any integral value of* r. From this result it follows of course that *Le Roy summability includes* (Cr) *summability for every positive value of* r. The result of § 3 naturally implies that of § 2; since the case $r = 1$ is so much simpler than the general case, the author decided to reproduce the particular case as well.

In § 4 we prove that *Mittag-Leffler summability includes* (Cr) *summability for any positive* r, and in § 5 we obtain a family of Dirichlet's series definitions which include (Cr) summability.

It is fitting that this introduction be terminated with an expression of the author's indebtedness to Professor Einar Hille for his helpful suggestions in the preparation of this paper.

2. **$(C1)\rightarrow LR$.** In this paragraph we prove that *if a series* (1.1) *is summable* $(C1)$ *it is also summable Le Roy.*

At the outset we state a theorem due to Bromwich.[10]

If a series (1.1) is summable (Cr) to the sum s and if $v_n(t)$ is a function of t such that

1. $\lim\limits_{t\to 0} v_n(t) = 1,$

2. $\lim\limits_{n\to\infty} n^r\, v_n(t) = 0, \quad t>0,$

3. $\sum\limits_{n=0}^{\infty} n^r\,|\Delta^{r+1} v_n(t)|$ converges for each $t>0$ and is bounded for all $t>0$, then $\sum\limits_{n=0}^{\infty} v_n(t)\,u_n$ converges, $t>0$, and

$$\lim_{t\to 0}\sum_{n=0}^{\infty} v_n(t)\,u_n = s.$$

To solve the problem at hand then it would be sufficient to prove that

1. $\lim\limits_{\alpha\to 1^-} \dfrac{\Gamma(n\alpha+1)}{\Gamma(n+1)} = 1,$

2. $\lim\limits_{n\to\infty} n\,\dfrac{\Gamma(n\alpha+1)}{\Gamma(n+1)} = 0, \quad \alpha<1,$

3. $\sum\limits_{n=0}^{\infty} n\left|\Delta^2\,\dfrac{\Gamma(n\alpha+1)}{\Gamma(n+1)}\right|$ converges for each $\alpha<1$ and is bounded for all $\alpha<1$.

We observe by inspection that the first condition is satisfied.

By Stirling's formula

$$\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} = n^{n(\alpha-1)}\,\alpha^{n\alpha+1/2}\,e^{-n(\alpha-1)}\,[1+o(1)],$$

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[10] Math. Annalen, 65 (1907–08), pp. 350–369, p. 359.

and

$$n\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)}=n^{n(\alpha-1)+1}\,\alpha^{n\alpha+1/2}\,e^{-n(\alpha-1)}\,[1+o(1)].$$

This expression can obviously be made less than a preassigned positive constant ε for n sufficiently large. Accordingly, condition 2 is fulfilled.

It remains to show that condition 3 is satisfied. Set

$$a_n=\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)}$$

and consider

$$\text{(2.1)}\quad K=\sum_{n=0}^{\infty}(n+1)\,|\,\Delta^2 a_n\,|=\sum_{n=0}^{3[\sigma]}(n+1)\,|\,\Delta^2 a_n\,|+\sum_{n=3[\sigma]+1}^{\infty}(n+1)\,|\,\Delta^2 a_n\,|$$
$$=K_1+K_2,$$

where $\sigma=\dfrac{1}{1-\alpha}$ and $[\sigma]$ is the integral part of σ.

Let us consider K_1. We have

$$\begin{aligned}K_1&=\sum_{n=0}^{3[\sigma]}(n+1)\,|\,\Delta^2 a_n\,|\\&=\sum_{n=0}^{3[\sigma]}(n+1)\left|\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)}-2\frac{\Gamma[(n+1)\alpha+1]}{\Gamma(n+2)}+\frac{\Gamma[(n+2)\alpha+1]}{\Gamma(n+3}\right|.\end{aligned}$$

Our next move is to express $\Delta^2 a_n$ as a polynomial in $1-\alpha$. This is done as follows.

$$\begin{aligned}\Delta^2 a_n&=\alpha\frac{\Gamma[(n+2)\alpha]}{\Gamma(n+2)}-2\alpha\frac{\Gamma[(n+1)\alpha]}{\Gamma(n+1)}+\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)}\\
&=\alpha\frac{(n+2)\alpha-1}{n+1}\,\frac{\Gamma[(n+2)\alpha-1]}{\Gamma(n+1)}-2\alpha\frac{\Gamma[(n+1)\alpha]}{\Gamma(n+1)}+\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)}\\
&=\frac{1}{\Gamma(n+1)}\left\{\alpha\left[\alpha-\frac{1-\alpha}{n+1}\right]\Gamma(n\alpha+2\alpha-1)-2\alpha\,\Gamma(n\alpha+\alpha)\right.\\
&\qquad\left.+\,\Gamma(n\alpha+1)\right\}\\
&=\frac{1}{\Gamma(n+1)}\left\{\Gamma(n\alpha+2\alpha-1)-2\,\Gamma(n\alpha+\alpha)+\Gamma(n\alpha+1)\right.\\
&\qquad+(1-\alpha)\left[2\,\Gamma(n\alpha+\alpha)-\left(2+\frac{1}{n+1}\right)\Gamma(n\alpha+2\alpha-1)\right]\\
&\qquad\left.+(1-\alpha)^2\frac{n+2}{n+1}\,\Gamma(n\alpha+2\alpha-1)\right\}\\
&=c_n-d_n,\end{aligned}$$

where

$$c_n = \frac{1}{\Gamma(n+1)}\Big\{\Gamma(n\alpha+2\alpha-1)-2\Gamma(n\alpha+\alpha)+\Gamma(n\alpha+1)$$
$$+2(1-\alpha)[\Gamma(n\alpha+\alpha)-\Gamma(n\alpha+2\alpha-1)]$$
$$+(1-\alpha)^2\frac{n+2}{n+1}\Gamma(n\alpha+2\alpha-1)\Big\}$$

and

$$d_n = \frac{(1-\alpha)\Gamma(n\alpha+2\alpha-1)}{(n+1)\Gamma(n+1)}.$$

The expression for c_n is a polynomial in $1-\alpha$ which is positive for $n \geqq 4$, $\frac{1}{2} \leqq \alpha < 1$, since the leading term is positive by virtue of the fact that $\Gamma^{(2)}(x) > 0$ for $x > 0$ and since the coefficients of the remaining terms are clearly positive. We have then

$$K_1 < \sum_{n=4}^{3[\sigma]}(n+1)\Delta^2 a_n + 2\sum_{n=4}^{3[\sigma]}(n+1)d_n + C = L_1 + L_2 + C. \tag{2.2}$$

We can write L_1 in the form

$$L_1 = \sum_{n=4}^{3[\sigma]}(n+1)\Delta^2 a_n = \sum_{n=4}^{3[\sigma]}(n+1)(\Delta a_n - \Delta a_{n+1})$$
$$= \sum_{n=4}^{3[\sigma]}(n+1)\Delta^2 a_n - \sum_{n=4}^{3[\sigma]-1}(n+1)\Delta a_{n+1} - (3[\sigma]+1)\Delta a_{3[\sigma]+1}$$
$$= \sum_{n=5}^{3[\sigma]}\Delta a_n + 5\Delta a_4 - (3[\sigma]+1)\Delta a_{3[\sigma]+1}.$$

Repeating this operation once more we get

$$L_1 = a_5 - a_{3[\sigma]+1} + 5\Delta a_4 - (3[\sigma]+1)\Delta a_{3[\sigma]+1}$$

or

$$L_1 = 5a_4 - 4a_5 - (3[\sigma]+2)a_{3[\sigma]+1} + (3[\sigma]+1)a_{3[\sigma]+2}.$$

But $0 < a_n < 1$, and $\frac{1}{\sigma}(3[\sigma]+2) > 2$ if $\sigma > 2$. Hence

$$0 < L_1 < 5a_4 + (3[\sigma]+1)a_{3[\sigma]+2} < 5 + \frac{1}{3[\sigma]+2} < 6. \tag{2.3}$$

We are now ready to consider

$$L_2 = 2\sum_{n=4}^{3[\sigma]}(n+1)d_n.$$

We have

$$d_n = \frac{1-\alpha}{n+1}\,\frac{\Gamma(n\alpha+2\alpha-1)}{\Gamma(n\alpha+1)} = \frac{1}{\sigma(n+1)}\,\frac{\Gamma\left(n+1-\frac{n+2}{\sigma}\right)}{\Gamma(n+1)}$$

and accordingly

$$d_n < \frac{1}{\sigma(n+1)}.$$

Hence

$$(2.4) \qquad L_2 = 2\sum_{n=4}^{3[\sigma]} (n+1) d_n < 6.$$

It follows from formulas (2.2), (2.3), and (2.4) that

$$(2.5) \qquad K_1 < L_1 + L_2 + C < C.$$

Finally let us consider

$$K_2 = \sum_{3[\sigma]+1}^{\infty} (n+1) \,|\, \Delta^2 a_n \,|.$$

The numerically largest term in $\Delta^2 a_n$ is a_n. Moreover, since $n\alpha+1 = n+1-\frac{n}{\sigma} < n+1-3 = n-2$, we have

$$(n+1)\,|\,\Delta^2 a_n\,| < C(n+1)\frac{\Gamma(n-2)}{\Gamma(n+1)} < \frac{C}{n^2}.$$

Hence

$$(2.6) \qquad K_2 < C \sum_{3[\sigma]+1}^{\infty} \frac{1}{n^2} < C.$$

Now, from (2.1), (2.5), and (2.6) we have $K < C$.

Since $K < C$, and since the boundedness of the series (2.1) implies its convergence, all its terms being positive, it is clear that condition 3 of p. 86 is satisfied.

We conclude that *any series summable* $(C1)$ *is also summable Le Roy to the same sum.*

In particular, it follows that *the Fourier series of any integrable function, $f(x)$, is summable Le Roy to the sum $f(x)$ for almost all values of x*, since the series is known to be summable $(C1)$ for almost all x.

3. **$(Cr) \to LR$.** In this paragraph we extend the methods of § 2 to show that *Le Roy summability includes* (Cr) *summability.* That a proof of this for integral values of r will be sufficient is readily indicated. We know that summability (Cs) always includes summability (Cr) if $s > r$. Now, take s equal to the least integer $\geqq r$. Then, if Le Roy summability includes summability (Cs) it will a fortiori include summability (Cr).

At the outset we give a number of lemmas to which we shall have occasion to make reference in the course of our proof.

LEMMA 1. *If we have*

$$\Delta^s \varphi(x) = \sum_{\nu=0}^{s} (-1)^{\nu} \binom{s}{\nu} \varphi(x+\nu\sigma)$$

or

$$\delta^s \varphi(x) = \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} \varphi(x+\nu\sigma),$$

and if $\varphi(x)$ *is* s *times differentiable, then*

$$\Delta^s \varphi(x) = (-\sigma)^s \varphi^{(s)}(x+\theta_s \sigma) \tag{3.1}$$

or

$$\delta^s \varphi(x) = \sigma^s \varphi^{(s)}(x+\theta_s \sigma), \tag{3.2}$$

where $0 < \theta_s < s$.

LEMMA 2.[11]

$$\lim_{x\to\infty} \frac{\Gamma(x-y)\, x^y}{\Gamma(x)} = 1 \tag{3.3}$$

if $\frac{y^2}{x} \to 0$ and $x \to \infty$ in such a manner that its distance from the negative real axis becomes infinite.

It follows from this lemma that the left hand side of (3.3) is bounded for large positive values of x when $\frac{y^2}{x}$ is bounded. We shall use merely this consequence of the lemma.

Using Stirling's formula we have

$$\frac{\Gamma(x-y)}{\Gamma(x)} = \frac{e^{-x+y}(x-y)^{x-y-1/2}}{e^{-x} x^{x-1/2}} [1+o(1)]$$

$$= e^y \left(1-\frac{y}{x}\right)^{x-y-1/2} x^{-y} [1+o(1)].$$

Then

$$\frac{\Gamma(x-y)\, x^y}{\Gamma(x)} = \left(1-\frac{y}{x}\right)^{x-y-1/2} e^y [1+o(1)].$$

Since we assumed that $\frac{y^2}{x} \to 0$ as $x \to \infty$, it follows that also $\frac{y}{x} \to 0$. Hence, if we set

$$z = \left(1-\frac{y}{x}\right)^{x-y-1/2},$$

then

$$\log z = \left(x-y-\frac{1}{2}\right) \log \left(1-\frac{y}{x}\right)$$

$$= \left(x-y-\frac{1}{2}\right) \left[-\frac{y}{x} + O\left(\left|\frac{y}{x}\right|^2\right)\right]$$

$$= -y + O\left(\left|\frac{y}{x}\right|^2\right).$$

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[11] Essentially the same theorem for the case that y is bounded is proved by N. Nielsen, Handbuch der Theorie der Gammafunktion, p. 96.

Accordingly, $e^{y+z} \to 1$, and

$$\lim_{x\to\infty} \frac{\Gamma(x-y)\,x^y}{\Gamma(x)} = 1.$$

LEMMA 3.

$$\begin{aligned}(3.4)\qquad \sum_{n=p}^{q} \frac{(n+r)!}{n} \Delta^{r+1} a_n = (p+r)! \sum_{\nu=0}^{r} \frac{r!}{(r-\nu)!} \frac{\Delta^{r-\nu} a_{p+\nu}}{(p+\nu)!} \\ - \frac{1}{q!} \sum_{\nu=0}^{r} \frac{r!}{(r-\nu)!} (q+r-\nu)!\, \Delta^{r-\nu} a_{q+1}.\end{aligned}$$

This formula is established as follows. We have

$$\begin{aligned}&\sum_{n=p}^{q} \frac{(n+r)!}{n!} \Delta^{r+1} a_n \\ &= \sum_{n=p}^{q} \frac{(n+r)!}{n!} \Delta^r a_n - \sum_{n=p}^{q} \frac{(n+r)!}{n!} \Delta^r a_{n+1} \\ &= \sum_{n=p}^{q} \frac{(n+r)!}{n!} \Delta^r a_n - \sum_{n=p}^{q-1} \frac{(n+r)!}{n!} \Delta^r a_{n+1} - \frac{(q+r)!}{q!} \Delta^r a_{q+1} \\ &= \sum_{n=p+1}^{q} \left[\frac{(n+r)!}{n!} - \frac{(n+r-1)!}{(n-1)!}\right] \Delta^r a_n - \frac{(q+r)!}{q!} \Delta^r a_{q+1} + \frac{(p+r)!}{p!} \Delta^r a_p \\ &= r \sum_{n=p+1}^{q} \frac{(n+r-1)!}{n!} \Delta^r a_n - \frac{(q+r)!}{q!} \Delta^r a_{q+1} + \frac{(p+r)!}{p!} \Delta^r a_p.\end{aligned}$$

Repeating this operation r times we obtain the formula (3.4).

LEMMA 4. *For large positive values of x the function $\Gamma^{(r)}(x)$ is positive and*

$$(3.5)\qquad \Gamma^{(r)}(x) = O[(\log x)^r] \cdot \Gamma(x).$$

We need the following formula for $\Gamma^{(r)}(x)$ which may be verified by complete induction:[12]

$$\Gamma^{(r)}(x) = \Gamma(x) \sum A^{(r)}_{\nu_1,\nu_2,\cdots,\nu_r} [\psi(x)]^{\nu_1} [\psi^{(1)}(x)]^{\nu_2} \cdots [\psi^{(r-1)}(x)]^{\nu_r},$$

where the summation is extended over all non-negative integers $\nu_1, \nu_2, \cdots, \nu_r$ such that

$$\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + r\nu_r = r,$$

and where the A's are non-negative integers. In particular,

$$A^{(r)}_{r,0,\cdots,0} = 1, \quad A^{(r)}_{r-2,1,0,\cdots,0} = \binom{r}{2}.$$

[12] Cf. formula (8) on p. 43 of N. Nielsen's Handbuch.

Further,

$$\psi^{(k)}(x) = \frac{d^{k+1}}{dx^{k+1}} \log \Gamma(x).$$

Now, we observe that for large values of $|x|$ in the sector $|\arg x| \leqq \pi - \varepsilon$[13]

$$\psi(x) = \log x + O\left(\frac{1}{x}\right), \tag{3.6}$$

$$\psi^{(k)}(x) = (-1)^{k-1}(k-1)!\, x^{-k}\left[1 + O\left(\frac{1}{x}\right)\right]. \tag{3.7}$$

It follows that

$$\Gamma^{(r)}(x) = \Gamma(x) \sum A^{(r)}_{\nu_1, \nu_2, \cdots \nu_r} (\log x)^{\nu_1} x^{-\nu_2 - 2\nu_3 - \cdots - (r-1)\nu_r}\left[1 + O\left(\frac{1}{x}\right)\right]$$

which implies the truth of our lemma.

Definition. Set

$$T_{n, \varrho, m} = \sum_{p=m}^{\varrho+1} (-1)^{m+p} \binom{\varrho+1}{p} (n+p)^m S^{(m)}_{n+1, n+p},$$

where

$$S^{(0)}_{n+1, n+k} = 1, \qquad k \geqq 0,$$

$$S^{(1)}_{n+1, n+k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k},$$

$$\cdots\cdots\cdots\cdots \tag{3.8}$$

$$S^{(m)}_{n+1, n+k} = \sum \frac{1}{(n+\nu_1)(n+\nu_2)\cdots(n+\nu_m)},$$

where the summation with respect to the ν's is to be extended over the range $1, 2, \cdots, k$, all the ν's are distinct, and no repetition is allowed. In other words, $S^{(m)}_{n+1, n+k}$ is the coefficient of x^m in the algebraic equation whose roots are $-\frac{1}{n+1}, -\frac{1}{n+2}, \cdots, -\frac{1}{n+k}$, the leading coefficient being unity. We set $S^{(m)}_{n+1, n+k} = 0$ if $k < m$.

LEMMA 5. *If $T_{n, \varrho, m}$ is defined as above we have*

$$T_{n, \varrho, 0} = 0, \qquad \varrho = 0, 1, 2, \cdots, \tag{3.9}$$

and

$$T_{n, -1, 0} = 1. \tag{3.10}$$

The truth of our lemma follows since

$$T_{n, \varrho, 0} = \sum_{p=0}^{\varrho+1} (-1)^p \binom{\varrho+1}{p}$$

[13] Cf. N. E. Nörlund, Vorlesungen über Differenzenrechnung, p. 106.

gives the coefficients in the expansion of $(1-x)^{\varrho+1}$, which is clearly equal to 0 for $x=1$, $\varrho=0, 1, 2, \cdots$, and to 1 for $x=1$, $\varrho=-1$.

LEMMA 6. *Set*

$$\delta^k S^{(m)}_{n+1,n} = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} S^{(m)}_{n+1,n+\nu}.$$

Then

$$\delta^k S^{(m)}_{n+1,n} = \begin{cases} O(n^{-m}), & k \leqq m, \\ O(n^{-k}), & k > m. \end{cases} \tag{3.11}$$

In order to prove this lemma we must first prove that

$$\delta S^{(m)}_{n+1,n+p} = S^{(m)}_{n+1,n+p+1} - S^{(m)}_{n+1,n+p} = \frac{1}{n+p+1} S^{(m-1)}_{n+1,n+p}. \tag{3.12}$$

We note that

$$\left(1-\frac{t}{n+p+1}\right) \prod_{k=1}^{p} \left(1-\frac{t}{n+k}\right) = \prod_{k=1}^{p+1} \left(1-\frac{t}{n+k}\right),$$

or

$$\left(1-\frac{t}{n+p+1}\right) \sum_{m=0}^{p} (-1)^m t^m S^{(m)}_{n+1,n+p} = \sum_{m=0}^{p+1} (-1)^m t^m S^{(m)}_{n+1,n+p+1}.$$

Equating coefficients of t^m in this identity gives us formula (3.12).

Next, we compute

$$\begin{aligned} \delta^2 S^{(m)}_{n+1,n+p} &= \frac{1}{n+p+2} S^{(m-1)}_{n+1,n+p+1} - \frac{1}{n+p+1} S^{(m-1)}_{n+1,n+p} \\ &= \frac{1}{n+p+2} \{S^{(m-1)}_{n+1,n+p+1} - S^{(m-1)}_{n+1,n+p}\} \\ &\qquad + \left\{\frac{1}{n+p+2} - \frac{1}{n+p+1}\right\} S^{(m-1)}_{n+1,n+p} \\ &= \frac{1}{(n+p+2)\,(n+p+1)} \{S^{(m-2)}_{n+1,n+p} - S^{(m-1)}_{n+1,n+p}\}. \end{aligned}$$

Repeating this operation $k-2$ times we get

$$\begin{aligned} \delta^k S^{(m)}_{n+1,n+p} &= \frac{1}{(n+p+1)\cdots(n+p+k)} \\ &\quad \times \{l_0^{(k)} S^{(m-k)}_{n+1,n+p} + l_1^{(k)} S^{(m-k+1)}_{n+1,n+p} + \cdots + l_{k-1}^{(k)} S^{(m-1)}_{n+1,n+p}\}, \end{aligned} \tag{3.13}$$

where the coefficients $l_i^{(k)}$, $i=0, 1, 2, \cdots, k-1$, are defined by the identity

$$x(x-1)\cdots(x-k+1) = l_0^{(k)} x^k + l_1^{(k)} x^{k-1} + \cdots + l_{k-1}^{(k)} x.$$

Multiplying both sides of this expression by $x-k$ we obtain the recursion formula

$$l_i^{(k+1)} = l_i^{(k)} - k\, l_{i-1}^{(k)}.$$

If $k > m$, $p \geqq 0$, we have clearly

$$\delta^k S_{n+1,n+p}^{(m)} = O(n^{-k}), \qquad m < k.$$

If $k \leqq m$, $m-k \leqq p$, we have at least $k+m-k$ terms in the denominator of the expression (3.13). Accordingly,

$$\delta^k S_{n+1,n+p}^{(m)} = O(n^{-m}), \qquad k \leqq m.$$

LEMMA 7. *The expression*

$$\delta^k n^m = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} (n+\nu)^m,$$

a polynomial in n, is of degree $m-k$ if $k \leqq m$, 0 if $k > m$.

From formula (3.2) we have

$$\delta^k n^m = \varphi^{(k)}(n+\theta_k), \qquad \varphi(n) = n^m,$$

and the truth of the lemma to be proved is immediately apparent.

LEMMA 8. *The expression for the kth difference of a product uv in terms of differences of u alone and v alone is given by the formula*[14]

$$(3.14) \quad \delta^k u_\mu v_\mu = u_{\mu+k}\, \delta^k v_\mu + \binom{k}{1} \delta u_{\mu+k-1}\, \delta^{k-1} v_\mu + \cdots + \binom{k}{k} v_\mu\, \delta^k u_\mu.$$

This formula is clearly the analogue of Leibnitz' formula for the kth derivative of a product.

In order to prove that Le Roy summability includes (Cr) summability it will be sufficient to prove that

1. $\lim\limits_{\alpha \to 1^-} \dfrac{\Gamma(n\alpha+1)}{\Gamma(n+1)} = 1,$

2. $\lim\limits_{n\to\infty} n^r \dfrac{\Gamma(n\alpha+1)}{\Gamma(n+1)} = 0, \qquad \alpha < 1,$

3. $\sum\limits_{n=0}^{\infty} n^r \left| \Delta^{r+1} \dfrac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \right|$ converges for each $\alpha < 1$ and is bounded for all $\alpha < 1$.

[14] G. Wallenberg und A. Guldberg, Theorie der Linearen Differenzengleichungen, p. 34.

Condition 1 is clearly satisfied, and condition 2 is easily shown to be fulfilled by use of Stirling's formula.

In order to show that condition 3 is fulfilled we first write

$$
\begin{aligned}
K = \sum_{n=0}^{\infty} n^r \,|\Delta^{r+1} a_n| &= \sum_{n=0}^{\nu} n^r \,|\Delta^{r+1} a_n| + \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r \,|\Delta^{r+1} a_n| \\
&\quad + \sum_{n=(r+2)[\sigma]+1}^{\infty} n^r \,|\Delta^{r+1} a_n| \\
&= K_1 + K_2 + K_3,
\end{aligned} \tag{3.15}
$$

where ν is a constant, independent of n or α, which will be specified later.

The first block of terms, K_1, is clearly bounded since K_1 consists merely of a finite number of terms. We write

$$K_1 < C, \tag{3.16}$$

where C depends only on r.

The most difficult task in this paragraph is to show that K_2 is uniformly bounded. We have

$$K_2 = \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r \,|\Delta^{r+1} a_n|,$$

where

$$\Delta^{r+1} a_n = \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} - \binom{r+1}{1} \frac{\Gamma[(n+1)\alpha+1]}{\Gamma(n+2)}$$

$$+ \cdots + (-1)^k \binom{r+1}{k} \frac{\Gamma[(n+k)\alpha+1]}{\Gamma(n+k+1)} + \cdots + (-1)^{r+1} \frac{\Gamma[(n+r+1)\alpha+1]}{\Gamma(n+r+2)}.$$

But

$$\frac{\Gamma[(n+k)\alpha+1]}{\Gamma(n+k+1)} = \frac{(n+k)\alpha}{n+k} \frac{\Gamma[(n+k)\alpha]}{\Gamma(n+k)} = \frac{(n+k)\alpha}{n+k} \frac{(n+k)\alpha-1}{n+k-1} \frac{\Gamma[(n+k)\alpha-1]}{\Gamma(n+k-1)}$$

$$= \cdots = \frac{(n+k)\alpha\,[(n+k)\alpha-1]\,[(n+k)\alpha-2]\cdots[(n+k)\alpha-k+1]}{(n+k)(n+k-1)(n+k-2)\cdots(n+1)} \frac{\Gamma[(n+k)\alpha-k+1]}{\Gamma(n+1)}$$

$$= R_{n+1,\,n+k}(\alpha) \frac{\Gamma[(n+k)\alpha-k+1]}{\Gamma(n+1)}.$$

Hence

$$\Delta^{r+1} a_n = \frac{1}{\Gamma(n+1)} \sum_{k=0}^{r+1} (-1)^k \binom{r+1}{k} R_{n+1,\,n+k}(\alpha)\,\Gamma[(n+k)\alpha-k+1].$$

We wish now to express $R_{n+1,\,n+k}(\alpha)$ as a polynomial in $1-\alpha$. Since

$$\frac{m\alpha-p}{m-p} = \frac{m-p-m(1-\alpha)}{m-p} = 1 - \frac{m}{m-p}(1-\alpha),$$

we have

$$R_{n+1,\,n+k}(\alpha) = \sum_{p=0}^{k-1}\left[1 - \frac{n+k}{n+k-p}(1-\alpha)\right]$$
$$= \sum_{m=0}^{k}(-1)^m (n+k)^m S_{n+1,\,n+k}^{(m)}(1-\alpha)^m,$$

where the $S_{n+1,\,n+k}^{(m)}$ are defined by the formulas (3.8).

Expressed in terms of the symmetric functions $S_{n+1,\,n+k}^{(m)}$ we have

$$\Delta^{r+1} a_n = \frac{1}{\Gamma(n+1)} \sum_{k=0}^{r+1} (-1)^k \binom{r+1}{k} \Gamma[(n+k)\alpha - k + 1]$$
$$\times \sum_{m=0}^{k} (-1)^m (n+k)^m S_{n+1,\,n+k}^{(m)} (1-\alpha)^m$$
$$= \frac{1}{\Gamma(n+1)} \sum_{m=0}^{r+1} (1-\alpha)^m \sum_{p=m}^{r+1} (-1)^{m+p} \binom{r+1}{p} (n+p)^m$$
$$\times S_{n+1,\,n+p}^{(m)} \Gamma[(n+p)\alpha - p + 1].$$

In this expression we want to replace Γ by an expression in terms of its own differences. We have

$$\Gamma[(n+p)\alpha - p + 1] = \Gamma[(n+r+1)\alpha - r + (r+1-p)(1-\alpha)]$$
$$= \gamma(n,\, r;\, r+1-p).$$

We define

$$\delta_{n,r}^{s} = \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} \gamma(n,\, r;\, \nu).$$

Then, conversely[15]

$$\Gamma[(n+p)\alpha - p + 1] = \sum_{s=0}^{r+1-p} \binom{r+1-p}{s} \delta_{n,r}^{s}.$$

Hence

$$\Delta^{r+1} a_n = \frac{1}{\Gamma(n+1)} \sum_{m=0}^{r+1} (1-\alpha)^m \sum_{p=m}^{r+1} (-1)^{m+p} \binom{r+1}{p} (n+p)^m$$
$$\times S_{n+1,\,n+p}^{(m)} \sum_{s=0}^{r+1-p} \binom{r+1-p}{s} \delta_{n,r}^{s}$$
$$= \frac{1}{\Gamma(n+1)} \sum_{m=0}^{r+1} (1-\alpha)^m \sum_{s=0}^{r+1-m} \delta_{n,r}^{s} \sum_{p=m}^{r+1-s} (-1)^{m+p}$$
$$\times \binom{r+1}{p} \binom{r+1-p}{s} (n+p)^m S_{n+1,\,n+p}^{(m)}$$

[15] Cf. Nörlund, Differenzenrechnung, p. 4.

$$= \frac{1}{\Gamma(n+1)} \sum_{m=0}^{r+1} (1-\alpha)^m \sum_{s=0}^{r+1-m} \delta_{n,r}^s \binom{r+1}{s} \sum_{p=m}^{r+1-s} (-1)^{m+p} \times \binom{r+1-s}{p} (n+p)^m S_{n+1,n+p}^{(m)},$$

or

$$\Delta^{r+1} a_n = \frac{1}{\Gamma(n+1)} \sum_{m=0}^{r+1} (1-\alpha)^m \sum_{s=0}^{r+1-m} \delta_{n,r}^s \binom{r+1}{s} T_{n,r-s,m}, \tag{3.17}$$

where

$$T_{n,\varrho,m} = \sum_{p=m}^{\varrho+1} (-1)^{m+p} \binom{\varrho+1}{p} (n+p)^m S_{n+1,n+p}^{(m)}.$$

Now, we write

$$\Delta^{r+1} a_n = c_n - d_n,$$

where

$$c_n = \frac{1}{\Gamma(n+1)} \delta_{n,r}^{r+1},$$

and

$$d_n = -\frac{1}{\Gamma(n+1)} \sum_{m=1}^{r+1} (1-\alpha)^m \sum_{s=0}^{r+1-m} \delta_{n,r}^s \binom{r+1}{s} T_{n,r-s,m}.$$

The expression for c_n, as may already have been observed, is obtained by setting $m = 0$ in (3.17). For, using formulas (3.9) and (3.10), we get

$$\begin{aligned} c_n &= \frac{1}{\Gamma(n+1)} \sum_{s=0}^{r+1} \binom{r+1}{s} \sum_{\nu=0}^{s} (-1)^{s-\nu} \gamma(n, r; \nu) \\ &= \frac{1}{\Gamma(n+1)} \sum_{\nu=0}^{r+1} (-1)^{r+1-\nu} \binom{r+1}{\nu} \gamma(n, r; \nu) \\ &= \frac{1}{\Gamma(n+1)} \delta_{n,r}^{r+1}. \end{aligned}$$

Moreover, by formula (3.2) and Lemma 4, c_n is positive for large values of n. We choose ν, the upper limit in K_1, large enough to insure that c_n be positive for $n \geqq \nu$.

To return to the expression for K_2 we have

$$\begin{aligned} K_2 &= \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r |\Delta^{r+1} a_n| = \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r |c_n - d_n| \\ &\leqq \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r \Delta^{r+1} a_n + 2 \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r |d_n| \qquad (3.18) \\ &= L_1 + L_2. \end{aligned}$$

To show that L_1 is bounded in absolute value it will be convenient to replace n^r in the expression for L_1 by $(n+r)!/n!$, a polynomial of r-th degree in n. The new expression

$$L_1' = \sum_{n=\nu+1}^{(r+2)[\sigma]} \frac{(n+r)!}{n!} \Delta^{r+1} a_n$$

will be bounded or fail to be bounded under the same conditions as the original expression. Using formula (3.4) we write L_1' in the form

$$L_1' = (p+r)! \sum_{\mu=0}^{r} \frac{r!}{(r-\mu)!} \frac{\Delta^{r-\mu} a_{p+\mu}}{(p+\mu)!} - \frac{1}{q!} \sum_{\mu=0}^{r} \frac{r!}{(r-\mu)!} (q+r-\mu)! \Delta^{r-\mu} a_{q+1}$$
$$= M_1 + M_2,$$

where $p = \nu + 1$ and $q = (r+2)[\sigma]$.

M_1 is clearly bounded in absolute value when r is fixed since it consists of a finite number of terms each of which is bounded. M_2 also consists of a finite number of terms of the type

$$\frac{(q+r-\mu)!}{q!} \Delta^{r-\mu} a_{q+1}. \tag{3.19}$$

Moreover, the $(r-\mu)$-th difference of a_{q+1} consists of a finite number of terms a_ν the numerically largest one of which is a_{q+1}. Let us then consider

$$\frac{(q+r-\mu)!}{q!} a_{q+1} \leqq \frac{\Gamma(q+r+1)\,\Gamma\left[q+2-\frac{1}{\sigma}(q+1)\right]}{\Gamma(q+1)\,\Gamma(q+2)}.$$

Now, recall that $q = (r+2)[\sigma]$, and suppose that $\sigma > 2r$; then $\frac{1}{\sigma}(q+1) \geqq r+1$. Hence

$$\frac{(q+r-\mu)!}{q!} < \frac{\Gamma(q+r+1)\,\Gamma(q-r+1)}{\Gamma(q+1)\,\Gamma(q+2)} < \frac{C}{\sigma},$$

where C depends only on r. Thus, the expression (3.19) is bounded in absolute value for $q = (r+2)[\sigma]$, $0 \leqq \mu \leqq r$. The reason for choosing $q = (r+2)[\sigma]$ will be evident when we have occasion to discuss the boundedness of K_3. For the present, we conclude that $|L_1'| \leqq |M_1| + |M_2| < C$ and hence that

$$|L_1| < C. \tag{3.20}$$

Now, consider

$$L_2 = 2 \sum_{n=\nu+1}^{(r+2)[\sigma]} n^r \, |d_n|.$$

In order to show that L_2 is bounded it is essential that we estimate the size of the expression

$$(1-\alpha)^m \delta_{n,r}^s T_{n,r-s,m}.$$

First of all we observe that

$$(-1)^m T_{n,\varrho,m} = \pm \delta^{\varrho+1} n^m S_{n+1,n}^{(m)}.$$

On the other hand, in the expansion of $\delta^{\varrho+1} n^m S_{n+1,n}^{(m)}$ by formula (3.14) it follows from (3.11) and Lemma 7 that each term is of the order of magnitude of $n^{m-\varrho-1}$, $\varrho+1 \geqq m$. Accordingly

$$\delta^{\varrho+1} n^m S_{n+1,n}^{(m)} = O[n^{m-\varrho-1}], \qquad \varrho+1 \geqq m,$$

and hence

$$(-1)^m T_{n,\varrho,m} = O[n^{m-\varrho-1}], \qquad \varrho+1 \geqq m.$$

From this result and formula (3.2) it follows that

$$(1-\alpha)^m \delta_{n,r}^s T_{n,r-s,m} = (1-\alpha)^{m+s} n^{m+s-r-1} \frac{d^s}{dx^s} \Gamma(x_{n,r,s}) \cdot C_{n,r,s,m},$$

where $|C_{n,r,s,m}| < C_r$, and where

$$x_{n,r,s} = (n+r+1)\alpha - r + \theta_s(1-\alpha), \qquad 0 < \theta_s < s.$$

Moreover,

$$x_{n,r,s} < (n+r+1)\alpha - r + s - s\alpha = n+1-(n+r-s+1)(1-\alpha).$$

Using these results and Lemma 4 we have

$$\frac{1}{\Gamma(n+1)} \left| (1-\alpha)^m \delta_{n,r}^s T_{n,r-s,m} \right|$$
$$< C(1-\alpha)^{m+s} n^{m+s-r-1} (\log n)^s \frac{\Gamma[n+1-(n+r-s+1)(1-\alpha)]}{\Gamma(n+1)},$$

and, from Lemma 2, it follows that

$$\frac{1}{\Gamma(n+1)} \left| (1-\alpha)^m \delta_{n,r}^s T_{n,r-s,m} \right|$$
$$< C(1-\alpha)^{m+s} n^{m+s-r-1-(n+r-s+1)(1-\alpha)} (\log n)^s.$$

We have now the estimate

$$n^r\,|\,d_n\,| < C\sum_{m=1}^{r+1}\sum_{s=0}^{r+1-m}\frac{n^{m+s-1}}{\sigma^{m+s}}\,\frac{(\log n)^s}{n^{n/\sigma}}$$
$$= C\sum_{m=1}^{r+1}\sum_{s=0}^{r+1-m}\frac{n^{m-1}}{\sigma^m}\,\frac{(n\log n)^s}{\sigma^s\,e^{n\log n/\sigma}}.$$

At this juncture we must study the function

$$z = (x\log x)^s e^{-x\log x/\sigma} = y^s e^{-y/\sigma},$$

where $y = x\log x$. Now, z has a maximum, for $y = \sigma s$, equal to $s^s e^{-s}\sigma^s = C_s\sigma^s$. Hence, we have

$$n^r\,|\,d_n\,| < C\sum_{m=1}^{r+1}\frac{n^{m-1}}{\sigma^m}.$$

Moreover, if $m \geqq 1$, $n \leqq (r+2)[\sigma]$, we have

$$n^r\,|\,d_n\,| < \frac{C}{\sigma}$$

and

(3.21) $$0 < L_2 < C\sum_{n=\nu+1}^{(r+2)[\sigma]}\frac{1}{\sigma} < C.$$

Thus, from (3.15), (3.17), and (3.18),

(3.22) $$0 < K_2 < |\,L_1\,| + |\,L_2\,| < C.$$

Finally, we consider

$$K_3 = \sum_{n=(r+2)[\sigma]+1}^{\infty} n^r\,|\,\varDelta^{r+1}a_n\,|.$$

Now, there are a finite number of terms in $\varDelta^{r+1}a_n$ the numerically largest one of which is a_n. Moreover, since

$$n\alpha+1 = n+1-\frac{n}{\sigma} < n+1-r-2 = n-r-1,$$

we have

$$n^r\,|\,\varDelta^{r+1}a_n\,| < Cn^r\frac{\Gamma(n-r-1)}{\Gamma(n+1)} < \frac{C}{n^2}.$$

Accordingly, we have

(3.23) $$K_3 < C\sum_{n=(r+2)[\sigma]+1}^{\infty}\frac{1}{n^2} < C.$$

Assembling our several results in formulas (3.16), (3.22), and (3.23) we see that $K = K_1 + K_2 + K_3 < C$. It follows that condition 3 of p. 94 is fulfilled.

We conclude that *every series summable* (Cr) *is also summable Le Roy.*

4. $(Cr) \to ML$. In this paragraph we propose to show that every series summable (Cr) is also summable by the method of Mittag-Leffler. We can again restrict ourselves to integral values of r (vide supra p. 89).

We state the set of sufficient conditions due to Bromwich as applied to the problem in this paragraph. It will be sufficient to prove that

1. $\lim\limits_{s \to 0} \dfrac{1}{\Gamma(1+sn)} = 1,$

2. $\lim\limits_{n \to \infty} \dfrac{n^r}{\Gamma(1+sn)} = 0,\ s > 0,$

3. $\sum\limits_{n=0}^{\infty} n^r \left| \Delta^{r+1} \dfrac{1}{\Gamma(1+sn)} \right|$ converges for each $s > 0$ and is bounded for all $s > 0$.

We see at a glance that condition 1 is fulfilled. From Stirling's formula we have

$$\frac{n^r}{\Gamma(1+sn)} = n^r e^{ns} (ns)^{-ns-1/2} (2\pi)^{-1/2} [1 + o(1)].$$

For $s > 0$, r fixed, this expression can readily be made less than a preassigned positive constant ε for n sufficiently large. Accordingly, condition 2 is also satisfied. It remains to show that condition 3 is fulfilled.

We set

$$a_n = \frac{1}{\Gamma(1+sn)}$$

and consider

$$K = \sum_{n=0}^{\infty} n^r |\Delta^{r+1} a_n| = \sum_{n=0}^{p-1} n^r |\Delta^{r+1} a_n| + \sum_{n=p}^{\infty} n^r |\Delta^{r+1} a_n| = I + J, \tag{4.1}$$

where the integer p will be specified later on.

First of all we examine

$$I = \sum_{n=0}^{p-1} n^r |\Delta^{r+1} a_n|.$$

From formula (3.1) of the preceding paragraph we have

$$\Delta^{r+1} a_n = (-1)^{r+1} s^{r+1} \frac{d^{r+1}}{dz^{r+1}} \frac{1}{\Gamma(z_{n,r})}, \tag{4.2}$$

where $z_{n,r} = ns+1+\theta_r s,\ 0<\theta_r<r$. We are concerned with the expression

$$\frac{d^{r+1}}{dz^{r+1}}\frac{1}{\Gamma(z)} \tag{4.3}$$

which is clearly bounded for $z \geqq 1$, r fixed.

Let us set $s = \frac{1}{\sigma}$, and let $[\sigma]$ equal the integral part of σ. If we set $p = \nu[\sigma]$, where ν is an integer, independent of σ or n, to be specified later, we have

$$z_{n,r} = ns+1+\theta_r s = \frac{n}{\sigma}+1+\frac{\theta_r}{\sigma},$$

and, for $n \leqq p-1$,

$$z_{n,r} = \frac{n}{\sigma}+1+\frac{\theta_r}{\sigma}<\nu+1+\frac{\theta_r}{\sigma}<C_r.$$

Since the expression (4.3) is bounded for $z \geqq 1$ we have from formula (4.2) the inequality

$$n^r\,|\Delta^{r+1}a_n|<C\frac{n^r}{\sigma^{r+1}},$$

and, for $n \leqq p-1 = \nu[\sigma]-1$,

$$n^r\,|\Delta^{r+1}a_n|<\frac{C}{\sigma}.$$

Then, we have

$$I = \sum_{n=0}^{p-1} n^r\,|\Delta^{r+1}a_n|<C\sum_{n=0}^{\nu[\sigma]-1}\frac{1}{\sigma}<C. \tag{4.4}$$

Next, we examine

$$J = \sum_{n=p}^{\infty} n^r\,|\Delta^{r+1}a_n|.$$

In this expression we find it convenient, as in the preceding paragraph, to replace n^r by $\frac{(n+r)!}{n!}$. It follows that if the expression

$$J' = \sum_{n=p}^{\infty}\frac{(n+r)!}{n!}\,|\Delta^{r+1}a_n| \tag{4.5}$$

is bounded then J will also be bounded.

Before we embark upon a detailed discussion of the expression (4.5) it will be necessary to prove a lemma similar to Lemma 4 of the preceding paragraph.

LEMMA 9. *For large positive values of x the function* $\frac{d^r}{dx^r}\frac{1}{\Gamma(x)}$ *is positive or negative according as r is even or odd, and*

$$\frac{d^r}{dx^r}\frac{1}{\Gamma(x)} = O[(\log x)^r]\cdot\frac{1}{\Gamma(x)}.$$

In establishing this lemma we make use of the formula

$$(4.6)\quad \frac{d^r}{dx^r}\frac{1}{\Gamma(x)} = \frac{1}{\Gamma(x)}\sum B^{(r)}_{\nu_1,\nu_2,\cdots,\nu_r}[\psi(x)]^{\nu_1}[\psi^{(1)}(x)]^{\nu_2}\cdots[\psi^{(r-1)}(x)]^{\nu_r},$$

where the summation is extended over all non-negative integers ν_1, $\nu_2, \cdots, \nu_r$ such that

$$\nu_1 + 2\nu_2 + \cdots + r\nu_r = r,$$

and where the B's are numerical constants. In particular

$$B^{(r)}_{r,0,\cdots,0} = (-1)^r,\; B^{(r)}_{r-2,1,0,\cdots,0} = (-1)^{r+1}\binom{r}{2}.$$

The proof follows directly from formulas (3.6), (3.7), and (4.6).

It follows from formula (3.1) and Lemma 9 that $|\Delta^{r+1}a_n| = \Delta^{r+1}a_n$ for $n \geqq p = \nu[\sigma]$ provided that ν is made large enough so that Lemma 9 obtains. Then, if we set

$$J'_q = \sum_{n=p}^{q}\frac{(n+r)!}{n!}\Delta^{r+1}a_n,$$

we have, using formula (3.4),

$$\begin{aligned} J'_q &= (p+r)!\sum_{\mu=0}^{r}\frac{r!}{(r-\mu)!}\frac{\Delta^{r-\mu}a_{p+\mu}}{(p+\mu)!} - \frac{1}{q!}\sum_{\mu=0}^{r}\frac{r!}{(r-\mu)!}(q+r-\mu)!\,\Delta^{r-\mu}a_{q+1} \\ &= L + M_q, \end{aligned}$$

where $p = \nu[\sigma]$ and L is independent of q.

Now

$$L = (p+r)!\sum_{\mu=0}^{r}\frac{r!}{(r-\mu)!}\frac{\Delta^{r-\mu}a_{p+\mu}}{(p+\mu)!}$$

consists of a finite number of terms of the type

$$\frac{(p+r)!}{(p+\mu)!}\Delta^{r-\mu}a_{p+\mu}, \qquad \mu = 0,\,1,\,2,\,\cdots,\,r.$$

From formula (3.1) we have

$$\frac{(p+r)!}{(p+\mu)!}\,\Delta^{r-\mu}\,a_{p+\mu} = O\,[p^{r-\mu}]\cdot(-s)^{r-\mu}\,\frac{d^{r-\mu}}{dx^{r-\mu}}\,\frac{1}{\Gamma(x_{p,r,\mu})},$$

and using Lemma 9 we obtain

$$\frac{(p+r)!}{(p+\mu)!}\,\Delta^{r-\mu}\,a_{p+\mu} = O\left[p^{r-\mu}\,s^{r-\mu}\,\frac{(\log x_{p,r,\mu})^{r-\mu}}{\Gamma(x_{p,r,\mu})}\right],$$

where

$$x_{p,r,\mu} = (p+\mu)\,s+1+\theta_r\,s = (\nu[\sigma]+\mu)/\sigma+1+\theta_r/\sigma < C.$$

Hence

$$\frac{(p+r)!}{(p+\mu)!}\,|\,\Delta^{n-\mu}\,a_{p+\mu}\,| = O\left[\frac{p^{r-\mu}}{\sigma^{r-\mu}}\right] < C,$$

whence it follows that $|\,L\,| < C$.

We must now consider

$$M_q = -\frac{1}{q!}\sum_{\mu=0}^{r}\frac{r!}{(r-\mu)!}\,(q+r-\mu)!\;\Delta^{r-\mu}\,a_{q+1},$$

which consists of a finite number of terms of the type

$$\frac{(q+r-\mu)!}{q!}\,\Delta^{r-\mu}\,a_{q+1}, \qquad \mu = 0,\,1,\,2,\,\cdots,\,r,$$

which in turn consists of a finite number of terms the numerically largest one of which is

$$\frac{(q+r)!}{q!}\,a_{q+1}.$$

Moreover, by condition 2, this expression tends to zero as q tends to infinity. Then, M_q tends to zero; and since J'_q tends to J' we observe that J' is bounded and hence

(4.7) $$O < J < C.$$

From (4.1), (4.4), and (4.7) it follows that $K < C$ and hence that condition 3 is fulfilled.

We conclude that *every series summable* (Cr) *is also summable by the method of Mittag-Leffler.*

5. **$(Cr)\to D$.** In this paragraph we propose to investigate under what conditions

$$\lim_{s\to 0}\sum_{n=0}^{\infty} u_n\,e^{-\lambda_n s}$$

exists (where $s = \sigma+i\tau$ approaches zero over a point set lying within an angle with vertex at the origin such that $|am|s \leqq a < \pi/2$) and is

equal to A, under the assumption that the series (1.1) is summable (Cr) to the sum A. In point of fact, we prove the following theorem.

THEOREM. *The Dirichlet's series definitions of summability include* (Cr) *summability* $(r > 0)$ *provided that* λ_n *is a logarithmico-exponential function of* n *which tends to infinity with* n *but not as slowly as* $\log n$ *nor faster than* n^{Δ}, *where* Δ *is any constant however large.*

Before proceeding with our proof we need to state a definition of summability due to M. Riesz.[16]

We write

$$C_\lambda^r(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^r u_n,$$

($r > 0$, not necessarily integral) where λ_n is a sequence of real increasing numbers whose limit is infinite, $\lambda_0 \geqq 0$. If

$$\omega^{-r} C_\lambda^r(\omega) \to A$$

as $\omega \to \infty$ it is said that the series (1.1) is summable (λ, r) to the sum A.

It has been shown by Riesz[17] that this definition is completely equivalent to (Cr) summability for the case that $\lambda_n = n$.

The solution of the problem proposed in this paragraph is based on several theorems due to Hardy[18] and to Hardy and M. Riesz[19] to which we shall refer as the occasion arises.

We first state that

(i) If a series (1.1) is summable (n, r), and if λ_n is a logarithmico-exponential function of n which tends to infinity with n but not faster than n^{Δ}, for every $\Delta > 1$, then the series is also summable (λ, r).[20]

Moreover

(ii) Since by assumption

$$\sum_{n=0}^{\infty} u_n e^{-\lambda_n s} \tag{5.1}$$

is summable (n, r), for $s = 0$, to the sum A, it is also summable (λ, r) to the same value.

This statement clearly follows from statement (i).

(iii) Since (1.1) is summable (λ, r), then (5.1) is uniformly summable (λ, r) throughout the angle a to the sum $f(s)$.[21]

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[16] Comptes Rendus, 149 (1909), pp. 909–912, p. 910.

[17] Comptes Rendus, 152 (1911), pp. 1651–1654, p. 1651.

[18] Proc. London Math. Soc., (2) 15 (1916), pp. 72–88.

[19] Hardy and Riesz, The General Theory of Dirichlet's Series (Cambridge Tracts in Mathematics and Mathematical Physics, No. 18).

[20] Hardy, loc. cit., p. 72.

[21] Hardy and Riesz, loc. cit., Theorem 23.

(iv) Since (5.1) is summable (λ, r) to the sum A for $s = 0$, and since (5.1) is uniformly summable (λ, r) to the sum $f(s)$ throughout the angle a, then $f(s) \to A$ as $s \to 0$ along any path lying entirely within the angle a. This follows from another theorem due to Hardy and Riesz.[22]

At this juncture it is necessary to discuss the convergence of the series (5.1). The abscissa of convergence of a Dirichlet's series is given by the formula[23]

$$\sigma_0 = \overline{\lim} \frac{\log |s_n|}{\lambda_n}. \tag{5.2}$$

Now, it is well known that if a series (1.1) is summable (Cr) then $s_n = o(n^r)$. Thus,

$$\sigma_0 \leqq \overline{\lim} \frac{r \log n}{\lambda_n}.$$

In order that $\sigma_0 \leqq 0$, in which case the series will converge in the right half of the s-plane, it is sufficient that λ_n tend to infinity faster than $\log n$. Hence, with this restriction on λ_n the series converges for $\Re(s) > 0$.

Now, the series converges to the same value $f(s)$ to which it is summable (λ, r). It follows that the value approached as $s \to 0$ must be the same in both cases. Thus, with the restrictions on λ_n which have been stated,

$$\lim_{s \to 0} \sum_{n=0}^{\infty} u_n e^{-\lambda_n s} = A,$$

and our theorem is proved.[24]

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[22] Loc. cit., Theorem 28.

[23] Cf. Hardy and Riesz, loc. cit., Theorem 7.

[24] A similar theorem has recently been announced by W. H. Durfee. See Bull. Amer. Math. Soc., 36 (1930), p. 634 (abstract No. 36-9-338). Added in proof.

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THE LOCUS DEFINED BY PARAMETRIC EQUATIONS.[1]

By William F. Osgood.

Let a configuration $\mathfrak{M}$ in the space of the complex variables $(x_1, \cdots, x_m)$ be defined by the equations

$$\text{(A)} \qquad x_\alpha = f_\alpha(u_1, \cdots, u_n), \qquad \alpha = 1, \cdots, m, \quad n \gtreqless m,$$

where $f_\alpha(u_1, \cdots, u_n)$ is analytic in a given point, taken for convenience as the origin, $(u) = 0$, and vanishes there, but does not vanish identically.

Consider the matrix

$$\text{(B)} \qquad \left\| \begin{matrix} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \dfrac{\partial f_m}{\partial u_1} & \cdots & \dfrac{\partial f_m}{\partial u_n} \end{matrix} \right\|.$$

Denote its rank with respect to identical vanishing by ϱ. Then $1 \leqq \varrho \leqq n, m$. We exclude the case $m = \varrho$, $n > \varrho$; for here a point (x) of $\mathfrak{M}$ would correspond to an infinite number of points (u), though $\mathfrak{M}$ would not be embedded in a space of higher dimensions.

More generally, let m functions $x_1, \cdots, x_m$ be given on a *configuration* $\mathfrak{g}$. A configuration $\mathfrak{g}$ is defined as follows.[2] Let

$$\mathfrak{G}: \quad w^l + A_1(u_1, \cdots, u_n) w^{l-1} + \cdots\cdots + A_l(u_1, \cdots, u_n) = 0$$

be an irreducible algebroid configuration, where $A_k(u_1, \cdots, u_n)$ is analytic in the origin and vanishes there. Let $w_1, \cdots, w_\sigma$ be functions each single-valued and continuous on $\mathfrak{G}$, vanishing at the origin, and analytic in the ordinary points. Then the points $(w, u) = (w_1, \cdots, w_\sigma, u_1, \cdots, u_n)$ constitute a *configuration* $\mathfrak{g}$ of the n-th *Stufe*, or *index* in the complex $(\sigma + n)$-dimensional space of the variables (w, u). And the definition is extended to any configuration into which the above is carried by a transformation

$$\text{(i)} \qquad v_k = \varphi_k(w_1, \cdots, w_\sigma, u_1, \cdots, u_n), \quad k = 1, \cdots, s = \sigma + n,$$

[1] Received July 30, 1930. — Lecture delivered at Stanford University, January 8, 1930.

[2] Cf. the author's Funktionentheorie, vol. II, 2d edition, 1929, Chap. 2, § 17.

where φ_k is analytic at the origin and

$$\frac{\partial(\varphi_1, \cdots, \varphi_s)}{\partial(w_1, \cdots, u_n)} \neq 0.$$

The problem with which this paper deals is the determination of the character of $\mathfrak{M}$.

§ 1. Definitions and the Case $\varrho = 1$.

Definition. By a *canonical element of surface* is meant the configuration of points (x) in the neighborhood of the origin, for which

$$\begin{aligned} x_1 &= u_1^\mu \\ x_\gamma &= u_\gamma, & \gamma &= 2, \cdots, r, \\ x_{r+\beta} &= f_{r+\beta}(u_1, \cdots, u_r), & \beta &= 1, \cdots, m-r, \end{aligned}$$

where μ is a natural number and $f_{r+\beta}(u_1, \cdots, u_r)$ is analytic at the origin and vanishes there. In particular, $f_{r+\beta}$ may vanish identically. The element is, moreover, said to be of *index* r. In the case $r = 1$, no equations $x_\gamma = u_\gamma$ appear. We speak here of a *canonical element of arc.* We will include also the case $r = n$, $m = n$, the equations $x_{r+\beta} = f_{r+\beta}$ being lacking, and speak here of a *canonical element of volume.*

More generally, the (x) may be subjected to any regular transformation, of form like (i), and likewise, in their turn, the (u). It is to be observed that the number, n, of the u_k may be greater than r, $n \geqq r$, $(u) = (u_1, \cdots, u_n)$.

We shall denote a canonical element of surface of index r by $\mathfrak{m}_r$, and understand by $\mathfrak{m}_0$ a point.

The regions $\mathfrak{U}$, $\overline{\mathfrak{U}}$, R. We denote by $\mathfrak{U}_n$ the neighborhood of a point (a) defined by the inequalities

$$|u_\gamma - a_\gamma| < \delta, \qquad \gamma = 1, \cdots, n.$$

In particular, if $(a) = (0)$, we have

$$\mathfrak{U}_n: \quad |u_\gamma| < \delta, \qquad \gamma = 1, \cdots, n.$$

We furthermore denote by $\overline{\mathfrak{U}}_n$ the bounded region,

$$\overline{\mathfrak{U}}_n: \quad |u_\gamma| \leqq \delta, \qquad \gamma = 1, \cdots, n;$$

and similarly for the point $(u) = (a)$.

Let H be so chosen that the points (x) which correspond to the points of $\mathfrak{U}_n$ or $\mathfrak{g}$ lie in the region

$$R: \quad |x_\alpha| \leqq H, \qquad \alpha = 1, \cdots, m.$$

THE CASE $\varrho = 1$.

We begin with equations (A) and assume $m > 1$. It follows, then, in case $n = 1$, that $\mathfrak{M}$ consists of a single canonical element of arc $\mathfrak{m}_1$, which, however, may be multiply covered as in the example

$$x_1 = u_1^2, \qquad x_2 = u_1^2.$$

If $n > 1$, then in particular all the two-rowed Jacobians of the matrix (B) vanish identically, and hence any two of the functions (A) are connected by an identical relation,

$$Q_{\alpha\beta}(x_\alpha, x_\beta) = 0,$$

where $Q_{\alpha\beta}(x_\alpha, x_\beta)$ is analytic at the origin and vanishes there.[3] Thus again it appears that $\mathfrak{M}$ consists of a single canonical element of arc $\mathfrak{m}_1$.

If $m = 1$, then $n = 1$, and $\mathfrak{U}_1$ is transformed in a $(1, k)$-manner on a region of R. Such a region is, according to our definition, a canonical "element of volume", though here of modest dimensionality.

Summing up, then, we have in all cases (the notation is explained just below)

(1) $$\mathfrak{M}_1(\mathfrak{U}_n) = \mathfrak{m}_1.$$

The x_α defined on a $\mathfrak{g}_n$. Turning now to the case that x_α is defined on a configuration $\mathfrak{g} = \mathfrak{g}_n$, we see that $n = 1$ yields as before $\mathfrak{M} = \mathfrak{m}_1$, and we proceed to the case $n = 2$. The singular points of $\mathfrak{g}_2$ fill a finite number of configurations of rank $r = 1$, $\{\mathfrak{g}_1\}$, or reduce to a point, $(u) = (0)$, when $r = 0$, or, finally, are altogether absent. In case a $\mathfrak{g}_1$ is present, x_α is not only single-valued and continuous on $\mathfrak{g}_1$, but also analytic there.[4] Hence $\mathfrak{g}_1$ goes over into a canonical element of arc $\mathfrak{m}_1$, or into a point $\mathfrak{m}_0$ — here the origin.

Denote the image of $\mathfrak{g}_1$ in the first case by $\mathfrak{M}_1(\mathfrak{g}_1)$, in the second by $\mathfrak{M}_0(\mathfrak{g}_1)$, the notation being, generally, that, when the rank of the matrix (B) on a configuration $\mathfrak{g}_\mu$ is r, the image shall be denoted by $\mathfrak{M}_r(\mathfrak{g}_\mu)$, and similarly for $\mathfrak{M}_r(\mathfrak{U}_n)$. If $r = 0$, the image is a point. Thus the complete image of $\{\mathfrak{g}_1\}$ is seen to be

(2) $$\sum \mathfrak{M}_r(\mathfrak{g}_1), \qquad r = 0, 1.$$

We can write a more general formula that will include the case that the singular points reduce to the origin, $(u) = (0)$, if we agree to denote a point as a configuration $\mathfrak{g}_0$. We then have

[3] Ibid. § 24.
[4] Ibid. pp. 122–23.

$$\sum \mathfrak{M}_r(\mathfrak{g}_\mu), \qquad \mu = 0, 1, \quad 0 \leqq r \leqq \mu, \tag{3}$$

where now, in particular, μ may $= 0$, the sum then reducing to the single term $\mathfrak{M}_0(\mathfrak{g}_0)$.

Embed the configuration (3), which we shall denote for convenience by $\mathfrak{N}$, in an arbitrarily small neighborhood, S_1, and choose a neighborhood $\mathfrak{S}_1$ of $\{\mathfrak{g}_1\}$ such that, when (u) lies in $\mathfrak{S}_1$, its image lies in S_1. Consider now a point (x^1) of the closed region $R_1 = R - S_1$, which is the image of a point (u^1) of $\mathfrak{g}_2$. Then (u^1) does not lie on $\{\mathfrak{g}_1\}$, and hence the neighborhood of (u^1) is transformed as in the earlier case on a canonical element of arc in the (x)-space, or conceivably into a point. The latter situation is, however, impossible, since the point would have to be the origin, and so each x_α would vanish identically on $\mathfrak{g}_2$.

It follows now from a familiar principle of analysis that the whole of the image of $\mathfrak{g}_2$ which lies in the bounded region R_1 can be covered by a finite number of overlapping canonical elements of arc. As S_1 shrinks down on $\mathfrak{N}$, it is conceivable that the image arcs in R_1 condense on $\mathfrak{N}$ in the neighborhood of every point of $\mathfrak{N}$ — why not? Thus $\mathfrak{M}$ is seen to be the configuration

$$\mathfrak{N}\,\mathfrak{M}_1(\mathfrak{U}_2), \tag{4}$$

where the notation shall mean $\mathfrak{N}$, and arcs arising each time as a certain $\mathfrak{M}_1(\mathfrak{U}_2)$, these latter arcs condensing conceivably on $\mathfrak{N}$.

Finally $\mathfrak{M}$ may consist of a single $\mathfrak{m}_1$, in case $\mathfrak{g}_2$ has no singular points.

We can express $\mathfrak{M}$ in all cases by the formula

$$\mathfrak{M}_1(\mathfrak{g}_2) = \{\varepsilon + \alpha \sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\,\mathfrak{M}_1(\mathfrak{U}_2), \quad \mu = 0, 1, \quad 0 \leqq r \leqq 1, \mu. \tag{5}$$

Here, ε and α have the values $0, 1$; and $\varepsilon + \alpha = 1$. Thus, when $\varepsilon = 1$, $\alpha = 0$, $\mathfrak{M}$ reduces to a single $\mathfrak{M}_1(\mathfrak{U}_2)$. But, in general, the notation means that an infinite number of $\mathfrak{M}_1(\mathfrak{U}_2)$'s may be present, condensing on the configuration $\{\ \}$.

The case $n = n$. It is clear from reasoning similar to the foregoing that in the general case

$$\mathfrak{M}_1(\mathfrak{g}_n) = \{\varepsilon + \alpha \sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\,\mathfrak{M}_1(\mathfrak{U}_n), \quad 0 \leqq \mu \leqq n-1, \quad 0 \leqq r \leqq 1, \mu, \tag{6}$$

where $\varepsilon, \alpha = 0, 1$ and $\varepsilon + \alpha = 1$. For, the singular points of $\mathfrak{g}_n$ may fill a set of configurations $\mathfrak{g}_\mu$, where μ takes on some or all the values from 0 to $n-1$; and for a given value of μ there may be several $\mathfrak{g}_\mu$, distinct or coincident. If no $\mathfrak{g}_\mu$ are present, then $\alpha = 0$, $\varepsilon = 1$, and $\mathfrak{M}$ reduces to a single $\mathfrak{M}_1(\mathfrak{U}_n)$. Otherwise, $\varepsilon = 0$ and $\alpha = 1$.

The final result can be expressed in the

THEOREM. *If* $\varrho = 1$, $\mathfrak{M}$ *is given in the case of equations* (A) *by the formula*

$$\mathfrak{M}_1(\mathfrak{U}_n) = \mathfrak{m}_1.$$

When the x_α *are defined on a configuration* $\mathfrak{g}_n$, $\mathfrak{M}$ *is given by* (6):

$$\mathfrak{M}_1(\mathfrak{g}_n) = \{\varepsilon + \alpha \sum \mathfrak{M}_r(\mathfrak{g}_\mu)\} \mathfrak{M}_1(\mathfrak{U}_n), \qquad 0 \leqq \mu \leqq n-1, \quad 0 \leqq r \leqq 1, \mu;$$
$$\varepsilon, \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

§ 2. THE CASE $\varrho = 2$.

We begin with equations (A) and assume that $n = 2$, $m = 2$. Then

(7) $$J(u_1, u_2) = \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} \neq 0.$$

If $J(0) \neq 0$, $\mathfrak{M}$ consists of a single canonical element of volume, $\mathfrak{m}_2$.

If $J(0) = 0$, let

(8) $$J = J_1^{\lambda_1} J_2^{\lambda_2} \cdots J_q^{\lambda_q},$$

where $J_k(u_1, u_2)$ is irreducible[5] in the origin. Denote any J_k by $\mathfrak{J}$ and consider the curve

(9) $$\mathfrak{J}(u_1, u_2) = 0.$$

Let (u^0) be a non-specialized point of this curve. Then the complex two-dimensional neighborhood of (u^0) goes over into a canonical element of volume, of index 2, namely, $\mathfrak{m}_2$; or else the complex one-dimensional neighborhood of (u^0) pertaining to the locus (9) goes into a point,[6] and since this point is seen to be the same for all points (u^0) on (9) it must be the origin.

By suitably restricting $\mathfrak{U}_2$ the exceptional points can be reduced to the origin, $\mathfrak{g}_0$, or to such $\mathfrak{g}_1$, given by (9), as are transformed into a point. In either case their image $\mathfrak{M}_0(\mathfrak{g}_\mu)$, $\mu = 0, 1$, is a point, namely, the origin in the (x)-space. Denote this point by $\mathfrak{N}$:

$$\mathfrak{N} = \sum \mathfrak{M}_0(\mathfrak{g}_\mu), \qquad \mu = 0, 1.$$

Embed $\mathfrak{N}$ in a neighborhood S_0, and remove S_0 from R. If (x^1) be a point of $\mathfrak{M}$ lying in the closed region $R_0 = R - S_0$, then its image (u^1) is a point of $\mathfrak{U}_2$, distinct from $\mathfrak{g}_\mu$, and thus $\mathfrak{M}$ consists in part of a canonical element of volume, of index 2, namely, $\mathfrak{m}_2$, about (x^1), and the

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[5] Ibid. § 4.

[6] Ibid. § 20.

part of $\mathfrak{M}$ lying in R_0 can be covered by a finite number of such elements. Hence

$$\mathfrak{M} = \mathfrak{N}\,\mathfrak{m}_2,$$

which means that $\mathfrak{M}$ consists of $\mathfrak{N}$ and canonical elements of volume, $\mathfrak{m}_2$, which condense on $\mathfrak{N}$.

In particular, however, as above pointed out, $\mathfrak{M}$ may consist of a single canonical element of volume, $\mathfrak{m}_2$. We can comprise all cases in the single formula

$$\mathfrak{M}_2(\mathfrak{U}_2) = \{\varepsilon + \alpha \sum \mathfrak{M}_0(\mathfrak{g}_\mu)\}\,\mathfrak{m}_2, \qquad \mu = 0, 1, \tag{10}$$

where ε, α have the values $0, 1$ and $\varepsilon + \alpha = 1$.

The functions x_α on a $\mathfrak{g}_2$. The singular points of the given configuration $\mathfrak{g}_2$ of rank 2, which forms the defining element, consist at most of a finite number of analytic curves, and reduce in particular to the origin, or are altogether absent. They yield, therefore, at most a finite number of canonical elements of arc passing through the origin. Denote these by $\mathfrak{N}$, when there are any.

Embed $\mathfrak{N}$ in a neighborhood S_1. Then the part of $\mathfrak{M}$ which lies in the closed region $R_1 = R - S_1$ can be covered by a finite number of overlapping regions $\mathfrak{M}_2(\mathfrak{U}_2)$. Hence

$$\mathfrak{M}_2(\mathfrak{g}_2) = \{\varepsilon + \alpha \sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\,\mathfrak{M}_2(\mathfrak{U}_2), \quad 0 \leqq \mu \leqq 1, \quad 0 \leqq r \leqq 1, \mu, \tag{11}$$

where as usual $\varepsilon, \alpha = 0, 1$ and $\varepsilon + \alpha = 1$.

Equations (A) for $n \geqq 3$, $m \geqq 3$. Here the common roots of all the 2-rowed determinants out of the matrix (B) form a finite number of configurations $\mathfrak{g}$, of index $\leqq n-1$. We will treat first the case $n = 3$.

Begin with a $\mathfrak{g}_2$, if one exists, and let (u^0) be an ordinary point on it, i. e. one such that

(i) (u^0) is not a singular point of $\mathfrak{g}_2$, and

(ii) some 2-rowed determinant $J(u_1, u_2, u_3)$ vanishes, in the neighborhood of (u^0), only in the points of $\mathfrak{g}_2$.

By means of regular transformations we can throw (u^0) to the origin and carry $\mathfrak{g}_2$ in the neighborhood of (u^0) over into the plane $u_1 = 0$. After the transformation and a suitable interchange of the indices of the x_α, the determinant

$$J(u_1, u_2, u_3) \equiv \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} = u_1^\lambda\, \mathfrak{J}(u_1, u_2, u_3), \tag{12}$$

where $\mathfrak{J}(0) \neq 0$.

Consider next the transformation of that part of $\mathfrak{U}_3$ near the origin which lies in the plane $u_3 = 0$, namely the locus (we take $m = 3$; the extension to $m > 3$ is then immediate)

$$(13) \qquad x_\alpha = f_\alpha(u_1, u_2, 0), \qquad \alpha = 1, 2, 3.$$

The first two of these equations carry

$$\mathfrak{U}_2\colon \quad |u_1| < \delta, \quad |u_2| < \delta,$$

into a single canonical element of volume in the (x_1, x_2)-space, provided Case I[7] obtains, and all three carry $\mathfrak{U}_2$ into a single canonical element of surface,

$$(14) \qquad x_3 = \Phi(x_1, x_2),$$

where $\Phi(x_1, x_2)$ is a k-valued function, continuous throughout a certain region,

$$T\colon \quad |x_1| < h, \quad |x_2| < h,$$

and consisting of k branches, each analytic at all points of T except those of certain analytic curves, Γ. Hence x_3 satisfies an algebroid equation of the form

$$(15) \quad Q(x_1, x_2, x_3) \equiv x_3^k + A_1(x_1, x_2)\, x_3^{k-1} + \cdots + A_k(x_1, x_2) = 0,$$

where $A_i(x_1, x_2)$, $i = 1, \cdots, k$, is continuous in T, and analytic except possibly along Γ. It follows, then, that these excepted points are removeable singularities, and $A_i(x_1, x_2)$ is thus seen to be analytic throughout the whole of T. Hence $Q(x_1, x_2, x_3)$ is analytic in the three independent variables x_1, x_2, x_3 at the origin.

We now proceed to show that

$$Q(f_1, f_2, f_3) \equiv 0,$$

[7] Ibid. p. 143. If on the other hand Case II is present, then the whole of $\mathfrak{g}_2$ goes over into a point (and this must be the origin). For if there were a second point $(a) = (0, a_2, a_3)$ near $(u) = 0$ leading to a distinct point (x), then the u_2-axis could have been so chosen as to go through this point, and we are thus back on Case I again.

In detail the proof is as follows. Let

$$u_1' = u_1, \quad u_2' = \alpha_2 u_2 + \alpha_3 u_3, \quad u_3' = a_3 u_2 - a_2 u_3, \quad a_2 \alpha_2 + a_3 \alpha_3 \neq 0,$$

where $|a_2| < \varepsilon$, $|a_3| < \varepsilon$. Each point

$$u_1 = 0, \quad u_2 = t a_2, \quad u_3 = t a_3, \quad 0 \leqq t \leqq 1,$$

lies in the new axis, $u_1' = 0$, $u_3' = 0$. At each of these points we must have either Case I or Case II, provided ε was suitably restricted. We wish to show that Case II prevails throughout. Let $\{t\}$ be the assemblage of values for which Case I prevails, and let $\bar{t}$ be the lower limit of these values of t. Then at $\bar{t}$ either Case I or Case II must prevail, and each assumption leads to a contradiction.

where (u_1, u_2, u_3) is any point of a certain neighborhood of the origin. Hence it follows that the complete neighborhood of $(u) = 0$ is mapped on the canonical element of surface (14). To this end let a point $(v) = (v_1, v_2, 0)$ be so chosen that

(i) $$0 < |v_1| < \delta, \qquad |v_2| < \delta,$$

and (ii) the image point (x_1, x_2) lies in T:

(ii) $$|f_1(v_1, v_2, 0)| < h, \qquad |f(v_1, v_2, 0)| < h.$$

Then $J(v_1, v_2, 0) \neq 0$, and hence[8] the complete neighborhood of (v) will be mapped on a piece of surface

$$x_3 = \Psi(x_1, x_2).$$

In particular, the part of the neighborhood of (v) which lies in the plane $u_3 = 0$ will go over into a nappe of (14):

$$x_3 = \Phi_1(x_1, x_2).$$

Consequently $\Psi(x_1, x_2)$ and $\Phi_1(x_1, x_2)$ coincide in the neighborhood of the point (v). Since, however,

(16) $$f_3(u_1, u_2, u_3) \equiv \Psi[f_1(u_1, u_2, u_3), f_2(u_1, u_2, u_3)]$$

when (u) lies in the neighborhood of (v), it follows that

$$Q[f_1(u_1, u_2, u_3), f_2(u_1, u_2, u_3), f_3(u_1, u_2, u_3)] \equiv 0$$

when (u) lies there. And now we restrict the point (v) still further, requiring (iii) that it be so chosen that the image (x_1, x_2) of every point

$$u_1 = tv_1, \qquad u_2 = tv_2, \qquad u_3 = 0, \qquad 0 \leqq t \leqq 1,$$

shall lie in the region T. It follows, then, in continuing the functions f_1, f_2, f_3 analytically along this path that the relation (16) holds in the neighborhood of each of these points for which $0 < t$, and hence, in particular, since these neighborhoods remain uniformly above a certain size,

$$|u_1 - tv_1| < \eta, \qquad |u_2 - tv_2| < \eta, \qquad |u_3| < \eta,$$

where η is a positive constant, in the neighborhood of the origin, q. e. d.

We turn next to the configurations $\mathfrak{g}_1$. These consist in part of the exceptional points of the earlier $\mathfrak{g}_2$; in part, too, of the common roots of

[8] Ibid. § 23.

all the two-rowed determinants, which do not lie on the $\mathfrak{g}_2$. Let (u') be an ordinary point of a $\mathfrak{g}_1$. If $\mathfrak{U}_3$ be properly restricted, the only point of $\mathfrak{g}_1$ to be avoided will be the origin, $(u) = 0$. Then it is possible, by means of a regular transformation of all three variables (u_1, u_2, u_3), to carry an arc of $\mathfrak{g}_1$ containing (u') into a segment of the new u_2-axis, (u') going into the new origin, in such a way that some two-rowed determinant vanishes in the plane $u_3 = 0$ near the origin only in the points of the above segment.

The proof will be given directly. From here on the reasoning is the same as in the earlier case, with the final result that the complete neighborhood of (u') goes over into a canonical element of surface, $\mathfrak{m}_2$. We thus obtain the final result,

$$\begin{aligned} &\mathfrak{M}_2(\mathfrak{U}_3) = \{\varepsilon + \alpha \textstyle\sum \mathfrak{M}_0(\mathfrak{g}_\mu)\}\, \mathfrak{m}_2, \quad 0 \leqq \mu \leqq 2, \quad 3 \leqq m; \\ &\varepsilon, \alpha = 0, 1; \qquad \varepsilon + \alpha = 1. \end{aligned} \tag{17}$$

The outstanding proof is, in short, as follows. Each of the surfaces under consideration has a continuously turning tangent plane near (u'), and consequently, after the transformation, we can find a plane

$$c_1 u_1 + c_3 u_3 = 0$$

which cuts the tangent planes of the new surfaces, and hence the surfaces themselves only in the points of the u_2-axis.

The details are as follows. A given one of the original surfaces can be represented in the neighborhood of the old origin by an algebroid equation,

$$u_3^q + A_1(u_1, u_2)\, u_3^{q-1} + \cdots + A_q(u_1, u_2) = 0, \tag{18}$$

where $A_k(u_1, u_2)$ is analytic in the origin. Let

$$D(u_1, u_2) = D_1^{\lambda_1} \cdots D_p^{\lambda_p}$$

be the discriminant of (18). The curves

$$D_i(u_1, u_2) = 0, \qquad\qquad i = 1, \cdots, p,$$

meet only at the origin, if $\mathfrak{U}_2$ be suitably restricted, and have, furthermore, no other singularity in $\mathfrak{U}_2$. Let (u') be a point on $\mathfrak{g}_1$ such that (u_1', u_2') lies in $\mathfrak{U}_2$ and $\neq (0, 0)$, and let $D_i(u_1', u_2') = 0$, $\partial D_i/\partial u_1 \neq 0$. Make the transformation

$$v_1 = D_i(u_1, u_2), \qquad v_2 = u_2 - u_2', \qquad v_3 = u_3.$$

On returning to the u-notation (i. e. replacing the v after the transformation by u) the transformed equation is of the form (18), where q may, however, have a smaller value; and now, in particular, the new

$$D(u_1, u_2) = u_1^\lambda \mathfrak{J}(u_1, u_2), \qquad \mathfrak{J}(0, 0) \neq 0.$$

From this it follows that the new surface (18) can be uniformized in the neighborhood of the new origin by the transformation

$$u_1 = t^q, \qquad u_2 = u_2.$$

Thus

$$(19) \qquad u_3 = \varphi_0(u_2) + t^K \varphi_K(u_2) + t^{K+1} \varphi_{K+1}(u_2) + \cdots,$$

where now $\varphi_K(0) \neq 0$. For, the discriminant is

$$\prod_{\alpha, \beta} (u_3^{(\alpha)} - u_3^{(\beta)})^2, \qquad \alpha < \beta \leqq q,$$

and consequently has the form

$$t^N \{[\varphi_K(u_2)]^\mu + t\mathfrak{A}(t, u)\}.$$

But D vanishes only when $u_1 = 0$, i. e. $t = 0$. Hence $\varphi_K(0) \neq 0$.

It remains merely to make the final substitution

$$v_3 = u_3 - \varphi_0(u_2),$$

and to return to the u-notation:

$$(20) \qquad u_3 = t^K \varphi_K(u_2) + t^{K+1} \varphi_{K+1}(u_2) + \cdots.$$

The desired proof follows now at once from (20). For it appears from this formula that, when $u_3 = 0$, t, and hence u_1, must be 0 also.

The case that, in the original equation, $q = 1$ can be dealt with immediately.

The case $n > 3$, $m \geqq 3$. Let $n = 4$. After performing, if necessary, a linear transformation on the $u_1, \cdots, u_4$ we can make sure that no one of the functions

$$x_\alpha = f_\alpha(u_1, u_2, u_3, 0) \qquad \alpha = 1, \cdots, m,$$

vanishes identically, and also (on interchanging, if necessary, the indices of the x_α) that

$$J(u_1, u_2, u_3, u_4) \equiv \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} \not\equiv 0, \qquad J(u_1, u_2, u_3, 0) \not\equiv 0.$$

The image $\mathfrak{M}'$ defined by the equations

$$x_\alpha = f_\alpha(u_1, u_2, u_3, 0), \qquad \alpha = 1, \cdots, m,$$

comes under the case just treated. Let $(u') = (u'_1, u'_2, u'_3, 0)$ be a point in which $J(u'_1, u'_2, u'_3, 0) \neq 0$. Then the complete neighborhood of (u') in the (u_1, u_2, u_3, u_4)-space is mapped on a piece of the manifold $\mathfrak{M}'$. In particular, let

$$|u_\alpha - u'_\alpha| < \delta, \qquad \alpha = 1, 2, 3; \qquad |u_4| < \delta$$

be such a neighborhood.

Let $f_\alpha(u_1, \cdots, u_4)$, $\alpha = 1, 2, 3$, be analytic in the domain

$$|u_\beta| \leqq h_\beta, \qquad \beta = 1, \cdots, 4,$$

and let

$$J(u_1, u_2, u_3, u_4) \neq 0, \qquad |v_4| \leqq h_4,$$

where v_4 is constant and u_1, u_2, u_3 are the variables. In the neighborhood of each point of the domain

$$|u_\gamma| \leqq h_\gamma, \qquad \gamma = 1, 2, 3,$$

the solution is given by the earlier result, and since the domain is closed, the total image $\mathfrak{M}_v$ can be covered completely by a finite number of such overlapping images. But, in the neighborhood of the point (u'_1, u'_2, u'_3, v_4), $\mathfrak{M}'$ and $\mathfrak{M}_v$ coincide. Hence they coincide throughout.

From this fact it follows, in particular, that the complete neighborhood of the origin in the $(u_1, \cdots, u_4)$-space goes over into the same configuration $\mathfrak{M} = \mathfrak{M}'$ as that portion of this neighborhood which lies in the plane $u_4 = 0$.

The same reasoning can be used in all the higher cases, and we thus arrive at the result:

$$\text{(21)} \qquad \mathfrak{M}_2(\mathfrak{U}_n) = \{\varepsilon + \alpha \sum \mathfrak{M}_0(\mathfrak{g}_\mu)\}\, \mathfrak{m}_2, \quad 0 \leqq \mu \leqq n-1, \quad m \geqq 3;$$
$$\varepsilon, \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

The functions x_α on a $\mathfrak{g}_n$. In case $\mathfrak{g}_n$ is singular at the origin, the singular manifold consists of one or more configurations $\mathfrak{g}_\mu$, $0 \leqq \mu \leqq n-1$. We apply the method of induction and regard their images in the (x)-space as known. The result is as follows.

$$\text{(22)} \quad \mathfrak{M}_2(\mathfrak{g}_n) = \{\varepsilon + \alpha \sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\, \mathfrak{M}_2(\mathfrak{U}_n), \quad 0 \leqq r \leqq 2, \mu; \quad 0 \leqq \mu \leqq n-1;$$
$$\varepsilon, \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

The final result can be expressed in the

THEOREM. *If* $\varrho = 2$, $\mathfrak{M}$ *is given in the case of equations* (A) *by the formula* (21):

$$\mathfrak{M}_2(\mathfrak{U}_n) = \{\varepsilon + \alpha \sum \mathfrak{M}_0(\mathfrak{g}_\mu)\}\, \mathfrak{m}_2, \quad 0 \leqq \mu \leqq n-1, \quad m \geqq 3;$$
$$\varepsilon, \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

When the x_α are defined on a configuration $\mathfrak{g}_n$, $\mathfrak{M}$ is given by (22):

$$\mathfrak{M}_2(\mathfrak{g}_n) = \{\varepsilon + \alpha \sum \mathfrak{M}_r(\mathfrak{g}_\mu)\} \mathfrak{M}_2(\mathfrak{U}_n), \qquad 0 \leqq r \leqq 2, \mu; \qquad 0 \leqq \mu \leqq n-1;$$
$$\varepsilon, \alpha = 0, 1; \qquad \varepsilon + \alpha = 1.$$

§ 3. The General Case, $\varrho = \varrho$.

We can now give the result in the general case, $\varrho = \varrho$. Consider first **The case** $\mathfrak{M}_n(\mathfrak{U}_n)$, i. e. $n = \varrho$, $m \geqq \varrho$. Let $m = \varrho$.

$$(23) \qquad J(u_1, \cdots, u_n) \equiv \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} \not\equiv 0.$$

If $J(0) = 0$, let

$$(24) \qquad J = J_1^{\lambda_1} \cdots J_q^{\lambda_q}.$$

Consider the configuration $\{\mathfrak{g}\}$ made up (i) of the points (u) which are simultaneous roots of the equations

$$(25) \qquad J_k = 0, \qquad k = 1, \cdots, q;$$

and (ii) of the singular points of the individual surfaces (25). Let $\mathfrak{g}_\mu$ be any one of the constituent configurations $\mathfrak{g}$. Then $0 \leqq \mu \leqq n-1$.

On $\mathfrak{g}_\mu$ the functions x_α are not only continuous; they are analytic in the ordinary points.[9] Let r be the rank of the matrix (B) of these functions near an ordinary point of $\mathfrak{g}_\mu$. Then $0 \leqq r \leqq \mu$.

Consider a point of a surface (25) not lying on any $\mathfrak{g}_\mu$. Here, either Case I or Case II of the Funktionentheorie, ibid., p. 143, obtains. In Case I the neighborhood goes into a canonical element of volume and thus gives rise to no exception, provided the point does not lie on certain manifolds $\mathfrak{g}_\mu$ of lower order. These latter $\mathfrak{g}_\mu$ must be added to the earlier $\mathfrak{g}_\mu$. In Case II the rank r of the matrix (B) is $< n-1$. We will include such surfaces (25) among the $\mathfrak{g}_\mu$. Thus the image $\mathfrak{M}_r(\mathfrak{g}_\mu)$ always has $r \leqq n-2$, though μ may be $= n-1$.

We now apply the method of induction, assuming the theorem true for values of ϱ less than the one under consideration. Hence $\mathfrak{M}_r(\mathfrak{g}_\mu)$ is a known configuration.

Plot, then, in the (x)-space — more particularly in R — the images of all the $\mathfrak{g}_\mu$, for which, as we know, $0 \leqq r \leqq n-2$.

$$\sum \mathfrak{M}_r(\mathfrak{g}_\mu), \qquad 0 \leqq r \leqq \mu, n-2; \ 0 \leqq \mu \leqq n-1.$$

Embed this configuration in an arbitrarily small neighborhood S_1. The part of $\mathfrak{M}$ that lies outside of S_1, or in the closed space $R_1 = R - S_1$, can be covered by a finite number of canonical elements of volume, $\mathfrak{m}_n$.

[9] Ibid. pp. 122–23.

As S_1 shrinks down, these may condense on the above configuration. Hence $\mathfrak{M}$ can be represented in the form

$$\sum \mathfrak{M}_r(\mathfrak{g}_\mu)\,\mathfrak{m}_n .$$

We can include the excepted case by writing

$$(26)\quad \mathfrak{M}_n(\mathfrak{U}_n) = \{\varepsilon + \alpha \textstyle\sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\,\mathfrak{m}_n, \quad 1 \leqq r \leqq \mu,\ n-2;\ 0 \leqq \mu \leqq n-1;$$
$$\varepsilon,\ \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

This result holds without modification of the proof when $n = \varrho$, $m > \varrho$.

The case $\mathfrak{M}_\varrho(\mathfrak{g}_n)$, i. e. $\varrho = n$, $m \geqq n$; or $n > \varrho$, $m > \varrho$. Consider first the case $n = \varrho$. The singular points of the given $\mathfrak{g}_n$ constitute a finite number of configurations $\{\mathfrak{g}\}$, where a given $\mathfrak{g}_\mu$ has its $0 \leqq \mu \leqq n-1$; or they are altogether absent. In the last case, $\mathfrak{M}$ consists of a single canonical element of surface or volume, $\mathfrak{m}_n$.

Consider seriatim the $\mathfrak{g}_\mu$. Let r be the rank of the matrix (B) for $\mathfrak{g}_\mu$; then surely is $0 \leqq r \leqq n-1$. But we know such a $\mathfrak{M}_r(\mathfrak{g}_\mu)$ by hypothesis. Hence

$$(27)\quad \mathfrak{M}_n(\mathfrak{g}_n) = \{\varepsilon + \alpha \textstyle\sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\,\mathfrak{M}_n(\mathfrak{U}_n), \quad 0 \leqq r \leqq \mu \leqq n-1;$$
$$\varepsilon,\ \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

Equations (A) for $n \geqq \varrho + 1$, $m \geqq \varrho + 1$. The treatment here is the same as in the case $\varrho = 2$, $n \geqq 3$, $m \geqq 3$. The result is

$$(28)\quad \mathfrak{M}_\varrho(\mathfrak{U}_n) = \{\varepsilon + \alpha \textstyle\sum \mathfrak{M}_r\, \mathfrak{g}_\mu)\}\,\mathfrak{m}_\varrho, \quad 0 \leqq r \leqq \mu,\ \varrho-2;\ 0 \leqq \mu \leqq n-1;$$
$$\varepsilon,\ \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

The functions x_α on a $\mathfrak{g}_n$. In case $n > \varrho$, $m > \varrho$, the result is:

$$(29)\quad \mathfrak{M}_\varrho(\mathfrak{g}_n) = \{\varepsilon + \alpha \textstyle\sum \mathfrak{M}_r(\mathfrak{g}_\mu)\}\,\mathfrak{M}_\varrho(\mathfrak{U}_n), \quad 0 \leqq r \leqq \mu,\ \varrho;\ 0 \leqq \mu \leqq n-1;$$
$$\varepsilon,\ \alpha = 0, 1; \quad \varepsilon + \alpha = 1.$$

§ 4. Application.

The theorem which gave rise to the foregoing study is the following.

THEOREM. *Let a transformation be defined by Equations* (A) *and let* $m = n = \varrho$. *Furthermore let it be* $(1, k)$.

Let $w = \Phi(u_1, \cdots, u_n)$ *be analytic (meromorphic) in the origin. Then* Φ *goes over into a function of* $(x_1, \cdots, x_n)$ *which satisfies an equation of the form,*

$$(30)\quad w^k + C_1(x_1, \cdots, x_n)\, w^{k-1} + \cdots + C_k(x_1, \cdots, x_n) = 0,$$

in which the coefficients are analytic (meromorphic) in the origin.

The function $w = \Phi(u_1, \cdots, u_n)$ goes over into a k-valued function $w = \Psi(x_1, \cdots, x_n)$, continuous and in general analytic — to restrict ourselves first to the analytic case. Form the symmetric functions

$$w_1^s + w_2^s + \cdots + w_k^s, \qquad s = 1, \cdots, k.$$

These are single-valued and analytic in the (x)-space, save possibly (i) along the singular manifold of a canonical element of volume; (ii) in the points of $\mathfrak{M}$.

The points of (i) lie on a regular configuration two real spaces down, and the function is continuous there. Hence it is analytic there, also.[10] And now the exceptional points under (ii) crumble away seriatim in like manner, till all disappear.

In case $\Phi(u_1, \cdots, u_n)$ is meromorphic, but not analytic,

$$\Phi(u_1, \cdots, u_n) = \frac{G(u_1, \cdots, u_n)}{H(u_1, \cdots, u_n)},$$

the proof can be given in a similar manner by the aid of Hartogs's Theorem.[11] Or we may notice that

$$w_1 = G(u_1, \cdots, u_n), \qquad w_2 = H(u_1, \cdots, u_n)$$

satisfy algebroid equations of the form (30), and now, setting

$$w = \frac{w_1}{w_2}, \quad \text{or} \quad w w_2 - w_1 = 0,$$

eliminate w_1, w_2 between the three equations.

[10] Ibid. Chap. 3, § 3.
[11] Ibid. Chap. 3, § 15.

RINGS OF IDEALS.[1]

By E. T. Bell.

Introduction. Addition, $[+]$, subtraction, $[-]$, multiplication, $[\cdot]$, and equality, $[=]$, are defined for ideals so that with respect to $[+]$, $[-]$, $[\cdot]$, $[=]$, the set of all (integral and fractional) ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots$ of an arbitrary algebraic number field K of degree n form a ring R. Interpreted in K, equality in R is equality of ideals in the usual sense; addition in R is multiplication of ideals in K, subtraction in R is division of ideals in K, and multiplication in R is a new process on ideals in K related to the G. C. D. Arithmetical divisibility is defined for elements of R and unique factorization into prime elements of R (in general not prime ideals of K) is established.

The elements of R (all the ideals of K) are ordered with respect to $[>]$, $[=]$, $[<]$, in abstract identity with the ordering of the rational integers with respect to $>$, $=$, $<$.

A number α of K determines an ideal of K, but not conversely. Thus if α is an integer of K, the ideal determined is the principal ideal $[\alpha]$. An ideal of K which is determined by a number of K will be called an *existent number* of K; all other ideals of K will be called *fictive numbers* of K. The *elements* of R are all the existent and fictive numbers of K.

The elements of R are separated into three classes, called respectively *positive, mixed,* and *negative,* which will be defined in the proper place. A positive element of R is an integral ideal in K; a mixed element of R is in K a fractional ideal $\mathfrak{a}/\mathfrak{b}$, where $\mathfrak{a}$, $\mathfrak{b}$ are coprime integral ideals of K both different from $[1]$; a negative element of R is the reciprocal of an integral ideal in K. A positive (negative) element of R is uniquely a product of prime positive (negative) elements of R, with an abstractly identical theorem for mixed elements which will be stated. A prime ideal in K is a unitary element in R, but not conversely; unitary elements in R play the part of units in K.

As unique factorization in K holds only in the sense of unique prime ideal decomposition of principal ideals of K, it seems reasonable to adjoin to K its fictive numbers after having replaced its numbers by the corresponding existent numbers (as above defined), and to refer to R instead of K as the domain in which unique factorization holds. If this be done, the laws of rational arithmetic persist.

[1] Received June 11, 1930.

Before R can be constructed certain properties, developed next, of real one-rowed matrices are required.

1. **Ordered matrices.** A one-rowed matrix $(x_1, \cdots, x_m)$ $(m \geqq 1)$ whose m *coördinates* $x_1, \cdots, x_m$ are finite real numbers, is called *real* of *order* m. Henceforth *matrix* of *order* m shall mean real matrix of order m.

We say that $(x_1, \cdots, x_m)$ is *greater than* $(y_1, \cdots, y_m)$, and write $(x_1, \cdots, x_m) (>) (y_1, \cdots, y_m)$, if and only if one, and necessarily not more than one, of the following m sets of $1, 2, \cdots, m$ conditions respectively is satisfied:

$$\begin{array}{l}
x_1 > y_1; \\
x_1 = y_1, \quad x_2 > y_2; \\
x_1 = y_1, \quad x_2 = y_2, \quad x_3 > y_3; \\
\cdots\cdots\cdots\cdots\cdots \\
x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3, \cdots, x_{m-1} = y_{m-1}, \quad x_m > y_m.
\end{array}$$

Similarly, we say that $(x_1, \cdots, x_m)$ is *less than* $(y_1, \cdots, y_m)$, and write $(x_1, \cdots, x_m) (<) (y_1, \cdots, y_m)$ if one, and necessarily not more than one, of the m sets of conditions obtained from the above on replacing $>$ by $<$ is satisfied.

Equality, $(x_1, \cdots, x_m) (=) (y_1, \cdots, y_m)$, holds if and only if $x_j = y_j (j = 1, \cdots, m)$.

The matrix $\zeta_m \equiv (0, \cdots, 0)$, each of whose m coördinates is zero, is called the *zero matrix* (of order m).

It is convenient to say that the real number x is *positive* when and only when $x > 0$. This convention will be observed henceforth.

The real matrix $(x_1, \cdots, x_m)$, provided it is not ζ_m, is said to be *positive, mixed* or *negative* according as none, at least one but not all, or all of its non-zero coördinates are negative.

In all questions of divisibility of matrices to be discussed, the concept of regularity is central. We say that $(x_1, \cdots, x_m)$ is *regular* if and only if $x_j \neq 0$ $(j = 1, \cdots, m)$. If $(x_1, \cdots, x_m) \equiv X$ is irregular and is not ζ_m, let x_j $(j = i_1, \cdots, i_s)$ be all the zero coördinates of X, and let $i_h < i_k$ when $h < k$. Then the regular matrix $(i_1, i_2, \cdots, i_s)$ is called the *index of irregularity* of X, and matrices having equal indices of irregularity are said to be *co-irregular*. By definition the *index of irregularity* of a regular matrix is (0).

A matrix other than the zero matrix is called *unitary* if and only if the absolute value of each of its non-zero coördinates is unity.

The matrix (x_1) of order 1 is defined to be identical with the real number x_1. In what follows the special notations $(>)$, $(=)$, $(<)$, $(+)$, $\cdots$ referring to matrices of order $m \geqq 1$ may be replaced by the usual $>, =, <, +, \cdots$ when $m = 1$.

1.1 THEOREMS. *Between any two matrices of order m one and only one of the relations* $(>)$, $(=)$, $(<)$ *holds.*

If $(x_1, \cdots, x_m)(>)(y_1, \cdots, y_m)$, *then* $(y_1, \cdots, y_m)(<)(x_1, \cdots, x_m)$, *and conversely.*

If $(x_1, \cdots, x_m)(>)(y_1, \cdots, y_m)$, *and* $(y_1, \cdots, y_m)(>)(z_1, \cdots, z_m)$, *then* $(x_1, \cdots, x_m)(>)(z_1, \cdots, z_m)$.

To proceed with the ordering of matrices we lay down the following definitions.

The *sum* $(x_1, \cdots, x_m)(+)(y_1, \cdots, y_m)$ of two matrices of order m is the matrix $(x_1 + y_1, \cdots, x_m + y_m)$; their *difference* $(x_1, \cdots, x_m)(-)(y_1, \cdots, y_m)$ is $(x_1 - y_1, \cdots, x_m - y_m)$; their *product* $(x_1, \cdots, x_m)(\cdot)(y_1, \cdots, y_m)$ is $(x_1 y_1, \cdots, x_m y_m)$.

1.2 THEOREM. *With respect to addition,* $(+)$, *multiplication,* $(\cdot)$, *subtraction,* $(-)$, *and equality,* $(=)$, *the set of all matrices of order m is a ring; the unique zero element of the ring is the zero matrix* ζ_m *of order* m.

1.3 THEOREMS. *If* $(x_1, \cdots, x_m)$, $(y_1, \cdots, y_m)$ *are any matrices of order m such that* $(x_1, \cdots, x_m)(>)(y_1, \cdots, y_m)$, *and if* $(z_1, \cdots, z_m)$ *is any matrix of order* m, *then*

$$(x_1, \cdots, x_m)(+)(z_1, \cdots, z_m)(>)(y_1, \cdots, y_m)(+)(z_1, \cdots, z_m).$$

If $(x_1, \cdots, x_m)$, $(y_1, \cdots, y_m)$ *are any co-irregular matrices of order m such that* $(x_1, \cdots, x_m)(>)(y_1, \cdots, y_m)$ *and if* $(z_1, \cdots, z_m)$ *is any positive matrix co-irregular with these two, then*

$$(x_1, \cdots, x_m)(\cdot)(z_1, \cdots, z_m)(>)(y_1, \cdots, y_m)(\cdot)(z_1, \cdots, z_m).$$

For $m = 1$ we have in 1.1, 1.3 the usual fundamental theorems for inequality of real numbers.

1.4 THEOREM. *If X is any non-zero matrix of order m, there exists precisely one unitary matrix U of order m, co-irregular with X, such that* $U(\cdot)X$ *is positive.*

2. Ordered prime ideals of K; Exponent of an ideal. The following elementary theorems on ideals will be required. A prime ideal $\mathfrak{p}$ of K divides precisely one positive rational prime p. Any integral ideal $\mathfrak{a}$ of K has a two-term basis, $\mathfrak{a} = [\alpha, \beta]$, where α, β are integers of K, one of which may be an arbitrarily chosen number of $\mathfrak{a}$ other than zero. Hence $\mathfrak{p}$ has the representation $\mathfrak{p} = [p, \omega]$, where ω is an integer of K. We shall assume given a fixed canonical basis $\omega_1, \cdots, \omega_n$ of the integers of K. Thus finally

$$\mathfrak{p} = [p, a_1 \omega_1 + \cdots + a_n \omega_n],$$

where $0 \leqq a_j < p$ $(j = 1, \cdots, n)$ without loss of generality. We proceed to order the prime ideals of K.

The n basis numbers $\omega_1, \cdots, \omega_n$ give rise to $n!$ distinct arrangements. Select any one of these, say that indicated by $(\omega_1, \cdots, \omega_n)$ as fixed. Then, if q is a positive rational prime, the *index* $I(\mathfrak{b})$ of the ideal $\mathfrak{b} = [q, b_1\omega_1 + \cdots + b_n\omega_n]$ is the matrix $(q, b_1, \cdots, b_n)$ of order $n+1$, where, without loss of generality, $0 \leqq b_j < q$ $(j = 1, \cdots, n)$. As the b_j range over their values, q^n ideals $\mathfrak{b}$, not necessarily distinct, are generated, of which at most n are prime. Let $\mathfrak{b}_{k,q}$ $(k = 1, \cdots, s_q)$ be all the distinct prime ideals in the set of q^n. Now let q range over all the positive rational primes $2, 3, 5, \cdots, p, \cdots$. The sets

2.1 $$\mathfrak{b}_{k,q} \quad (k = 1, \cdots, s_q;\ q = 2, 3, \cdots, p, \cdots)$$

exhaust the prime ideals of K, and each prime ideal of K occurs only once in 2.1.

Let $\mathfrak{p}_j$ $(j = 1, 2, 3, \cdots)$ be all the prime ideals of K. Then $\mathfrak{p}_j$ is defined to be *greater than, equal to* or *less* than $\mathfrak{p}_k$, and we write

$$\mathfrak{p}_j \mid > \mid \mathfrak{p}_k, \qquad \mathfrak{p}_j \mid = \mid \mathfrak{p}_k, \qquad \mathfrak{p}_j \mid < \mid \mathfrak{p}_k,$$

according as

$$I(\mathfrak{p}_j)\,(>)\,I(\mathfrak{p}_k), \qquad I(\mathfrak{p}_j)\,(=)\,I(\mathfrak{p}_k), \qquad I(\mathfrak{p}_j)\,(<)\,I(\mathfrak{p}_k).$$

The notation may be chosen so that $\mathfrak{p}_j \mid \gtreqless \mid \mathfrak{p}_k$ according as $j \gtreqless k$. When this is done (as henceforth), $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \cdots, \mathfrak{p}_n, \cdots$ is called the *natural order* of the prime ideals of K.

Any integral ideal $\mathfrak{a}$ other than the zero ideal of K is uniquely expressible in the form

2.2 $$\mathfrak{a} = \mathfrak{p}_1^{x_1}\,\mathfrak{p}_2^{x_2} \cdots \mathfrak{p}_n^{x_n} \cdots,$$

where each of the numbers x_j $(j = 1, \cdots, n, \cdots)$ is zero or a finite positive rational integer, and where moreover there exists a finite integer m such that $x_n = 0$ for all $n > m$ (since otherwise $\mathfrak{a}$ would not have a finite norm).

The matrix $(x_1, x_2, \cdots, x_n, \cdots)$ of infinite order defined as in 2.2 by the ideal $\mathfrak{a}$ of K is called the *exponent* $E(\mathfrak{a})$ of $\mathfrak{a}$. The exponent is defined uniquely since $\mathfrak{p}_1, \mathfrak{p}_2, \cdots$ in 2.2 are in natural order.

To the zero ideal of K is assigned the conventional exponent $(-\infty, -\infty, \cdots, -\infty, \cdots)$, all of whose coördinates are $-\infty$. This may be denoted by $E(0)$. If necessary its properties can be defined; they will be obvious from those of exponents in general. To proceed to R, we must consider the algebra and arithmetic of exponents. These depend upon the like for what are called N-numbers; we continue with R in Section 5.

Evidently $\mathfrak{p}_j \mid = \mid \mathfrak{p}_k$ when and only when $\mathfrak{p}_j = \mathfrak{p}_k$.

If $E(\mathfrak{a}) \equiv (x_1, \cdots, x_n, \cdots)$, $E(\mathfrak{b}) \equiv (y_1, \cdots, y_n, \cdots)$ and $\mathfrak{a}, \mathfrak{b}$ are coprime, the *exponent* of $\mathfrak{a}/\mathfrak{b}$ is $(x_1 - y_1, \cdots, x_n - y_n, \cdots)$.

3. **N-numbers.** A real matrix $(x_1, \cdots, x_m)$ of finite order $m \geqq 1$ in which $x_m \neq 0$ is called an *N-number* of *order* m; the *zero N-number*, or the *N-number of order zero*, is (0).

The *conjoint* of any two real matrices $(x_1, \cdots, x_r)$, $(y_1, \cdots, y_s)$, in this order, is the matrix $(x_1, \cdots, x_r, y_1, \cdots, y_s)$. The symbol Ω_j, where j is a finite integer > 0, denotes a sequence of precisely j terms each of which is zero; Ω_0 is defined by

$$(x_1, \cdots, x_r, \Omega_0) \equiv (x_1, \cdots, x_r)$$

for all real matrices $(x_1, \cdots, x_r)$ of all finite orders r.

Let k be any finite integer > 0, and let $X_1, \cdots, X_k$ be any N-numbers of the respective orders $m_1, \cdots, m_k$. Let m be the least integer such that $m \geqq m_j$ $(j = 1, \cdots, k)$. Then the conjoints

$$(X_j, \Omega_{m-m_j}) \qquad (j = 1, \cdots, k)$$

are k real matrices (not all N-numbers unless $m_1 = \cdots = m_k$) of order m. These k matrices are called the *least equivalent set* of the N-numbers $X_1, \cdots, X_k$.

Let $(x_1, \cdots, x_n)$ be any real matrix of (finite) order n different from ζ_n (the zero matrix of order n). Then there exists an integer $m > 0$, called the *rank* of $(x_1, \cdots, x_n)$, such that $x_m \neq 0$, $x_h = 0$ for $h > m$, and we call $(x_1, \cdots, x_m)$ the *reduced equivalent* of $(x_1, \cdots, x_n)$; the *reduced equivalent* of (Ω_j), $j > 0$, is (0).

The fundamental operations and equality for N-numbers are now defined on referring to the like for real matrices of the same order. Let X, Y be any N-numbers, and let X', Y' be their least equivalent set. Then X, Y are said to be *N-equal*, $X ((=)) Y$, if and only if $X' (=) Y'$, and hence if and only if $X (=) Y$; the *N-sum*, $X ((+)) Y$ of X, Y is the N-number Z, where Z is the reduced equivalent of $X' (+) Y'$; the *N-difference* $X ((-)) Y$ of X, Y is the N-number W, where W is the reduced equivalent of $X' (-) Y'$; the *N-product* $X ((\cdot)) Y$ of X, Y is the N-number V, where V is the reduced equivalent of $X (\cdot) Y$.

3.1 THEOREM. *With respect to N-equality, $((=))$, N-addition, $((+))$, N-subtraction, $((-))$, the set of all N-numbers is a ring; the unique modulus of addition in the ring is the zero N-number* (0).

As always in a ring the question of a possible inverse of multiplication is irrelevant.

An N-number of order m being a real matrix, the definitions of positive, mixed and negative matrices in Section 1 apply to N-numbers.

Let X, Y be any N-numbers and let X', Y' be their least equivalent set. Then we define X to be *N-greater than* or *N-less than* Y, and write in the respective cases

$$X\,((>))\,Y, \quad X\,((<))\,Y,$$

according as $X'\,(>)\,Y'$ or $X'\,(<)\,Y'$.

3.2 THEOREMS. *Between any two N-numbers one and only one of the relations $((>))$, $((=))$, $((<))$ holds.*

If $X\,((>))\,Y$, then $Y\,((<))\,X$, and conversely.

If $X\,((>))\,Y$ and $Y\,((>))\,Z$, then $X\,((>))\,Z$.

If $X\,((>))\,Y$, then $X\,((+))\,Z\,((>))\,Y\,((+))\,Z$, where Z is an arbitrary N-number.

If X, Y are any N-numbers of the respective orders r, s and if t denotes the greater of r, s if $r \neq s$, or either if $r = s$, such that $X\,((>))\,Y$, then $X\,((\cdot))\,Z\,((>))\,Y\,((\cdot))\,Z$, where Z is any positive N-number of order t.

The above become 1.1, 1.3 if all the N-numbers are restricted to be of the same order, in which case 3.1 becomes 1.2.

4. **Divisibility of N-numbers.** An N-number of order 1 is a real number $\neq 0$; what follows becomes the usual theory of divisibility for real numbers, including arithmetical divisibility for rational integers, if all the N-numbers are restricted to be of order 1.

4.1 THEOREM. *Let A, B be any given N-numbers both different from* (0). *Then, in order that the N-number X be uniquely defined by the equation $A\,((\cdot))\,X\,((=))\,B$, it is necessary and sufficient that A, B be restricted to be co-irregular and that X be restricted to be co-irregular with A, B. If these conditions are violated, either X does not exist, or an infinity of N-numbers X exist satisfying the equation.*

When the conditions are satisfied, namely, when A, B are co-irregular, and X is restricted to be co-irregular with A, B, neither of which is (0), we call X the *N-quotient* of B by A and write $X\,((=))\,B\,((\div))\,A$, or $X\,((=))\,((B/A))$.

4.2 THEOREM. *If X is any N-number other than* (0), *there exists precisely one unitary N-number U such that $((X/U))$ is positive.*

The N-quotient $((X/U))$ in 4.2 is called the *N-absolute* of X, and we write, as a definition, $\|X\|\,((=))\,((X/U))$.

4.3 THEOREM. *If X is any N-number other than* (0), $X\,((=))\,U\,((\cdot))\,\|X\|$, *where U is a uniquely determined unitary N-number.*

An N-number other than (0) each of whose non-zero coördinates is a rational integer is called an *N-integer*; the *zero N-integer* is (0).

If A, B are N-integers other than (0), we say that A *N-divides* B, and write $A \| B$, when and only when $((B/A))$ exists and is an N-integer. By definition we say that the N-quotient of (0) by any N-number other than (0) exists and is (0).

If P is a non-unitary N-integer other than (0) such that the only N-integers X for which $X \| P$ are $X((=))P$, $X((=))U$, where U is unitary, P is called an *N-prime.*

By 4.3 divisibility of N-integers may be discussed only for positive N-integers.

4.4 THEOREM. *If P is a positive N-prime, not all of its coördinates are* 0, 1, *and all of its coördinates that are not* 0 *or* 1 *are positive rational primes.*

4.5 THEOREM. *If A, B are positive N-integers, neither* (0), *and not both unitary, and if P is an N-prime such that $P \| A((\cdot))B$, then either $P \| A$, or $P \| B$, or both.*

4.6 THEOREM. *A positive non-zero non-unitary N-integer is divisible by only a finite number of N-primes.*

4.7 THEOREM. *If A, B, C are N-integers such that $A \| B$ and $B \| C$, then $A \| C$.*

4.8 THEOREM. *A non-unitary N-integer is uniquely the product (up to permutations of the factors) of a unitary N-integer and a finite number of positive N-primes.*

The foregoing is sufficient for the development of the divisibility properties of N-integers. For example, if $A \| B$ and $A \| C$, A is called a *common divisor* of B, C; if $B \| A$ and $C \| A$, A is called a *common multiple* of B, C; if B, C are positive, their N. G. C. D. is that common positive divisor which is N-divisible by every common positive divisor of B, C, and their N. L. C. M. is that positive common multiple which N-divides every common multiple of B, C. The N. G. C. D. of the positive N-integers B, C is their N-greatest common divisor; their N. L. C. M. is their N-least positive common multiple; the N-product of the N. G. C. D. and N. L. C. M. is N-equal to the N-product of B, C. If B, C are coprime N-integers such that $B \| A$ and $C \| A$, then $B((\cdot))C \| A$.

From the definitions there is no loss of generality in discussing the divisibility of N-integers if we make the following restrictions: in a given context all the N-integers are to have the same order, say n, and each N-integer other than ζ_n is to have all of its coördinates different from zero. Having developed the abstract properties of divisibility for such a set, each of whose non-zero integers has index of irregularity (0), we pass at once to the corresponding properties of a set each of whose integers has index of irregularity $(i_1, \cdots, i_s)$ by inserting zeros in the indicated coördinate places of the first set. This follows since division is defined only

for co-irregular N-numbers. The further development of divisibility requires new considerations into which we shall not enter here.

5. **Exponents.** Returning to Section 2 we now apply to exponents $E(\mathfrak{a})$, $E(\mathfrak{b})$, $\cdots$ all the properties of N-numbers and N-integers as in Sections 3, 4. We shall exclude $E(0)$. Hence if $\mathfrak{a}$ is any integral or fractional ideal, $E(\mathfrak{a})$ is of the form $(x_1, \cdots, x_n, \cdots)$ in which $x_1 = \cdots = x_n = \cdots = 0$ if $\mathfrak{a}$ is the unit ideal, and if $\mathfrak{a}$ is not the unit ideal, there exists a finite integer $m > 0$ such that $x_m \neq 0$, $x_h = 0$ for all integers $h > m$, each of $x_1, \cdots, x_m$ is a finite rational integer, and at least one of $x_1, \cdots, x_m$ is not zero. The unique exponent Ω all of whose coördinates are zero is called the *zero exponent.* If $E(\mathfrak{a})$ is not Ω, and if m is the integer just defined, we write $E(\mathfrak{a}) \sim (x_1, \cdots, x_m)$, and say that $E(\mathfrak{a})$ is *equivalent*, $\sim$, to the N-integer $(x_1, \cdots, x_m)$. By definition Ω is equivalent to (0), $\Omega \sim (0)$.

If $E(\mathfrak{x}) \sim X$, the ideal $\mathfrak{x}$ is uniquely determined as that ideal whose exponent is the conjoint (Section 3, beginning) (X, Ω_∞), where Ω_∞ denotes a sequence of an infinity of terms, each zero. We shall write symbolically $\mathfrak{x} = E^{-1}(X)$, that is, $\mathfrak{x}$ is the ideal whose exponent is (X, Ω_∞), where X is a given N-integer.

A 1,1 correspondence between exponents and N-integers is now established by means of the following definitions.

Let $E(\mathfrak{a})$, $E(\mathfrak{b})$ be any exponents, and let $E(\mathfrak{a}) \sim A$, $E(\mathfrak{b}) \sim B$. Then $E(\mathfrak{a})$, $E(\mathfrak{b})$ are said to be *E-equal*, $E(\mathfrak{a}) \{=\} E(\mathfrak{b})$, if and only if $A ((=)) B$, and hence if and only if $\mathfrak{a} = \mathfrak{b}$; the *E-sum* $E(\mathfrak{a}) \{+\} E(\mathfrak{b})$ of $E(\mathfrak{a})$, $E(\mathfrak{b})$ is defined by

$$E(\mathfrak{a}) \{+\} E(\mathfrak{b}) \{=\} E(\mathfrak{s}), \quad \mathfrak{s} = E^{-1}(S), \quad S((=)) A ((+)) B;$$

the *E-difference* and *E-product* are similarly defined by

$$E(\mathfrak{a}) \{-\} E(\mathfrak{b}) \{=\} E(\mathfrak{d}), \quad \mathfrak{d} = E^{-1}(D), \quad D((=)) A ((-)) B;$$
$$E(\mathfrak{a}) \{\cdot\} E(\mathfrak{b}) \{=\} E(\mathfrak{p}), \quad \mathfrak{p} = E^{-1}(P), \quad P((=)) A ((\cdot)) B.$$

5.1 Theorem. *With respect to E-equality, $\{=\}$, E-addition, $\{+\}$, E-subtraction, $\{-\}$, E-multiplication, $\{\cdot\}$, the set of all exponents is a ring (the E-ring); the unique modulus of addition in the ring is the zero exponent (Ω_∞).*

The relations $\{>\}$, *E-greater than*, $\{<\}$, *E-less than*, are defined as follows. Let $E(\mathfrak{a}) \sim A$, $E(\mathfrak{b}) \sim B$; then $E(\mathfrak{a}) \{>\} E(\mathfrak{b})$ if and only if $A ((>)) B$, and $E(\mathfrak{a}) \{<\} E(\mathfrak{b})$ if and only if $A ((<)) B$. Further, $E(\mathfrak{a})$ is said to be *positive, mixed* or *negative* according as A is positive, mixed or negative, and the *order* of $E(\mathfrak{a})$ is defined to be identical with the order of A; $E(\mathfrak{a})$ is called *unitary* if and only if A is unitary.

5.2 THEOREMS. *The Theorems* 3.2 *hold with N-number replaced throughout by exponent.*

Let $E(\mathfrak{a}) \sim A$, $E(\mathfrak{b}) \sim B$. Then we say that $E(\mathfrak{a})$ *E-divides* $E(\mathfrak{b})$, and write $E(\mathfrak{a}) \uparrow E(\mathfrak{b})$, if and only if $A \| B$.

5.3 THEOREMS. *The theorems of Sec.* 4 *hold with the same replacement as in* 5.2.

6. **The ring R.** The *elements* of R are all the ideals of K with the zero ideal omitted. (The zero ideal can be included by defining the properties of $E(0)$ in an obvious way, but for simplicity we shall ignore it). The ring R is defined by a 1,1 correspondence between its elements and operations and those of the E-ring (Section 5).

Let $\mathfrak{a}$, $\mathfrak{b}$ be any elements of R, and let $\mathfrak{a} = E^{-1}(A)$, $\mathfrak{b} = E^{-1}(B)$. Then $\mathfrak{a}$, $\mathfrak{b}$ are said to be *R-equal*, $\mathfrak{a}\,[=]\,\mathfrak{b}$, when and only when $A\,\{=\}\,B$, and hence $\mathfrak{a} = \mathfrak{b}$. The *R-sum, difference, product* are defined by

$$\begin{aligned} &\mathfrak{a}\,[+]\,\mathfrak{b}\,[=]\,\mathfrak{s}, && \mathfrak{s} = E^{-1}(A\,\{+\}\,B);\\ &\mathfrak{a}\,[-]\,\mathfrak{b}\,[=]\,\mathfrak{d}, && \mathfrak{d} = E^{-1}(A\,\{-\}\,B);\\ &\mathfrak{a}\,[\cdot]\,\mathfrak{b}\,[=]\,\mathfrak{p}, && \mathfrak{p} = E^{-1}(A\,\{\cdot\}\,B). \end{aligned}$$

6.1 THEOREM. *With respect to R-equality,* $[=]$, *R-addition,* $[+]$, *R-subtraction,* $[-]$, *R-multiplication,* $[\cdot]$, *the set of all ideals of K is a ring, R; the elements of R are all the ideals of K, and the unique modulus of addition in R (the zero element of R) is the unit ideal of K.*

We say that $\mathfrak{a}\,[>]\,\mathfrak{b}$, where $\mathfrak{a}$, $\mathfrak{b}$ are elements of R, if and only if $A\,\{>\}\,B$; similarly, $\mathfrak{a}\,[<]\,\mathfrak{b}$ if and only if $A\,\{<\}\,B$. An element of R and its exponent are said to *correspond.*

6.2 THEOREMS. *The Theorems* 5.2 *hold with exponents replaced throughout by their corresponding elements of R, an element of R being positive, mixed, negative, or unitary (by definition) according as its exponent is positive, mixed, negative or unitary.*

We say that the element $\mathfrak{a}$ of R *R-divides* the element $\mathfrak{b}$ of R, and write $\mathfrak{a} \mid \mathfrak{b}$, if and only if $E(\mathfrak{a}) \uparrow E(\mathfrak{b})$.

6.3 THEOREMS. *The Theorems* 5.3 *hold with the same replacement as in* 6.2.

The existent and fictive numbers of K may be distinguished, as in the next section, by means of the following considerations in R.

Let $e_j\,(j = 1, \cdots, n, \cdots)$ be a sequence of constant finite positive rational integers, and let $c_j\,(j = 1, \cdots, n, \cdots)$ be arbitrary finite rational integers. Write

$$\epsilon_j \equiv (c_1 e_1, c_2 e_2, \cdots, c_j e_j) \qquad (j = 1, \cdots, n, \cdots).$$

6.4 THEOREM. *With respect to* $\{+\}$, $\{-\}$, $\{\cdot\}$, $\{=\}$, *the set of all exponents* $(\epsilon_j, \Omega_\infty)$ *is a ring; with respect to* $\{+\}$, $\{-\}$, $\{=\}$ *the set is a*

module, the E-module, of the E-ring; the E-product of an element of the E-ring and an element of the E-module is an element of the E-module, and hence the E-module is an ideal, the E-ideal, of the E-ring, and is uniquely determined by $(e_1, \cdots, e_n, \cdots)$.

6.5 THEOREM. *In* 6.4 *replace* $\{\ \}$ *by* $[\ ]$, $(\varepsilon_j, \Omega_\infty)$ by $E^{-1}(\varepsilon_j)$ *and* E *by* R.

The *base* $(e_1, \cdots, e_n, \cdots)$ of the R-ideal is named merely for reference; it is not abstractly identical with the basis of an ideal in an algebraic number field. There is no occasion here to discuss the basis of an R-ideal.

7. The derived field K' of K. The elements of K' coincide with those of the R-ring, and are hence (see introduction) all the existent and fictive numbers of K. If to K we adjoin its fictive numbers we obtain K', in which the fundamental operations and relations are defined to be identical with those of the R-ring. Thus K' is identical with the R-ring, and K is contained in K'. Unique factorization holds in K', but not in terms of existent numbers only of K.

The existent numbers of K coincide with the elements of the following ideal J of K$'$. Let $\mathfrak{p}_j$ be any prime ideal (in the ordinary sense) of K, and let π_j be the least positive rational integer (necessarily equal to or less than the class number of K) such that $\mathfrak{p}_j^{\pi_j}$ is equivalent (in the ordinary sense of ideals) to a principal ideal. Then the base of J is $(\pi_1, \cdots, \pi_n, \cdots)$. The existent numbers of K form a ring in K'.

THE STRUCTURE OF MATRICES WITH ANY NORMAL DIVISION ALGEBRA OF MULTIPLICATIONS.[1]

By A. Adrian Albert.

1. **Introduction.** The outstanding problem in the theory of Riemann matrices[2] is the determination of all pure Riemann matrices. The chief sub-problem has been that of finding the structure of a pure Riemann matrix with a given multiplication algebra. This problem was solved for the case of fields by S. Lefschetz,[2] was reduced essentially to the case of normal division algebras by the author,[3] and was solved for the case of "known"[4] normal division algebras by the author.[3]

In the present paper the author defines *algebras of multiplications* of matrices not necessarily Riemann matrices and finds necessary and sufficient conditions that a matrix have a given algebra of multiplications. By adding the conditions that a given matrix be a Riemann matrix having no multiplication not in the given algebra, the author completely determines the structure of all pure Riemann matrices with a given multiplication algebra.

2. **New properties of normal division algebras.** Let $\mathfrak{B}$ be a normal division algebra in n^2 units over any non-modular field F and let a be in $\mathfrak{B}$ and have minimum equation

$$(1) \qquad \varphi(\xi) \equiv \xi^n + \lambda_1 \xi^{n-1} + \cdots + \lambda_n = 0 \qquad (\lambda_1, \cdots, \lambda_n \text{ in } F).$$

We shall use the notation $\alpha_1, \cdots, \alpha_n$ for the scalar roots of (1) and shall choose a basis of $\mathfrak{B}$

$$(2) \qquad u_1 = 1, \qquad u_{(j-1)n+k} = a^{n-1} u_k \qquad (j, k = 1, \cdots, n).$$

The author has proved[5] that $\mathfrak{B}$ is representable as an algebra $\mathfrak{B}_1$ of m-rowed square matrices with elements in F if and only if m is divisible

[1] Received June 16, 1930.

[2] For references see the report of the National Research Committee on Rational Transformations, Chapters 15 and 17.

[3] In a paper presented for publication to the Circolo Matematico di Palermo. For a summary of this paper see the author's *On the Structure of Pure Riemann Matrices with Non-commutative Multiplication Algebras*, Proceedings of the National Academy of Sciences, April 1930.

[4] That is, the algebras of L. E. Dickson, Transactions of the American Mathematical Society, vol. 28 (1926), pp. 207–234.

[5] In an as yet unpublished paper *On Direct Products, Cyclic Division Algebras, and Pure Riemann Matrices*, for a summary of which see the Proceedings of the National Academy of Sciences, April (1930).

by n^2, and that when such a representation exists so that $m = n^2 q = n m'$, then $\mathfrak{B}$ has a representation in which the modulus of $\mathfrak{B}$ is represented by $I(m)$, the zero element by $0(m)$,[6] and a by

$$A = \|A_{jk}\|, \quad A_{j,j-1} = I(m'), \quad A_{jk} = 0(m') \quad (k \neq j-1), \tag{3}$$
$$A_{jn} = -\lambda_{n+1-j}\, I(m'), \quad (j, k = 1, \cdots n).$$

The author has also shown the truth of

THEOREM 1. *There exist scalars* π_{sjk}, σ_{jks} *in* $K = F(\alpha_1, \cdots, \alpha_n)$ *such that if*[7]

$$e_{jk} = \sum_{s=1}^{n^2} \sigma_{jks}\, u_s, \quad e_{jj} = \sum_{t=(r-1)n+1}^{r=1,\cdots,n} \sigma_{jjt}\, a^{r-1}, \tag{4}$$

then

$$e_{jk}\, e_{kt} = e_{jt}, \quad e_{jk}\, e_{rt} = 0 \quad (k \neq r;\ j, k, r, t = 1, \cdots, n), \tag{5}$$
$$u_s = \sum \pi_{sjk}\, e_{jk}, \quad a = \sum \alpha_j\, e_{jj}.$$

Consider the generalized Vandermonde matrix

$$V = \|\alpha_j^{k-1} I(m')\| \qquad (j, k = 1, \cdots, n), \tag{6}$$

an m-rowed non-singular square matrix. Let U_k be the representation of u_k in $\mathfrak{B}_1$, a representation of $\mathfrak{B}$ as in (3) and the statement preceding (3). Define

$$E_{jk} = \sum_{s=1}^{n^2} \sigma_{jks}\, U_s. \tag{7}$$

Since $\mathfrak{B}_1$ is equivalent to $\mathfrak{B}$ under the correspondence $u_k \cong U_k$, we have

$$E_{jk}\, E_{kt} = E_{jt}, \quad E_{jk}\, E_{rt} = 0\,(m) \quad (k \neq r;\ j, k, r, t = 1, \cdots, n). \tag{8}$$

Let ε_{jk} be an m-rowed square matrix which, when considered as an n-rowed square matrix whose elements are m'-rowed square matrices, has $I(m')$ in the jth row and kth column and $0\,(m')$ elsewhere. Then the algebra (ε_{jk}) over K has

$$\varepsilon_{jk}\, \varepsilon_{kt} = \varepsilon_{jt}, \quad \varepsilon_{jk}\, \varepsilon_{rt} = 0\,(m) \quad (k \neq r;\ j, k, r, t = 1, \cdots, n), \tag{9}$$

has the matrix $I(m)$ as its identity matrix, and is equivalent to (E_{jk}) over K. But both are sub-algebras of M, the algebra of all m-rowed square matrices with elements in K. It follows (loc. cit. On the Structure

[6] We shall use the notation $I(k)$ to represent a k-rowed identity matrix, $0(k)$ to represent a k-rowed zero matrix, and $0(k_1, k_2)$ to represent a matrix with k_1 rows and k_2 columns of zeros.

[7] In fact

$$e_{jj} = \frac{(a-\alpha_1)\cdots(a-\alpha_{j-1})\,(a-\alpha_{j+1})\cdots(a-\alpha_n)}{(\alpha_j-\alpha_1)\cdots(\alpha_j-\alpha_{j-1})\,(\alpha_j-\alpha_{j+1})\cdots(\alpha_j-\alpha_n)}.$$

of Pure Riemann Matrices, etc.) that there exists a non-singular m-rowed square matrix Q with elements in K such that

$$Q\,E_{jk}\,Q^{-1} = \epsilon_{jk}. \tag{10}$$

Now, in particular,

$$Q\,A\,Q^{-1} = Q \sum \alpha_j\,E_{jj}\,Q^{-1} = \sum \alpha_j\,\epsilon_{jj} = \alpha.$$

But $VA = \alpha V$ as may be easily verified by direct computation. Hence if we let $Q\,V^{-1} = P$ we have

$$P\,V A\,V^{-1} P^{-1} = P \alpha P^{-1} = \alpha.$$

But, from $(4)_2$

$$E_{jj} = \sum_{r,t} \sigma_{jjt}\,A^{r-1},$$

so that

$$\epsilon_{jj} = \sum_{r,t} \sigma_{jjt}\,\alpha^{r-1}$$

and, since $P\alpha = \alpha P$ we have $P\epsilon_{jj} = \epsilon_{jj} P$. Hence

$$P = \| P_{jk} \|, \quad P_{jk} = 0\,(m'), \ (j \neq k), \quad P_{jj} = P_j \quad (j, k = 1, \cdots, n).$$

We have stated that Q is non-singular, so that so is P and, from the diagonal form of P, so are the P_j. In particular

$$\Delta = \| \Delta_{jk} \|, \quad \Delta_{jk} = 0\,(m'), \ (j \neq k), \quad \Delta_{jj} = P_{11}^{-1}$$

is non-singular. It is obvious that

$$\Delta\,\epsilon_{jk} = \epsilon_{jk}\,\Delta \cdot \qquad (j, k = 1, \cdots, n),$$

so that when we write $\tilde{Q} = \Delta Q$ we have

$$\tilde{Q}\,E_{jk}\,\tilde{Q}^{-1} = \Delta\,Q\,E_{jk}\,Q^{-1}\Delta^{-1} = \epsilon_{jk}.$$

Hence we may replace Q by $\tilde{Q}$ and take the new P_{11} to be $I(m')$.

THEOREM 2. *There exists a set of n non-singular m'-rowed square matrices P_j, $P_1 = I(m')$, with elements in $K = F(\alpha_1, \cdots, \alpha_n)$ such that the matrix P whose jth diagonal element is P_j and whose matrix elements off the diagonal are $0(m')$ has the property that*

$$(PV)\,E_{jk}\,(PV)^{-1} = \epsilon_{jk}, \qquad (PV)\,A(PV)^{-1} = \sum \alpha_j\,\epsilon_{jj} = \alpha. \tag{11}$$

In the above equations (11) *V is the generalized Vandermonde matrix, and the E_{jk} are the m-rowed representations of the e_{jk} of* (4).

We have chosen the α_j to be the scalar roots of $\varphi(\xi) = 0$. Hence if η is a scalar variable $\varphi(\eta) \equiv \psi(\alpha_j, \eta)\,(\eta - \alpha_j)$ where $\psi(\alpha_j, \alpha_j) \neq 0$,

$\psi(\alpha_j, \alpha_k) = 0$ $(j \neq k)$, since $\varphi(\xi) = 0$ has no multiple roots. Let $\Gamma(\alpha_j)$ be any polynomial in α_j with coefficients m'-rowed square matrices with elements in F. We may define a matrix

$$\Gamma(\alpha_j, \eta) \equiv \Gamma(\alpha_j) [\psi(\alpha_j, \alpha_j)]^{-1} \cdot \psi(\alpha_j, \eta).$$

Then $\Gamma(\alpha_j, \alpha_j) = \Gamma(\alpha_j)$, $\Gamma(\alpha_j, \alpha_k) = 0(m')$ $(j \neq k)$. Let

$$\Gamma(\alpha_j, \eta) = \sum_{r,s}^{1,\cdots,n} \alpha_j^{r-1} \Psi_{rs} \eta^{s-1},$$

where the Ψ_{rs} have elements in F. Then if V' is the transpose of V and

$$\Psi = \| \Psi_{rs} \|, \qquad (r, s = 1, \cdots, n),$$

we have the relations

$$V \Psi V' = \| \Gamma(\alpha_j, \alpha_k) \|, \qquad (j, k = 1, \cdots, n),$$

so that we have proved

THEOREM 3. *Let $\Gamma(\alpha_j)$ be any m'-rowed square matrix with elements in $F(\alpha_j)$. Then there exists an $m = nm'$-rowed square matrix Ψ with elements in F such that $V \Psi V' = \Gamma$ has as its jth diagonal matrix element $\Gamma(\alpha_j)$ and as matrices off the diagonal $0(m')$.*

Let us apply the above theorem to the matrix $\Gamma(\alpha_j) = I(m')$. We have thus shown the existence of an m-rowed square matrix T with elements in F such that $VTV' = I(m)$. Let now Γ be any matrix with $\Gamma(\alpha_j)$ as its jth diagonal element and $0(m')$ elsewhere. Then

$$\Gamma = V \Psi V' = (V \Psi V') (VTV')^{-1} = V(\Psi T^{-1}) V^{-1}$$

since $(VTV')^{-1} = (V')^{-1} T^{-1} V^{-1} = I(m)$. Hence we have

THEOREM 4. *Let $\Gamma(\alpha_j)$ be any m'-rowed square matrix with elements in $F(\alpha_j)$. Then there exists an m-rowed square matrix H with elements in F such that*

$$VHV^{-1} = \Gamma,$$

where Γ has $\Gamma(\alpha_j)$ as its jth diagonal element and zero matrices off the diagonal.

Since $\mathfrak{B}$, a normal division algebra over F is a sub-algebra of $\mathfrak{M}$, the algebra of all m-rowed square matrices with elements in F and, since $\mathfrak{M}$ has the same modulus and zero element as $\mathfrak{B}$, it is known that $\mathfrak{M}$ is the direct product of $\mathfrak{B}$ and another algebra $\mathfrak{C}$ over F having the same modulus and zero element as $\mathfrak{M}$. Let X be any quantity of $\mathfrak{C}$. Then $\mathrm{X} \mathrm{U}_s = \mathrm{U}_s \mathrm{X}$ $(s = 1, \cdots, n^2)$. Hence $\mathrm{X} E_{jk} = E_{jk} \mathrm{X}$ and

$$(11) \qquad (PV) \mathrm{X} (PV)^{-1} \varepsilon_{jk} = \varepsilon_{jk} (PV) \mathrm{X} (PV)^{-1}.$$

But, from the form of the ϵ_{jk} it is evident that

$$(12)\qquad \begin{aligned}(PV)\,\mathrm{X}\,(PV)^{-1} = \zeta = \| \zeta_{jk} \|, \quad &\zeta_{jk} = 0(m'), \qquad (j \neq k),\\ &\zeta_{jj} = \zeta_{11} = \xi_1, \qquad (j, k = 1, \cdots, n).\end{aligned}$$

Consider the matrix

$$(13)\qquad V\mathrm{X}V^{-1} = \| P_j^{-1}\zeta_{jk}P_k \| = \| \eta_{jk} \|.$$

Let Ψ be chosen, as in the proof of Theorem 4, so that $V\Psi V' = I(m)$ and

$$(14)\qquad V\mathrm{X}V^{-1}V\Psi V' = V(\mathrm{X}\Psi)V' = V\mathrm{X}V^{-1} = \| \eta_{jk} \|.$$

Hence $V(\mathrm{X}\Psi)V'$ is a matrix with elements off the diagonal all zero matrices. But if $G = \mathrm{X}\Psi = \| G_{rs} \|$ then

$$(15)\qquad VGV' = \left\| \sum_{r,s}^{1,\cdots,n} \alpha_j^{r-1} G_{rs} \alpha_k^{s-1} \right\|.$$

Hence, from (13) and (14), $\sum \alpha_j^{r-1} G_{rs} \alpha_k^{s-1} = 0(m')$ for $j \neq k$, while

$$(16)\qquad P_j(\sum \alpha_j^{r-1} G_{rs} \alpha_j^{s-1})P_j^{-1} = \xi_1.$$

But $P_1 = I(m')$ so that (16) becomes

$$P_j\,\xi_1(\alpha_j)P_j^{-1} = \xi_1(\alpha_1) = \xi_1,$$

where $\xi_1(\alpha_1)$ is an m'-rowed square matrix with elements in $F(\alpha_1)$.

Conversely let (16) be satisfied. By Theorem 4 we can define a matrix X satisfying (12). It follows that $(PV)\,\mathrm{X}\,(PV)^{-1}$ is commutative with all of the ϵ_{jk} and hence that X, an m-rowed square matrix with elements in F, is commutative with all of the E_{jk}. But then X is commutative with all of the quantities of $\mathfrak{B}$ and, since both $\mathfrak{B}$ and $\mathfrak{M}$ are normal simple algebras, X is in $\mathfrak{C}$.

THEOREM 5. *Let $\mathfrak{M}_1$, the algebra of all m-rowed square matrices with elements in F be expressed as the direct product of $\mathfrak{B}_1$ and another algebra $\mathfrak{C}_1$ over F which has the same modulus and zero element as algebras $\mathfrak{B}_1$ and $\mathfrak{M}_1$ and is normal simple. Then, if X is any quantity of $\mathfrak{C}_1$, $(PV)\,\mathrm{X}\,(PV)^{-1}$ is a matrix whose non-diagonal elements are $0(m')$ and whose diagonal elements are all the same matrix $\xi(\alpha_1)$, an m'-rowed square matrix with elements in $F(\alpha_1)$ such that*

$$(17)\qquad P_j\,\xi(\alpha_j)\,P_j^{-1} = \xi(\alpha_1).$$

Conversely every $\xi(\alpha_1)$ satisfying (17) defines a quantity

$$\mathrm{X} = (PV)^{-1}\cdot \| \xi_{jk} \| \cdot (PV), \qquad \xi_{jk} = 0(m'), \qquad \xi_{jj} = \xi(\alpha_1),$$

which is in $\mathfrak{C}_1$.

3. **The algebras of multiplications of a matrix.** Let ω be a p-rowed and q-columned matrix of complex elements and suppose that now F is a field of complex numbers. We call a normal division algebra $\mathfrak{B}$ of order n^2 an algebra of multiplications of ω if there exists a representation $\mathfrak{B}_2$ of $\mathfrak{B}$ as an algebra of p-rowed square matrices with complex elements and a q-rowed representation $\mathfrak{B}_1$ of $\mathfrak{B}$ as an algebra of square matrices with elements in F such that if ξ in $\mathfrak{B}_2$ corresponds to X of $\mathfrak{B}_1$, then, for all pairs of corresponding elements

$$(18) \qquad \xi\omega = \omega X.$$

Two p-rowed and q-columned matrices ω_1 and ω_2 are called isomorphic in F if there exists a non-singular p-rowed complex matrix γ and a non-singular q-rowed matrix G with elements in F such that

$$(19) \qquad \omega_1 = \gamma\omega_2 G.$$

It is obvious that if $\mathfrak{B}$ is an algebra of multiplications of ω it is an algebra of multiplications of any ω_1 which is isomorphic in F to ω, and that the representations $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in passing from ω to ω_1 are replaced by similar representations in which (18) becomes

$$(\gamma\xi\gamma^{-1})\omega_1 = \omega_1(G^{-1}XG).$$

Hence, by passing to an isomorphic matrix, we may reduce the question as to what restrictions are imposed on an ω when it has an algebra $\mathfrak{B}$ of multiplications to an equivalent question when $\mathfrak{B}_1$ is replaced by a similar representation in F and $\mathfrak{B}_2$ by a similar representation in the field of all complex numbers. Now the author has shown that necessarily q is divisible by n^2, p is divisible by n, and any two representations $\mathfrak{B}_1$ of $\mathfrak{B}$ are similar in F, any two representations $\mathfrak{B}_2$ are similar in the field of all complex numbers. Hence we may take as the representation $\mathfrak{B}_2$ of $\mathfrak{B}$, that of section 2, a representation as in (5), and have

$$\mu_s = \sum_{j,k} \pi_{sjk}\,\epsilon_{jk}^{(p)}, \qquad \alpha^{(p)} = \sum \alpha_j\,\epsilon_{jj}^{(p)},$$

where μ_s corresponds to u_s, the π_{sjk} are in $K = F(\alpha_1, \cdots, \alpha_n)$ now a field of complex numbers, $p = np'$, $\epsilon_{jk}^{(p)}$ is an np'-rowed square matrix with $I(p')$ in the jth row and kth column and $0(p')$ elsewhere, and $\alpha^{(p)}$ corresponds to a. We similarly take the $\mathfrak{B}_1$ of (3) and statements preceding (3) for $m = q$. Then $\alpha^{(p)}\omega = \omega A$ so that

$$(20) \qquad \omega = \| \tau_j\,\alpha_j^{k-1} \| \qquad (j, k = 1, \cdots, n),$$

where τ_j is a p'-rowed and q'-columned matrix and $q = nq'$. Let P be defined as in Theorem 2, and let $Q = PV^{(q)}$ where $V^{(q)}$ is the q-rowed generalized Vandermonde matrix for the $n-\alpha_j$'s. We also write $\tilde{\omega} = \omega Q^{-1}$. Now

$$\epsilon_{jk}^{(p)} \omega = \omega E_{jk}$$

since $\mu_s \omega = \omega U_s$. Hence

$$\epsilon_{jk}^{(p)} \omega = \epsilon_{jk}^{(p)} \omega Q^{-1} = \omega E_{jk} Q^{-1} = \tilde{\omega} \epsilon_{jk}^{(q)}.$$

It follows that

$$\tilde{\omega} = \| \tilde{\omega}_{jk} \|, \qquad \tilde{\omega}_{jk} = 0(p', q') \ (j \neq k), \qquad \tilde{\omega}_{jj} = \tilde{\omega}_{11} \ (j, k = 1, \cdots, n),$$

where $0(p', q')$ is a p'-rowed and q'-columned zero matrix. But

$$\omega = \tilde{\omega} Q = \tilde{\omega} P V = \| \tilde{\omega}_{jk} P_k \| \, V,$$

while, as we have seen before,

$$\omega = \| \omega_{jk} \| \, V, \qquad \omega_{jk} = 0(p', q'), \qquad \omega_{jj} = \tau_j.$$

It follows that $\tau_1 = \tilde{\omega}_1$ and that $\tau_j = \tau_1 P_j$. Writing τ for τ_1 we have

$$\omega = \| \tau P_j \alpha_j^{k-1} \| \qquad (j, k = 1, \cdots, n). \tag{21}$$

Conversely if ω has the form (21) then where $\tilde{\omega}$ is a matrix whose diagonal elements are the matrices τ and whose further elements are zero p'-rowed and q'-columned matrices. It is obvious that in this case $\epsilon_{jk}^{(p)} \tilde{\omega} = \tilde{\omega} \epsilon_{jk}^{(q)}$ and that $\epsilon_{jk}^{(p)} \omega = \omega E_{jk}$ so that $\mu_k \omega = \omega U_k$. Hence when ω has the form (21) it has $\mathfrak{B}$ as an algebra of multiplications.

THEOREM 6. *Let $\mathfrak{B}$ be a normal division algebra in n^2 units over a field F of complex numbers, let a be in $\mathfrak{B}$, have grade n with respect to F and $\alpha_1, \cdots, \alpha_n$ as complex roots of its minimum equation, and let the matrices P_j be defined as in Theorem 2. Then a p-rowed and q-columned matrix ω with complex elements has $\mathfrak{B}$ as an algebra of multiplications if and only if p is divisible by n, q is divisible by n^2, and ω is isomorphic to a matrix*

$$\| \tau P_j \alpha_j^{k-1} \| \qquad (j, k = 1, \cdots, n), \tag{21}$$

where τ is a p'-rowed and q'-columned matrix of complex elements, and $p = np'$, $q = nq'$.

4. Application to the theory of pure Riemann matrices. A matrix ω with p rows and $2p$ columns of complex elements is called a *Riemann matrix over a real field F* if there exists a $2p$-rowed alternate

matrix C with elements in F such that $\omega C \omega' = 0$ while the matrix $\sqrt{-1}\, \omega C \bar{\omega}'$ is a positive definite non-singular Hermitian matrix. Any alternate matrix C having the above properties for ω is called a principal matrix of ω. A $2p$-rowed square matrix A with elements in F is called a projectivity of a Riemann matrix ω if there exists a p-rowed square complex matrix α such that $\alpha\omega = \omega A$. It is easily shown that the matrix

$$\Omega = \begin{Vmatrix} \omega \\ \bar{\omega} \end{Vmatrix}$$

is non-singular and that if A is a projectivity of ω then

$$\begin{Vmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{Vmatrix} = \Omega A \Omega^{-1}.$$

If S is a matrix for which $\omega S \omega' = 0$ while S has elements in F, then S is called a matrix of ω. If C is a principal matrix of ω then, for every matrix S of ω, the matrix SC^{-1} is a projectivity of ω, while if A is a projectivity of ω then AC is a matrix of ω.

A Riemann matrix is called pure if all of its projectivities are non-singular. In this case the projectivities form a division algebra over F which is a $2p$-rowed representation in F of an algebra $\mathfrak{D}$ called the *multiplication algebra of* ω. The quantities a of the algebra $\mathfrak{D}$ are the multiplications

$$a: \qquad \alpha\omega = \omega A. \tag{22}$$

In particular the zero element of $\mathfrak{D}$ is

$$0: \quad 0(p)\,\omega = \omega\, 0(2p), \tag{23}$$

and the modulus of $\mathfrak{D}$ is

$$1: \quad I(p)\,\omega = \omega\, I(2p). \tag{24}$$

The matrix α in the above is uniquely determined whenever A is given, and these matrices α form a p-rowed complex representation of $\mathfrak{D}$. The author has proved (loc. cit. On the Structure of Pure Riemann Matrices, etc.).

THEOREM 7. *The structure of all pure Riemann matrices over any real field F is determined by the structure of all Riemann matrices over real fields F whose multiplication algebras are of one of the types:*

(i) *A normal division algebra $\mathfrak{B}$ of order n^2 over F, containing a quantity a of grade n with respect to F and with minimum equation having roots $\alpha_1, \cdots, \alpha_n$, all real.*

(ii) *The same as* (i) *but with $\alpha_1, \cdots, \alpha_n$ all imaginary, $n = 2n'$, and such that there exists a polynomial $\theta(\alpha_1)$ with coefficients in F and the property that*

$$\alpha_{j+n} \equiv \alpha_j, \qquad \bar{\alpha}_j = \alpha_{j+n'} = \theta(\alpha_j), \qquad \theta^2(\alpha_j) = \alpha_j \qquad (j = 1, \cdots, n).$$

(iii) *A division algebra* $\mathfrak{A}$ *of order* $2n^2$ *over* F *which may be considered as a normal division algebra of order* n^2 *over a quadratic field* $F(q)$, $q^2 = \epsilon < 0$ *in* F. *Moreover* $\mathfrak{A}$ *contains a quantity* a *of grade* n *with respect to both* F *and* $F(q)$ *and with its minimum equation with respect to these fields having all real roots* $\alpha_1, \cdots, \alpha_n$.

We also have the necessary condition

THEOREM 8. *The order* h *of the multiplication algebra of a pure Riemann matrix of genus* p *is a divisor of* $2p$, where, of course, the genus of a Riemann matrix is the number of its rows.

Consider first the case (i) of Theorem 7. We take $h = n^2$ in Theorem 8 and have $2p = rn^2$, whence $p = np'$, $2p = 2np'$. Since $\mathfrak{B}$ is obviously an algebra of multiplications of ω the matrix ω is isomorphic in F to

$$\| \tau P_j \alpha_j^{k-1} \| . \tag{21}$$

But it is known that, in the case we are considering, ω is a Riemann matrix over F if and only if τP_j is a Riemann matrix over $F(\alpha_j)$ with $\Gamma(\alpha_j)$ as principal matrix. Hence there must exist an alternate $2p'$-rowed square matrix $\Gamma(\alpha_1)$ with elements in $F(\alpha_1)$ such that

$$\tau P_j \Gamma(\alpha_j) P_j' \tau' = 0, \qquad H_j = \sqrt{-1}\, \tau P_j \Gamma(\alpha_j) P_j' \bar{\tau}' \tag{25}$$

are positive definite Hermitian matrices. Conversely when (25) are satisfied then ω of (21) is a Riemann matrix over F and has $\mathfrak{B}$ as an algebra of multiplications.

We wish $\mathfrak{B}$ to be the multiplication algebra of ω and hence that ω have no new multiplications. The algebra $\mathfrak{D}$ of all multiplications of ω has a normal division sub-algebra $\mathfrak{B}$ with the same modulus (24) as $\mathfrak{D}$ and hence is the direct product of $\mathfrak{B}$ and another algebra $\mathfrak{C}$ over F having (24) for its modulus. Hence ω has a multiplication not in $\mathfrak{B}$ if and only if there exists a projectivity X of ω, commutative with every quantity of $\mathfrak{B}_1$. But in Theorem 5 we showed that if X were any $2p$-rowed square matrix with elements in F which is commutative with all of the quantities of $\mathfrak{B}_1$ than $(PV)\mathrm{X}(PV)^{-1}$ is a matrix whose diagonal elements are the same $2p'$-rowed square matrix $\xi(\alpha_1)$ with elements in $F(\alpha_1)$ such that (17) is true. But if X is a projectivity of ω then $(PV)\mathrm{X}(PV)^{-1}$ is a projectivity of the Riemann matrix $\tilde{\omega} = \omega(PV)^{-1}$ over $F(\alpha_1, \cdots, \alpha_n)$, and hence $\xi(\alpha_1)$ is a projectivity of τ. Conversely when there exists a projectivity $\xi(\alpha_1)$ of τ such that (17) is satisfied, then it is obvious that Theorem 5 implies that the X of that theorem is a projectivity of ω not in $\mathfrak{B}$.

THEOREM 9. *Let* $\mathfrak{B}$ *be an algebra satisfying* (i) *of Theorem* 7, *and let* ω *be a* p*-rowed and* $2p$*-columned matrix of complex elements. Then* ω *is a pure Riemann matrix over* F *with* $\mathfrak{B}$ *as its multiplication algebra if and only if:*

(i) *The integer $2p$ is divisible by n^2 so that $2p = 2np'$ and we can define matrices P_j as in Theorem 2 for $m = 2p$.*

(ii) *The matrix ω is isomorphic to a matrix*

$$\| \tau\, P_j\, \alpha_j^{k-1} \| \tag{21}$$

where τ is a p'-rowed and $2p'$-columned matrix.

(iii) *There exists an alternate, $2p'$-rowed square matrix $\Gamma(\alpha_1)$ with elements in $F(\alpha_1)$ such that*

$$\tau\, P_j \Gamma(\alpha_j) P_j'\, \tau' = 0, \qquad H_j = \sqrt{-1}\, \tau\, P_j \Gamma(\alpha_j) P_j'\, \bar{\tau}'$$

are positive definite Hermitian matrices, and hence, in particular, τ is a Riemann matrix over $F(\alpha_1)$.

(iv) *The matrix τ has no projectivity $\xi(\alpha_1)$ such that*

$$P_j\, \xi(\alpha_j)\, P_j^{-1} = \xi(\alpha_1).$$

The condition (i) above evidently is necessary and sufficient that $\mathfrak{B}$ have a $2p$-rowed representation in F, the condition (ii) makes $\mathfrak{B}$ an algebra of multiplications of ω, (iii) makes ω a Riemann matrix, and finally (iv) insures that ω have no multiplication not in $\mathfrak{B}$.

We shall next consider algebras of type (ii) of Theorem 7. Again $2p = 2np'$ and ω is isomorphic to a matrix in the form (21). Consider now the equations (17). For every matrix $\xi(\alpha_1)$ which is an m'-rowed representation of a quantity of $\mathfrak{C}$, known to be a normal simple algebra of order m'^2 over F we have

$$P_j\, \xi(\alpha_j)\, P_j^{-1} = \xi(\alpha_1).$$

Passing to complex conjugates we obtain

$$\bar{P}_j\, \xi(\bar{\alpha}_j)\, \bar{P}_j^{-1} = \xi(\bar{\alpha}_1), \tag{26}$$

whence

$$(P_{1+n'} \bar{P}_j)\, \xi(\alpha_{j+n'})\, (P_{1+n'} \bar{P}_j)^{-1} = \xi(\alpha_1). \tag{27}$$

But (17) for the value $j + n'$ is

$$P_{j+n'}\, \xi(\alpha_{j+n'})\, P_{j+n'}^{-1} = \xi(\alpha_1). \tag{28}$$

Hence the matrix

$$(P_{j+n'})^{-1}\, (P_{1+n'} \bar{P}_j) \tag{29}$$

is a $2p'$-rowed matrix commutative with all of the $\xi(\alpha_1)$. The set of all $\xi(\alpha_1)$ is an m'-rowed representation of the algebra $\mathfrak{C}$ of order m'^2 and, as

a consequence, it is easily shown that the matrix (29) is commutative with all m'-rowed square matrices and is a scalar matrix. It follows that

$$P_{j+n'} = g_j P_{1+n'} \bar{P}_j \qquad (j = 1, \cdots, n), \tag{30}$$

where the g_j are in $K = F(\alpha_1, \cdots, \alpha_n)$.

It is known that ω in the form (21) is a Riemann matrix over F if and only if there exists a $2p'$-rowed square matrix $\Gamma(\alpha_1)$ with elements in $F(\alpha_1)$ such that if $\tau_j = \tau P_j$ then

$$\Gamma(\bar{\alpha}_1)' = -\Gamma(\alpha_1), \qquad \tau_j \Gamma(\alpha_j) \tau'_{j+n'} = 0, \qquad \sqrt{-1}\, \tau_j \Gamma(\alpha_j) \bar{\tau}_j'$$

is a positive definite Hermitian matrix. This becomes

$$\tau P_j \Gamma(\alpha_j) \bar{P}_j' \tau_{1+n'} = 0, \qquad \sqrt{-1}\, \tau P_j \Gamma(\alpha_j) \bar{P}_j' \bar{\tau}' \tag{31}$$

a positive definite Hermitian matrix when we use (30).

The matrix

$$\Omega_1 = \left\| \begin{matrix} \tau \\ \bar{\tau}_{1+n'} \end{matrix} \right\|$$

is an Omega matrix over $F(\alpha_1)$ (loc. cit. on the Structure, etc.) when (31) is satisfied. Suppose that X is in $\mathfrak{C}$ so that from $(PV)\,\mathrm{X}\,(PV)^{-1}$ we obtain a $2p'$-rowed square matrix $\xi(\alpha_1)$ satisfying (17). Then if X is a projectivity of ω there must exist a p'-rowed complex matrix ϱ such that

$$\varrho \tau = \tau \xi(\alpha_1). \tag{32}$$

But

$$\tau_{1+n'} \xi(\alpha_{1+n'}) = \tau P_{1+n'} \xi(\alpha_{1+n'}) = \tau \xi(\alpha_1) P_{1+n'} = \varrho \tau_{1+n'}, \tag{33}$$

whence

$$\bar{\tau}_{1+n'} \xi(\alpha_1) = \bar{\varrho}\, \bar{\tau}_{1+n'}, \tag{34}$$

so that $\xi(\alpha_1)$ defines a multiplication of Ω_1. Conversely let $\xi(\alpha_1)$ define a non-scalar multiplication (32), (34) of Ω_1 and let (17) be true. Then it is obvious that

$$\varrho \tau_j = \varrho \tau P_j = \tau \xi(\alpha_1) P_j = \tau_j \xi(\alpha_j) \tag{35}$$

and that we may define a multiplication of ω not in $\mathfrak{B}$.

THEOREM 10. *Let $\mathfrak{B}$ be an algebra satisfying* (ii) *of Theorem* 7, *and let ω be a p-rowed and $2p$-columned matrix of complex elements. Then ω is a pure Riemann matrix over F with $\mathfrak{B}$ as its multiplication algebra if and only if*

(i) *The integer $2p$ is divisible by n^2 so that $2p = 2np'$, and we can define matrices P_j as in Theorem* 2 *for $m = 2p$, but now satisfying* (30).

(ii) *The matrix* ω *is isomorphic to a matrix*

$$\| \tau P_j \alpha_j^{k-1} \| . \tag{21}$$

(iii) *There exists a* $2p'$*-rowed skew-hermitian matrix* $\Gamma(\alpha_1)$ *with elements in* $F(\alpha_1)$ *such that if* $\tau_{1+n'} = \tau P_{1+n'}$,

$$\tau (P_j \Gamma(\alpha_j) \bar{P}_j') \tau_{1+n'} = 0, \quad \sqrt{-1}\, \tau P_j \Gamma(\alpha_j) \bar{P}_j' \bar{\tau}'$$

are positive definite Hermitian matrices, so that in particular,

$$\Omega_1 = \left\| \begin{matrix} \tau \\ \bar{\tau}_{1+n'} \end{matrix} \right\| \tag{36}$$

is an Omega matrix over $F(\alpha_1)$.

(iv) *The Omega matrix* Ω_1 *has no non-scalar projectivity* $\xi(\alpha_1)$ *such that*

$$P_j \xi(\alpha_j) P_j^{-1} = \xi(\alpha_1). \tag{37}$$

We finally have case (iii) to consider. If ω has $\mathfrak{A}$ as an algebra of multiplications then p is divisible by n^2 so that $p = rn^2 = rp'$. We can take, by passing to a matrix isomorphic to ω, the projectivity corresponding to q to be

$$Q = \left\| \begin{matrix} 0(p) & \varepsilon I(p) \\ I(p) & 0(p) \end{matrix} \right\| . \tag{38}$$

For π, the p-rowed representation of q, we have

$$\pi = \left\| \begin{matrix} \sqrt{\varepsilon}\, I(r_1) & 0(r_1, r_2) \\ 0(r_2, r_1) & -\sqrt{\varepsilon}\, I(r_2) \end{matrix} \right\| , \tag{39}$$

where $r_1 + r_2 = p$ and, as the author has shown, necessarily $r_1 = r_1' n$, $r_2 = r_2' n$. We let $p = np'$ as before. Since $\pi\omega = \omega Q$

$$\omega = \left\| \begin{matrix} \omega_1 & \omega_1 \sqrt{\varepsilon} \\ \omega_2 & -\omega_2 \sqrt{\varepsilon} \end{matrix} \right\| , \tag{40}$$

where

$$\Omega = \left\| \begin{matrix} \omega_1 \\ \bar{\omega}_2 \end{matrix} \right\| \tag{41}$$

is an Omega matrix over $F_1 = F(\sqrt{\varepsilon})$. Conversely it is known that when Ω is an Omega matrix over $F(\sqrt{\varepsilon})$ then ω of (40) is a Riemann matrix over F with $\pi\omega = \omega Q$ as a multiplication.

Let h be any multiplication of ω. Then if $\mathfrak{A}$ is the multiplication algebra of ω we have $hq = qh$ so that if ζ is the representation of h in $\mathfrak{A}_2$ then

$\zeta\pi = \pi\zeta$, while if H is the representation of h in $\mathfrak{A}_1$ then $HQ = QH$. But it follows that

$$(42) \qquad \zeta = \begin{Vmatrix} \xi_1 & 0(r_1, r_2) \\ 0(r_2, r_1) & \xi_2 \end{Vmatrix}$$

from $\zeta\pi = \pi\zeta$, where ξ_1 is an r_1-rowed square matrix and ξ_2 is an r_2-rowed square matrix. Let

$$(43) \qquad W = \begin{Vmatrix} I(p) & \sqrt{\varepsilon}\, I(p) \\ I(p) & -\sqrt{\varepsilon}\, I(p) \end{Vmatrix},$$

whose inverse is

$$(44) \qquad W^{-1} = \begin{Vmatrix} \frac{1}{2} I(p) & \frac{1}{2} I(p) \\ \frac{1}{2}\varepsilon^{-1/2} I(p) & -\frac{1}{2}\varepsilon^{-1/2} I(p) \end{Vmatrix}.$$

It is easily shown that

$$(45) \qquad \tilde{Q} = WQW^{-1} = \begin{Vmatrix} \sqrt{\varepsilon}\, I(p) & 0 \\ 0 & -\sqrt{\varepsilon}\, I(p) \end{Vmatrix},$$

$$(46) \qquad \tilde{\omega} = \omega W^{-1} = \begin{Vmatrix} \omega_1 & 0(r_1, p) \\ 0(r_2, p) & \omega_2 \end{Vmatrix}.$$

But if $HQ = QH$ then WHW^{-1} is commutative with WQW^{-1} and in fact

$$(47) \qquad WHW^{-1} = \begin{Vmatrix} H_1(\sqrt{\varepsilon}) & 0 \\ 0 & H_1(-\sqrt{\varepsilon}) \end{Vmatrix}.$$

Hence

$$(48) \qquad \xi_1\omega_1 = \omega_1 H_1(\sqrt{\varepsilon}), \qquad \xi_2\omega_2 = \omega_2 H_1(-\sqrt{\varepsilon}).$$

The matrices $H_1(\sqrt{\varepsilon})$ form the quantities of an algebra $\mathfrak{A}^{(1)}$ over $F_1 = F(\sqrt{\varepsilon})$ equivalent to $\mathfrak{A}$ as over $F(q)$ when we replace q in the multiplication table of $\mathfrak{A}$ by $\sqrt{\varepsilon}$. Hence $\mathfrak{A}^{(1)}$ is a normal division algebra over F_1 and contains a quantity A_1 corresponding to a of $\mathfrak{A}$; the minimum equation of A_1 with respect to F_1 has coefficients in F, degree n and roots $\alpha_1, \cdots, \alpha_n$ all real. It follows that we may define the P_j of Theorem 2 and that ω_1 is isomorphic in F_1 to

$$(49) \qquad \omega_{11} = \| \tau^{(1)} P_j \alpha_j^{k-1} \| \qquad (j, k = 1, \cdots, n),$$

since ω_1 has $\mathfrak{A}^{(1)}$ as an algebra of multiplications. Let $\omega_{11} = \gamma_1\omega_1 G_1$ where G_1 has elements in F_1 and is the matrix which transforms any given representation $H_1(\sqrt{\varepsilon})$ of the quantities of $\mathfrak{A}^{(1)}$ into the canonical form

giving (49). But then $G_1(-\sqrt{\varepsilon})$ is the matrix transforming $H_1(-\sqrt{\varepsilon})$ into the canonical form conjugate to the above and there exists a complex matrix γ_2 with the property that $\omega_{2\Gamma} = \gamma_2\, \omega_2\, G_2(-\sqrt{\varepsilon})$ where

$$(51) \qquad \omega_{21} = \| \tau^{(2)} \bar{P}_j \alpha_j^{k-1} \| \qquad (j, k = 1, \cdots, n).$$

It follows that ω is isomorphic in F to

$$(52) \qquad \tilde{\omega} = \begin{Vmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{Vmatrix} \omega G = \begin{Vmatrix} \omega_{11} & 0 \\ 0 & \omega_{21} \end{Vmatrix} \cdot W,$$

where G is a matrix with elements in F determined so that

$$WGW^{-1} = \begin{Vmatrix} G_1(\sqrt{\varepsilon}) & 0 \\ 0 & G_1(-\sqrt{\varepsilon}) \end{Vmatrix},$$

so that G is non-singular when $G_1(\sqrt{\varepsilon})$ is non-singular.

Let us take ω as given in (52), (51), (49) for the left side of (52) and discuss the conditions that ω have no multiplications $\zeta\omega = \omega\mathrm{X}$ where X is not commutative with Q. If $\mathrm{X}Q - Q\mathrm{X} \neq 0$ then $W(\mathrm{X}Q - Q\mathrm{X})W^{-1}$ has matrix elements off the diagonal not zero matrices and is not commutative with WQW^{-1} so that $\mathrm{X}Q - Q\mathrm{X}$ is not commutative with Q. But $\mathrm{X}Q - Q\mathrm{X}$ is a projectivity of ω when X is, while $W(\mathrm{X}Q - Q\mathrm{X})W^{-1}$ has, as elements on the diagonal, zero matrices. Hence we may take X so that

$$(53) \qquad W\mathrm{X}W^{-1} = \begin{Vmatrix} 0 & \mathrm{X}_1(\sqrt{\varepsilon}) \\ \mathrm{X}_1(-\sqrt{\varepsilon}) & 0 \end{Vmatrix}, \qquad \zeta = \begin{Vmatrix} 0 & \zeta_2 \\ \bar{\zeta}_1 & 0 \end{Vmatrix} \neq 0(p),$$

and have $\mathrm{X}_1 = \mathrm{X}_1(\sqrt{\varepsilon})$,

$$(54) \qquad \zeta_2\, \omega_{21} = \omega_{11}\, \mathrm{X}_1, \qquad \zeta_1\, \bar{\omega}_{11} = \bar{\omega}_{21}\, \mathrm{X}_1.$$

Conversely when (54) are satisfied we can define a matrix X with elements in F which satisfies (53) and obviously defines a multiplication of ω not commutative with q. But write

$$(55) \qquad V = \| \alpha_j^{k-1} I(p') \|, \qquad \omega_{11} = \omega_{12} V, \qquad \omega_{21} = \omega_{22} V.$$

We have, from (54),

$$(56) \qquad \zeta_2\, \omega_{22} = \omega_{12}\, V\mathrm{X}_1 V^{-1}, \qquad \zeta_1\, \bar{\omega}_{12} = \bar{\omega}_{22}\, V\mathrm{X}_1 V^{-1}.$$

Let us write $Z = (PV)\mathrm{X}_1(PV)^{-1}$ so that if

$$(57) \qquad Z = \| Z_{rs} \| \qquad (r, s = 1, \cdots, n),$$

then at least one $Z_{rs} \neq 0(p')$. If in particular $Z_{kj} \neq 0(p')$ then the matrix

$$Z_1 = \varepsilon_{jk} Z,$$

where ε_{jk} has $I(p')$ in the jth row and kth column and $0(p')$ elsewhere, has $Z_{kj} \neq 0(p')$ in the jth row and jth column and hence has at least one diagonal matrix not $0(p')$. But, as we have seen,

$$\varepsilon_{jk} = (PV)\, E_{jk} (PV)^{-1} = (PV)\left(\sum \sigma_{jks} U_s\right)(PV)^{-1},$$

so that

$$Z_1 = \sum \sigma_{jks} (PV)\,(U_s X_1)\,(PV)^{-1}.$$

At least one of the matrices $(PV)\,(U_s X_1)\,(PV)^{-1}$ has not $0(p')$ in the jth row and jth column, since a linear combination with scalar coefficients of matrices each with $0(p')$ in the jth row and column has $0(p')$ in that row and column. Hence, for some s,

$$Z_2 = (PV)\,(U_s X_1)\,(PV)^{-1}$$

has at least one diagonal element not $0(p')$. Leaving the use of j to indicate a fixed element we now proceed with our argument.

The matrix P is a matrix of diagonal elements so that

$$\Phi = V(U_s X_1) V^{-1} = \| \Phi(\alpha_j, \alpha_k) \| \qquad (j, k = 1, \cdots, n),$$

has at least one element $\Theta(\alpha_j) = \Phi(\alpha_j, \alpha_j) \neq 0(p')$, and hence all such elements are not $0(p')$. Now

(58) $$\mu_{1s}\, \omega_{11} = \omega_{11}\, U_s, \qquad \bar{\mu}_{2s}\, \bar{\omega}_{12} = \bar{\omega}_{12}\, U_s,$$

so that, if (54) is true and $\mu_{1s}\, \zeta_2 = \eta_2$, $\mu_{2s}\, \zeta_1 = \eta_1$, we have

(59) $$\eta_2\, \omega_{22} = \omega_{12}\, \Phi, \qquad \eta_1\, \bar{\omega}_{12} = \bar{\omega}_{22}\, \Phi.$$

The element in the jth row and column of $\eta_2\, \omega_{22}$ is $\eta_{22j}\, \tau_j^{(2)}$, where

(60) $$\eta_1 = \| \eta_{1jk} \|, \qquad \eta_2 = \| \eta_{2jk} \|, \qquad \tau_j^{(2)} = \tau^{(2)} \bar{P}_j, \qquad \tau_j^{(1)} = \tau^{(1)} P_j,$$

while that in the jth row and column of $\omega_{12}\, \Phi$ is $\tau_j^{(1)}\, \Theta(\alpha_j)$. Similarly obtaining $\eta_{1jj}\, \bar{\tau}_j^{(1)}$ and $\bar{\tau}_j^{(2)}\, \Theta(\alpha_j)$ the equation (59) becomes

(61) $$\eta_{2jj}\, \tau_j^{(2)} = \tau_j^{(1)}\, \Theta(\alpha_j), \qquad \eta_{1jj}\, \bar{\tau}_j^{(1)} = \bar{\tau}_j^{(2)}\, \Theta(\alpha_j) \qquad (j = 1, \cdots, n),$$

for $\Theta(\alpha_j) \neq 0(p')$ which is true for all values of j when true for one value since the group of the equation of which the α_j are roots is a transitive group.

Conversely if (61) is true for $\Theta(\alpha_j) \neq 0(p')$ we can define X_1 satisfying

$$VX_1V^{-1} = \|\Phi(\alpha_j, \alpha_k)\|, \quad \Phi(\alpha_j, \alpha_k) = 0(p') \quad (j \neq k),$$
$$\Phi(\alpha_j, \alpha_j) = \Phi(\alpha_j) \quad (j, k = 1, \cdots, n), \tag{62}$$

and matrices

$$\eta_1 = \|\eta_{1jk}\|, \quad \eta_2 = \|\eta_{2jk}\|, \quad \eta_{1jk} = 0, \quad \eta_{2jk} = 0 \quad (j \neq k), \tag{63}$$

such that

$$\eta_2\omega_{21} = \omega_{11}X_1, \qquad \eta_1\bar{\omega}_{11} = \bar{\omega}_{22}X_1. \tag{64}$$

The matrix ω will then have a multiplication not commutative with q.

LEMMA. *The matrix ω in the form of the left side of* (52) *with* (51), (49), *has no multiplications not commutative with q if and only if* (61) *are impossible for $\Theta(\alpha_j) \neq 0(p')$ and η_{1jj}, η_{2jj} complex matrices.*

Suppose now that all multiplications of ω are commutative with q. Then all projectivities of ω have the form (47) and the multiplication algebra of ω is an algebra over $F(q)$ and contains a normal division sub-algebra $\mathfrak{A}$ over $F(q)$. The multiplication algebra $\mathfrak{D}$, by a well known theorem in the theory of linear associative algebras, is the direct product of $\mathfrak{A}$ and another algebra $\mathfrak{C}$ over $F(q)$. If x is a non-scalar quantity of $\mathfrak{C}$ then, since $qx = xq$,

$$x \cong X, \qquad WXW^{-1} = \begin{Vmatrix} X_1(\sqrt{\varepsilon}) & 0 \\ 0 & X_1(-\sqrt{\varepsilon}) \end{Vmatrix},$$

where $X_1(\sqrt{\varepsilon})$ is commutative with all of the quantities of $\mathfrak{A}^{(1)}$ over F_1. Hence, as we showed in section 2, if $X_1 = X_1(\sqrt{\varepsilon})$, then $(PV)X_1(PV)^{-1}$ is a matrix whose matrix elements off the diagonal are all $0(p')$ while those on the diagonal are all the same non-scalar matrix $\xi(\alpha_1)$ such that

$$P_j\,\xi(\alpha_j)P_j^{-1} = \xi(\alpha_1), \tag{65}$$

while there exist matrices λ_1 and λ_2 with complex elements and such that

$$\lambda_1\tau^{(1)} = \tau^{(1)}\xi(\alpha_1), \qquad \bar{\lambda}_2\bar{\tau}^{(2)} = \bar{\tau}^{(2)}\xi(\alpha_1), \tag{66}$$

when X defines a multiplication of ω. Conversely, if (66) is true for non-scalar $\xi(\alpha_1)$ satisfying (65), then a multiplication of ω exists which is not in $\mathfrak{A}$.

If C_1 is a principal matrix of Ω, the Omega matrix of (41), then VC_1V' has as its diagonal matrices $\Gamma(\alpha_j)$ where $\Gamma(\alpha_1)$ is a skew-hermitian matrix with elements in $F_1(\alpha_1)$ such that

$$\tau^{(1)}(P_j\,\Gamma(\alpha_1)\bar{P}_j')\tau^{(2)\prime} = 0 \tag{67}$$

from the principal minors of $\omega_{11} C_1 \omega_{21}' = 0$. Again, from principal minors of the positive definite Hermitian matrices defining the property that Ω is an Omega matrix,

$$\sqrt{-1}\, \tau^{(1)} (P_j \Gamma(\alpha_j) \overline{P_j'}) \overline{\tau^{(1)}}', \tag{68}$$

$$\sqrt{-1}\, \tau^{(2)} (\overline{P_j}\, \overline{\Gamma(\alpha_j)}\, P_j') \overline{\tau^{(2)}}', \tag{69}$$

are necessarily positive definite Hermitian, so that in particular the matrix

$$\Omega_1 = \left\| \begin{matrix} \tau^{(1)} \\ \overline{\tau^{(2)}} \end{matrix} \right\| \tag{70}$$

is an Omega matrix over $F_1(\alpha_1)$, which by (65), (66) must have no projectivity satisfying (65) when $\mathfrak{A}$ is the multiplication algebra of ω.

Conversely when (67), (68), (69) hold then we may define C_1 such that VC_1V' has $\Gamma(\alpha_j)$ as diagonal matrices and $0(p')$ elsewhere so that Ω is an Omega matrix over F_1, a necessary and sufficient condition that ω be a Riemann matrix over F.

THEOREM 11. *Let $\mathfrak{A}$ be an algebra satisfying* (iii) *of Theorem* 7 *and let ω be a matrix with p rows and $2p$ columns of complex elements. Then ω is a Riemann matrix over F with $\mathfrak{A}$ as its multiplication algebra if and only if*

(i) *The integer p is divisible by n^2 so that $p = np' = n(\mathrm{r}_1' + \mathrm{r}_2')$, and we can define p'-rowed square matrices P_j with elements in $F_1(\alpha_1, \cdots, \alpha_n)$, $F_1 = F(\alpha_1)$, as in Theorem* 2.

(ii) *The matrix ω is isomorphic in F to*

$$\left\| \begin{matrix} \omega_1 & \omega_1 \sqrt{\varepsilon} \\ \omega_2 & -\omega_2 \sqrt{\varepsilon} \end{matrix} \right\|,$$

where

$$\omega_1 = \|\tau^{(1)} P_j \alpha_j^{k-1}\|, \qquad \omega_2 = \|\tau^{(2)} P_j \alpha_j^{k-1}\| \qquad (j, k = 1, \cdots, n),$$

$\overline{P}_j$ is the complex conjugate matrix to P_j, and $\tau^{(1)}$ is an r_1'-rowed and p'-columned matrix of complex elements. $\tau^{(2)}$ is an r_2'-rowed and p'-columned matrix of complex elements.

(iii) *There exists a p'-rowed skew Hermitian matrix $\Gamma(\alpha_1)$ with elements in $F_1(\alpha_1)$ such that* (67), (68), (69) *are satisfied and hence, in particular, the matrix* (70) *is an Omega matrix over $F_1(\alpha_1)$.*

(iv) *There exists no p'-rowed square matrix $\Theta(\alpha_1) \neq 0(p)$ which has elements in $F_1(\alpha_1)$ and for which*

$$\eta_{2j} \tau^{(2)} \overline{P}_j = \tau^{(1)} P_j \Theta(\alpha_j), \qquad \eta_{1j} \overline{\tau^{(1)}} \overline{P}_j = \overline{\tau^{(2)}} P_j \Theta(\alpha_j), \qquad (j = 1, \cdots, n), \tag{71}$$

for complex matrices η_{1j}, η_{2j}.

(v) *The matrix* (70) *has no projectivity $\xi(\alpha_1)$ satisfying* (65).

In closing we shall show that while $\mathfrak{r}_1'$ and $\mathfrak{r}_2'$ seem arbitrary at least they are both not zero when $p > 1$ and ω is pure. For if either is zero, then without loss of generality we may take $\mathfrak{r}_2' = \mathfrak{r}_2 = 0$ and

$$\omega = \| \omega_1 \quad 0 \| W.$$

But then if T is a matrix so chosen that

$$WTW' = \begin{Vmatrix} 0 & A(\sqrt{\varepsilon}) \\ A(-\sqrt{\varepsilon}) & 0 \end{Vmatrix},$$

where $A = A(\sqrt{\varepsilon})$ has a single non-zero element which may be taken to be unity and hence has zero determinant for $p > 1$, we have

$$\omega T \omega' = \| \omega_1 \quad 0 \| \cdot \begin{Vmatrix} 0 & A(\sqrt{\varepsilon}) \\ A(-\sqrt{\varepsilon}) & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \omega_1' \\ 0 \end{Vmatrix} = 0.$$

The matrix T has zero determinant and is a matrix of ω. It defines a projectivity TC^{-1} of ω, where C is a principal matrix of ω. But TC^{-1} is singular when T is and then ω is not pure. As ω is pure when a division algebra is its multiplication algebra we have

THEOREM 12. *When ω has $\mathfrak{A}$ as its multiplication algebra and genus $p > 1$ then* $\mathfrak{r}_1' \mathfrak{r}_2' \neq 0$.

NEW YORK, N. Y.,
June 12, 1930.

ON THE AVERAGE NUMBER OF SIDES OF POLYGONS OF A NET.[1]

By W. C. Graustein.

A colleague in the Harvard Medical School, Professor Frederic T. Lewis, who is interested in quantitative problems connected with the structure of cells, is the instigator of this contribution to elementary topology.

In simple epithelia, for example, in the skin of the cucumber or in the pigment layer of the retina, the cells are in the form of prisms with essentially parallel axes and are so stacked that a section by a plane transverse to the axes presents a net of polygons. Though their origin is lost in obscurity, it may be assumed that in the early stages the polygons are, in general, all hexagons. During growth, division of the cells gives rise to polygons which vary in the number of their sides from four to twelve or perhaps more. One property of the net is, however, seldom violated: the number of sides which issue from each (interior) vertex is always three; a vertex common to four or more polygons is a rare occurence because of its instability.

The remarkable fact about this type of cell structure is that the average number of sides, per polygon, of the polygons of the net turns out always to be almost precisely six.[2] It was a desire for a theoretical explanation of this fact that gave rise to the present investigations.

The proposition suggested by the biological data is true. We state it, to begin with, in its simplest form.

Theorem 1. *If in a net of regular hexagons covering the plane a finite number of hexagons are removed and the regions thus vacated are covered by polygons in any way which leaves the number of sides issuing from each (new or old) vertex always equal to three, the average number of sides, per polygon, of the polygons which are inserted is precisely six.*

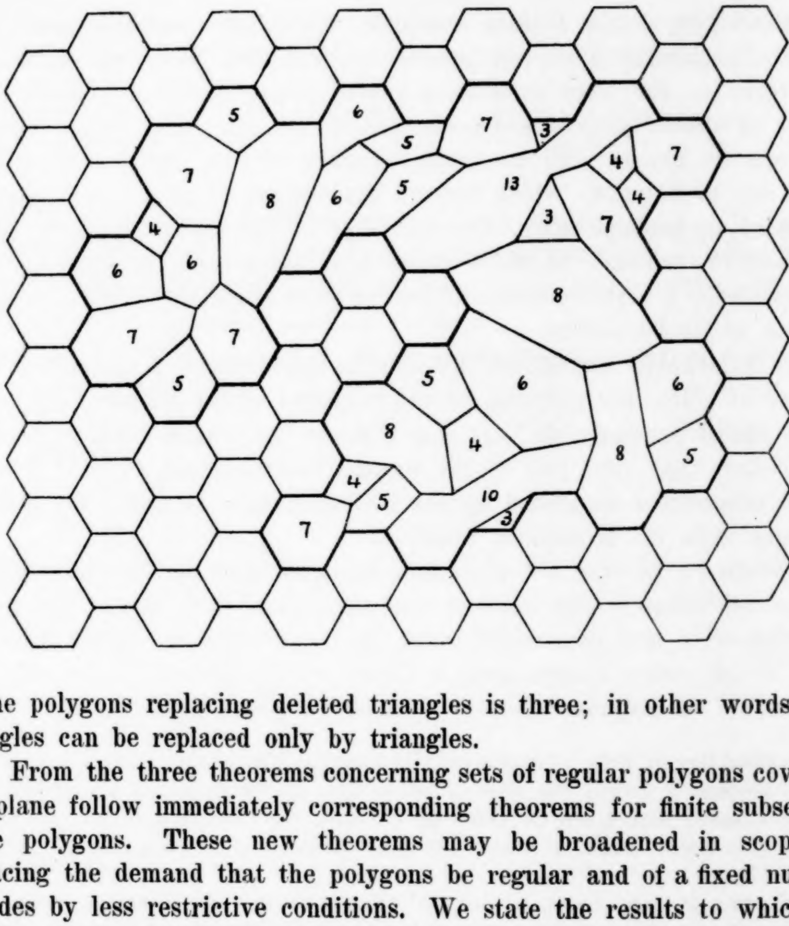
[1] Received May 10, 1930. — Presented to the American Mathematical Society, April 6, 1930.

[2] Cf. Frederic T. Lewis, *The effect of cell division on the shape and size of hexagonal cells*, Anatomical Record, vol. 33 (1926), pp. 331–355; *The correlation between cell division and the shapes and sizes of prismatic cells in the epidermis of Cucumis,* ibid. vol. 38 (1928), pp. 341–376.

In the first of these papers, Lewis cites earlier work by Georg Wetzel, in which mathematical support for the appearance in equal numbers of pentagons and heptagons — which with the hexagons predominate — is purported to be found in the dissertation of Reinhardt, *Über die Zerlegung der Ebene in Polygone*, Frankfurt a. M., 1918. The present writer could, however, find no comfort for the biologists in Reinhardt's paper or elsewhere in the literature on topology.

The theorem is illustrated by the accompanying figure.

Similar theorems hold for the other two nets of regular polygons of a given number of sides which cover the plane, namely for a net of squares and a net of equilateral triangles. For a net of squares whose alteration in the manner prescribed is subject to the condition that the number of sides issuing from each vertex remain always four, the average number of sides, per polygon, of the polygons inserted in the place of squares must be four. In the case of a net of equilateral triangles, the number of sides at each vertex is restricted to six, and the average number of sides



of the polygons replacing deleted triangles is three; in other words, the triangles can be replaced only by triangles.

From the three theorems concerning sets of regular polygons covering the plane follow immediately corresponding theorems for finite subsets of these polygons. These new theorems may be broadened in scope by replacing the demand that the polygons be regular and of a fixed number of sides by less restrictive conditions. We state the results to which we are thus led in a single theorem, restricting ourselves to the case in which the given set of polygons covers a simply connected finite region.

THEOREM 2. *A finite set of polygons covers a region of the plane bounded by a simple closed broken line and is so constituted that (a) n sides issue*

from each interior vertex and at most n sides from each vertex on the boundary, and (b) the average number of sides, per polygon, is m, *where* n *is to be given in turn the values* $3, 4, 6$ *and the corresponding values of* m *are* $6, 4, 3$. *If polygons not having a vertex on the boundary are removed and the regions thus vacated are covered by new polygons in any way which leaves the number of sides at each interior (new or old) vertex equal to* n, *then the average number of sides, per polygon, of the reconstructed net is* m.

The theorem is also true for any positive integral value of $n \geqq 3$, when $m = 2n/(n-2)$. But only for the values $n = 3, 4, 6$ associated with the regular polygons covering the plane is it of interest. For, it is only for these values of n that $m = 2n/(n-2)$ is an integer.

It is hardly necessary to remark on the generalizations of which the foregoing, and subsequent, theorems are capable because of their invariance with respect to continuous deformations.

The Cases $n = 3, 4, 6$. The criteria by means of which we propose to establish our propositions bear on the nature of the vertices on the boundary of a finite set of polygons. These vertices differ in type according to the number of sides of the net which issue from them. To distinguish readily between the various types, we lay down the following definition.

A vertex on the boundary of a finite set of polygons at which there are k *interior sides, or* $k+2$ *sides in all, shall be said to be of type* k.

If the number of sides at a vertex on the boundary is restricted to be $\leqq n$, as in the case of the net of Theorem 2, the boundary may have vertices of types ranging from 0 to $n-2$.

The number of vertices on the boundary which are of type i *we shall denote by* h_i, $i = 0, 1, \cdots, n-2$.

We are now in a position to state in simple form the proposed criteria. We begin with the case $n = 3$.

THEOREM 3. *If a finite set of polygons covers a region of the plane bounded by a simple closed broken line and is so constituted that* $n = 3$ *sides issue from each interior vertex and at most* $n = 3$ *sides from each vertex on the boundary, a necessary and sufficient condition that the average number of sides, per polygon, of the polygons of the net be equal to* $m = 6$ *is that*

$$h_0 - h_1 = 6. \tag{1}$$

Let F be the number of polygons in the net; E the number of distinct sides; V the number of distinct vertices; and T the number of sides or the number of vertices obtained by counting each side or vertex as many times as there are polygons to which it belongs.

In the total number of sides T each interior side is counted twice and each side belonging to the boundary once; and in the total number of

vertices T each interior vertex is counted three times, each vertex of type 1 on the boundary twice, and each vertex of type 0 on the boundary once. Hence:

$$E = \frac{T + h_1 + h_0}{2}, \qquad V = \frac{T + h_1 + 2h_0}{3}.$$

Substituting these values for E and V in Euler's identity,

$$F = E - V + 1,$$

we get

$$F = \frac{T + h_1 - h_0 + 6}{6}.$$

Hence

$$\frac{T}{F} = 6 + \frac{h_0 - h_1 - 6}{F},$$

and $T/F = 6$ if and only if $h_0 - h_1 = 6$.

Theorem 1 and Theorem 2 ($n = 3$) are now readily established. Theorem 2 ($n = 3$) follows immediately from Theorem 3, inasmuch as the criterion (1) is valid for the given set of polygons and is not affected by the process applied to transform the given set. Theorem 1 is rendered a consequence of Theorem 2 ($n = 3$) by restricting the attention to a finite subset of the given set of regular hexagons so chosen that it is bounded by a simple closed broken line which contains within its interior all the hexagons which are to be removed.

The theorems dealing with the net of squares and the net of equilateral triangles and Theorems 2 ($n = 4, 6$) are established in the same way, by application of theorems analogous to Theorem 3. When $n = 4$, $m = 4$, the criterion which takes the place of (1) is

$$h_0 - h_2 = 4. \tag{2}$$

When $n = 6$, $m = 3$, the criterion reads

$$2h_0 + h_1 - h_3 - 2h_4 = 6. \tag{3}$$

The elementary character of the criteria (1), (2), (3) is brought out by consideration of a finite set of regular polygons (regular hexagons, squares, equilateral triangles) bounded by a simple closed broken line. Trace the boundary once in the positive sense, rotating each directed side (produced) into the next directed side through the numerically smallest algebraic angle. In the case of squares, for example, the angle of rotation at a given vertex is $+1$, 0, or -1 times $\pi/2$ according as the vertex is of

type 0, 1, or 2, and the sum of the angles of rotation is 4 times $\pi/2$. Hence, $(+1)h_0+(0)h_1+(-1)h_2 = 4$ or $h_0-h_2 = 4$.

The case $n = n$. In seeking generalizations of our theorems, we assume a finite set of polygons bounded by a simple closed broken line and so constituted that there are n sides at each interior vertex and at most n sides at each vertex on the boundary, where n is an arbitrary integer $\geqq 3$. We have, then, on the boundary vertices of types $0, 1, 2, \cdots, n-2$, in numbers $h_0, h_1, h_2, \cdots, h_{n-2}$.

Let F, E, V, T have the meanings already defined. In T, as the total number of sides, a side of the boundary fails of being counted twice by unity, and in T, as the total number of vertices, a vertex on the boundary of type i fails of being counted n times by $n-1-i$. Consequently,

$$2E = T+\sum_{i=0}^{n-2} h_i, \qquad nV = T+\sum_{i=0}^{n-2}(n-1-i)h_i.$$

When these values of E and V are substituted in Euler's identity, the average number of sides, per polygon, of the polygons of the net is found to be

$$\frac{T}{F} = \frac{2n}{n-2}+\frac{\sum_{i=0}^{n-2}(n-2-2i)h_i-2n}{(n-2)F}.$$

It follows that Theorem 2 is true for $n \geqq 3$ and $m = 2n/(n-2)$. But, as already noted, the theorem is of importance only when $n = 3$, 4, or 6.

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THE UNIVERSAL QUANTIFIER IN COMBINATORY LOGIC.*

By H. B. Curry.

Contents.

1. **Introduction.** In a previous paper[1] I have defined the term combinatory logic and have carried through a preliminary investigation concerning it; viz. the study of substitution and related processes, in so far as these processes are considered in their formal relations to one another. In the present paper I shall attempt to go a step further, and to consider these processes in their formal relations to the universal quantifier, or Allzeichen, or sign or prefix of generality, as it is variously called. Knowledge of the main definitions and of certain results of the previous paper is assumed; but the form of the previous paper is, like that of the present one, such that the reader can obtain this information without going through the details.

Statement of the problem. The reader will recognize the truth of the following intuitive principle, which I shall call the principle of Substitution: "if we have any expression, say $\mathfrak{X}$, involving variables, say $x_1, x_2, \cdots, x_m$, which represents a true statement for all values of these variables, and if we form a new expression, say $\mathfrak{Y}$, by substituting other expressions which may again involve variables, for one or more variables in $\mathfrak{X}$, or by performing on the variables in $\mathfrak{X}$ a transformation[2] or by a succession of these processes; then the expression will also represent a true statement for all values of whatever variables appear therein." This principle lies at the bottom of most of the existing formal treatments of logic; but in spite of its fundamental character, it is nevertheless a principle of con-

* Received March 26, 1930. Presented to the American Mathematical Society, Dec. 27, 1929.

[1] Grundlagen der kombinatorischen Logik (Göttingen, Inauguraldissertation), American Journal of Mathematics, vol. 52, pp. 509-536, 789-834 (1930). Reference will be made to the two chapters of this as I and II respectively. (Cf. below under Preliminary Conventions.)

[2] I. e., what was called in my previous paper an "Umwandlung".

siderable complexity.[3] The primary object of the present paper is to establish this principle on the basis of a primitive frame of the same order of simplicity as that of my previous paper; in fact, of one which is identical with the previous one except for the addition of the five axioms given below.

The reader may will ask what it means to establish such an intuitive principle on the basis of an abstract primitive frame like that specified. To answer this question I shall first define the *interpretation* of a statement (or concept) about the present abstract theory to be the meaning which that statement (or concept) takes on, when the constituents of the primitive frame are given suitable meanings, specifically those set down for them in the formulation of the primitive frame.[4] Then the establishment of the above principle will naturally mean the demonstration of a theorem having that principle for its interpretation. This theorem will not, at least at the present stage, be a formula (i. e. statement of the form $\vdash Z$, where Z is an entity)[5], but will be a more complicated statement about the theory. Its demonstration will, as in the case if the previous theorems, consist in the exhibition of a definite process, whereby any special case arising under it may be abstractly proved.

Let us now turn to the precise formulation of this theorem. For this purpose we shall need to make note of the following facts.

1. The expression $\mathfrak{X}$, in the notation of the present theory can be put in the form

$$(X x_1\, x_2 \cdots x_m),$$

where X is a combination of constants, i. e. entities (Etwase).

2. The expression $\mathfrak{Y}$ is, from the manner of its formation, a normal combination of X, certain constants, say $u_1, u_2, \cdots, u_p$, and certain variables, which without loss of generality we may take to be $x_1, x_2, \cdots, x_n$; hence[6] there exists a regular combinator $\mathfrak{R}$, such that $\mathfrak{Y}$ can be expressed in the form

$$Y x_1\, x_2 \cdots x_n,$$

where

$$Y \equiv \mathfrak{R} X u_1\, u_2 \cdots u_p.$$

3. I shall show below that it is possible to define for each value of n, an entity Π_n such that for any entity Z the statement

[3] Cf. a still earlier paper by the author: "An Analysis of Logical Substitution", American Journal of Mathematics, vol. 51, pp. 363–384 (July, 1929).

[4] I, Abschnitt C.

[5] German "Etwas".

[6] Cf. II D 6 Satz 7.

$$\vdash \Pi_n Z$$

has the interpretation: the expression $(Z x_1 x_2 \cdots x_n)$ represents a true statement of arbitrary entities be substituted for $x_1, x_2, \cdots, x_n$. It will furthermore be shown that if Z is an entity such that

$$\vdash \Pi_{n+p} Z,$$

and if $u_1, u_2, \cdots, u_p$ are also entities, then

$$\vdash \Pi_n (Z u_1 u_2 \cdots u_p).$$

In view of these facts the formal Principle of Substitution may be stated as follows: If X is an entity such that

$$\vdash \Pi_m X$$

and $\mathfrak{R}$ is a regular combinator, then for a certain q determined by $\mathfrak{R}$ and m,

$$\vdash \Pi_q (\mathfrak{R} X).$$

This statement lacks precision only in that the value of q is not stated. The principle is proved below for q such that

$$\vdash \mathfrak{R} x_0 x_1 x_2 \cdots x_q = x_0 y_1 y_2 \cdots y_m,$$

where $y_1, y_2, \cdots, y_m$ are combinations of $x_1, x_2, \cdots, x_q$. The reader will readily see that this q is then equal to the $p + n$ of the above discussion. The proof of this theorem constitutes the establishing of the Principle of Substitution.

In order to obtain this result it seems to be necessary to consider properties of a different sort. These concern the relation of formal implication; i. e. that relation between the functions φ, ψ which is ordinarily[7] symbolized by

$$(x_1, x_2, \cdots, x_m) [\varphi(x_1, x_2, \cdots, x_m) \rightarrow \psi(x_1, x_2, \cdots, x_m)].$$

An important proposition concerning this relation when $m = 1$ is the following: $(f, g) [(x) (f(x) \rightarrow g(x)) \rightarrow ((x) f(x) \rightarrow (x) g(x))]$. The properties mentioned are certain generalizations of this proposition for $m > 1$. It is a by-product of the present investigation that when we assume as axiom

[7] This is essentially the notation used in Hilbert-Ackermann, Grundzüge der theoretischen Logik, 1928. These authors would, however, write the prefix $(x_1)(x_2)\cdots(x_n)$ instead of $(x_1, x_2, \cdots, x_n)$.

a formula having the above proposition for its interpretation, then all the generalizations referred to can be established in the same sense as the above Principle of Substitution. For the details see § 6, below.

In the last case, moreover, it is possible to establish the various generalizations as formulas, provided we make an addition to the rules at our disposal. This addition consists of the transitive property of certain kinds of formal implication. A more stringent theorem concerning the Principle of Substitution can also be proved if we make this same addition. Now this transitive property is presumably unproveable, either as a formula or as a theorem, without the syllogism and perhaps other such propositions, and therefore lies in a certain sense outside the domain of combinatory logic. Nevertheless the fact that certain propositions can be proved by its aid is of interest even at the present stage; and consequently a few theorems at the close of §§ 6 and 7 below have to do with such considerations. In these cases the fact that an additional assumption is, so far as known, necessary for the proof, is explicitly indicated in the hypotheses of the theorems mentioned.

Preliminary conventions. The two chapters of my previous paper will be referred to hereafter as I and II respectively. These numerals will be followed by the symbols for the various subdivisions of these chapters as indicated in the paper itself. Thus II B 4 Satz 7 means Kapitel II, Abschnitt B, Nummer 4, Satz 7.

The conventions and definitions of the previous paper, which were labelled as such, are presupposed. In the translation from the German, the following words shall be equivalent.

Definition = definition. Abbreviated Def.

Festsetzung = convention.

Etwas = entity. It has been called to my attention that this word has a philosophical connotation which is objectionable for this purpose. Nevertheless I shall continue to use it, because I cannot find an English word which is fully satisfactory. However it shall be emphasized here that the word is here used in a technical sense, and that no connection with metaphysical Being is intended. The word constant will also be used to some extent in the same connection.

Axiom = axiom.[8]

Regel = rule.

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[8] Objections have been raised by some writers to the use of the term axiom in this connection. There are nevertheless two reasons which have induced me to retain it: 1) it agrees with the nomenclature of my German paper, in which I naturally followed the usage of writers in that language; 2) such information as I can find inclines me to the opinion that these objections are unsound. This latter point I do not wish to urge

Other equivalents which are not self explanatory will be explained as they arise.

The following additional convention shall be made: the combination of symbols

$$\vdash X = \vdash Y$$

shall be construed as an abbreviation for the following argument

$$\vdash X \text{ and } \vdash X = Y.$$

$$\therefore \vdash Y, \qquad \text{(Rule } Q_1\text{)}.$$

2. **The quantifiers Π_n.** The entity Π_n, defined below, has for its interpretation the prefix $(x_1, x_2, \cdots, x_n)$ in such as expressions as

$$(x_1, x_2, \cdots, x_n) f(x_1, x_2, \cdots, x_n),$$

which means that f is true for all values of $x_1, x_2, \cdots, x_n$ (cf. Theorem 1, below).

DEFINITION 1.

$$\begin{aligned} \Pi_0 &\equiv I, \\ \Pi_1 &\equiv \Pi, \\ \Pi_{n+1} &\equiv \Pi \cdot B\Pi_n, \qquad n = 1, 2, 3, \cdots. \end{aligned}$$

THEOREM 1. *If X is an entity for which*

$$\vdash \Pi_n X;$$

then for any n entities $u_1, u_2, \cdots, u_n$,

$$\vdash X u_1 u_2 \cdots u_n, \qquad n = 0, 1, 2, \cdots.$$

here in detail. Suffice it to say that if we are to differentiate between these two words in the absolute at all, the distinction should be as far as possible invariant of changes in point of view or philosophic opinion—a requirement which the distinction, that an axiom is a self evident truth, while a postulate is an arbitrary assumption, certainly does not satisfy. On the other hand, if there is anything common to the various distinctions with which I am acquainted, it is that an axiom is an assumption of more fundamental and general character than a postulate. In this respect the primitive propositions of logic certainly stand on a different footing from those of the special branches of mathematics; and it is therefore consistent to call the assumptions of logic, and mathematics in general, axioms, whatever the name by which those of the special branches of mathematics may be designated. It may be added that if we should conceive of all mathematics and logic as a single abstract theory then the latter statements would not be represented in the theory by propositions at all, but by the hypotheses of certain formal implications or by definitions; consequently the term postulate might well be preferred for them; but the argument would not apply to the primitive propositions of the theory as a whole.

Proof. For $n = 0$, the theorem is trivial. For $n = 1$, the theorem is equivalent to Rule Π. Assume the theorem for $n = k \geqq 1$, then it follows for $n = k + 1$ as follows. Suppose

$$\vdash \Pi_{k+1} X$$

$$= \vdash \Pi(B \Pi_k X) \qquad \text{(Def. 1; II B 4 Satz 1)}.$$

$\therefore$ for any entity u_1,

$$\vdash B \Pi_k X u_1 \qquad \text{(Rule } \Pi\text{)},$$

$$= \vdash \Pi_k (X u_1) \qquad \text{(Rule } B\text{)}.$$

$\therefore$ for any k entities $u_2, u_3, \cdots, u_{k+1}$, by this theorem for $n = k$,

$$\vdash X u_1 u_2 \cdots u_{k+1}, \quad \text{q. e. d.}$$

THEOREM 2.

$$\vdash \Pi_n = \Pi \cdot B \Pi \cdot B_2 \Pi \cdot \cdots \cdot B_{n-1} \Pi.$$

Proof. Follows from Definition 1 by induction on n.

THEOREM 3.

$$\vdash \Pi_{m+n} = \Pi_m \cdot B_m \Pi_n, \qquad m, n = 0, 1, 2, \cdots.$$

Proof. For $m = 0$ or $n = 0$, follows from Definition 1 and II B 2 Satz 1, II B 4 Satz 4. For $m = 1$, $n \geqq 1$, follows immediately from Definition 1. Assume the theorem for $m = k$, $n \geqq 1$, then it follows for $m = k + 1$, $n \geqq 1$, as follows:

$$\begin{aligned}
\vdash \Pi_{n+k+1} &= \Pi \cdot B \Pi_{n+k} && \text{(this theorem, } m = 1\text{)},\\
&= \Pi \cdot B(\Pi_k \cdot B_k \Pi_n) && \text{(this theorem, } m = k\text{)},\\
&= \Pi \cdot B \Pi_k \cdot B_{k+1} \Pi_n && \text{(II B 4 Satz 2, II B 1 Satz 5)},\\
&= \Pi_{k+1} \cdot B_{k+1} \Pi_n && \text{(this theorem, } m = 1\text{; II B 4 Satz 3)}.
\end{aligned}$$

THEOREM 4. *If X is an entity such that*

$$\vdash \Pi_{m+n} X;$$

then for arbitrary entities $u_1, u_2, \cdots, u_m$,

$$\vdash \Pi_n (X u_1 u_2 \cdots u_m).$$

Proof.

$$\vdash \Pi_m (B_m \Pi_n X) \qquad \text{(Hp.; Theorem 3; II B 4 Satz 1; Rule } Q_1\text{)}.$$

$\therefore$ for arbitrary entities $u_1, u_2, \cdots, u_m$,

$$\vdash B_m \Pi_n X u_1 u_2 \cdots u_m \qquad \text{(Theorem 1)},$$

$$= \vdash \Pi_n (X u_1 u_2 \cdots u_m) \quad \text{q. e. d.} \qquad \text{(II B 1 Satz 3)}.$$

3. Some further theorems regarding combinators. In the course of the discussion we shall need various theorems on combinators in addition to those contained in the former paper. The proof of these theorems occupies the present section. Most of these theorems are not needed until the last section; the earlier ones will, however, be useful directly. In the discussion use will be made of variables in the same manner and subject to the same conditions as in II C.

CONVENTION 1. A regular combinator U shall be said to have the order m and degree n if the following equation involving variables holds:

$$\vdash U x_0 x_1 x_2 \cdots x_m = x_0 y_1 y_2 \cdots y_n, \tag{1}$$

where $y_1, y_2, \cdots, y_n$ are combinations of $x_1, x_2, \cdots, x_m$.[9]

THEOREM 1. *If U is a regular combinator of order m and degree n, then it is of order $m+k$ and degree $n+k$ for all $k \geqq 0$; furthermore if m and n are respectively the minimum order and degree of U, then U has no other order and degree than $m+k$, $n+k$, $k \geqq 0$.*

Proof. Clear.

THEOREM 2. *If U is a regular combinator, such that*

$$\vdash C_1 B_{m+1} U = BU \cdot B_{n+1};$$

then U is of order m and degree n.

Proof. Since U is regular it corresponds to a normal sequence (Folge). (II D 6 Satz 2.) Hence for proper values of the integers p and q,

$$\vdash U x_0 x_1 x_2 \cdots x_p = x_0 y_1 y_2 \cdots y_q, \tag{1}$$

where $y_1, y_2, \cdots, y_q$ are combinations of $x_1, x_2, \cdots, x_p$ (cf. II C 1 Festsetzung 7). Without loss of generality we may suppose $p \geqq m$ and $q \geqq n$.

This established it follows that

$$\begin{aligned} \vdash (B_m K \cdot U) x_0 x_1 x_2 \cdots x_p x_{p+1} &= K(U x_0 x_1 \cdots x_m) x_{m+1} \cdots x_{p+1} \\ &\qquad \text{(II B 4 Satz 1, II B 1 Satz 3)}, \\ &= U x_0 x_1 x_2 \cdots x_m x_{m+2} x_{m+3} \cdots x_{p+1} \\ &\qquad \text{(Rule } K\text{)}, \\ &= x_0 y_1' y_2' \cdots y_q', \end{aligned} \tag{2}$$

[9] This definition is to be understood as defining the pair of integers representing the order and degree respectively, and not as defining order and degree separately. It is of course true that to each order greater than a certain minimum there is a unique corresponding degree (cf. Theorem 1) and conversely.

where y'_i consists of y_i with x_{m+j} replaced by x_{m+j+1} $(j = 1, 2, \cdots, p-m)$. On the other hand

$$\begin{aligned} &\vdash (U \cdot B_n K) x_0 x_1 x_2 \cdots x_p x_{p+1} = U(B_n K x_0) x_1 x_2 \cdots x_p x_{p+1} \text{ (II B 4 Satz 1)}, \\ (3) \qquad &\qquad = B_n K x_0 y_1 y_2 \cdots y_q x_{p+1}, \\ &\qquad = x_0 y_1 y_2 \cdots y_n y_{n+2} \cdots y_q y_{q+1}, \\ &\qquad\qquad \text{where } y_{q+1} \equiv x_{p+1}. \end{aligned}$$

But

$$\vdash B_m K \cdot U = U \cdot B_n K \qquad \text{(Hp; II D 2 Satz 1)}$$

$$\therefore \vdash x_0 y'_1 y'_2 \cdots y'_q = x_0 y_1 y_2 \cdots y_n y_{n+2} \cdots y_q y_{q+1}.$$

But since these expressions involve variables only,

$$(4) \qquad y'_1 \equiv y_1,\ y'_2 \equiv y_2,\ \cdots,\ y'_n \equiv y_n;\quad y'_{n+1} \equiv y_{n+2} \cdots y'_q \equiv y_{q+1}.$$

The first n of these equations show that $y_1, y_2, \cdots, y_n$, can involve none of the variables $x_{m+1}, x_{m+2}, \cdots, x_p$; hence they are combinations of $x_1, x_2, \cdots, x_m$. On the other hand from the last equation, it follows, if $q > n$ that $y'_q \equiv x_{p+1}$. Then, by definition of y'_q, if $p > m$, it follows that $y_q \equiv x_p$. It then follows from the next to last equation (4), if $q-1 > n$, that $y'_{q-1} \equiv y_q \equiv x_p$, whence, if $p-1 > m$, $y_{q-1} \equiv x_{p-1}$. In this way we can continue, and show that

$$(5) \qquad y_{q-r} \equiv x_{p-r},$$

as long as $q-r > n$, $p-r > m$.

Suppose now we have reached a stage where $p-r = m+1$, but $q-r = s+1 > n+1$; then we have

$$y_{s+1} \equiv x_{m+1},$$

whence, since $s > n$,

$$y'_s \equiv y_{s+1} \equiv x_{m+1}, \qquad \text{(by (4))}.$$

This however is impossible, because (2) does not involve x_{m+1}.

Hence the equations (5) must be true down to a stage where $q-r = n+1$. Then let $p-r = t+1$, where $t \geqq m$. Substituting the evaluations of the y's thus obtained in (1) we have

$$(6) \qquad \vdash U x_0 x_1 x_2 \cdots x_p = x_0 y_1 y_2 \cdots y_n x_{t+1} x_{t+2} \cdots x_p.$$

Suppose now $t > m$. Then,

$$\begin{aligned} &\vdash (W_{m+1} \cdot U) x_0 x_1 x_2 \cdots x_p = U x_0 x_1 x_2 \cdots x_m x_{m+1} x_{m+1} x_{m+2} \cdots x_p \\ (7) \qquad &\qquad\qquad \text{(II B 4 Satz 1, II B 3 Satz 4)}, \\ &\qquad = x_0 y_1 y_2 \cdots y_n x_t x_{t+1} \cdots x_p; \end{aligned}$$

also,

$$\vdash (U \cdot W_{n+1})\, x_0\, x_1 \cdots x_p = W_{n+1}\, x_0\, y_1\, y_2 \cdots y_n\, x_{t+1}\, x_{t+2} \cdots x_p$$
$$(8) \qquad = x_0\, y_1\, y_2 \cdots y_n\, x_{t+1}\, x_{t+1}\, x_{t+2} \cdots x_p.$$

The expressions on the right hand sides of (7) and (8) must be identical (II D 2 Satz 5b). This is impossible, since the place after y_n is occupied in the one case by x_t, in the other case by x_{t+1}. This contradiction comes from assuming $t > m$. Hence $t = m$.

We now have

$$\vdash U x_0\, x_1\, x_2 \cdots x_p = x_0\, y_1\, y_2 \cdots y_n\, x_{m+1}\, x_{m+2} \cdots x_p.$$

Let U' be the normal form of U (II D 6 Satz 1). Then by II D 6 Satz 6 the expression $(U' x_0\, x_1 \cdots x_m)$ reduces to $(x_0\, y_1\, y_2 \cdots y_n)$. Hence

$$\vdash U x_0\, x_1 \cdots x_m = x_0\, y_1\, y_2 \cdots y_n, \qquad \text{q. e. d.}$$

THEOREM 3. *A necessary and sufficient condition that a regular combinator U be of order m and degree n is that*

$$\vdash C_1\, B_{m+1}\, U = B U \cdot B_n.$$

Proof. The condition is sufficient by Theorem 1.

To show it is necessary: if (1) under Convention 1 holds, then the two combinators $(C_1\, B_{m+1}\, U)$ and $(B U \cdot B_n)$ both correspond to the sequence

$$x_1\, (x_2\, z_1\, z_2 \cdots z_n)\, x_{m+3}\, x_{m+4} \cdots,$$

where z_i is what y_i becomes when x_j is replaced by x_{j+2}. Hence by II E 3 Satz 4 the two combinators are equal.

THEOREM 4. *If U is a regular combinator of order m and degree n; then $B_k U$ is of order $m+k$ and degree $n+k$* $(k \geqq 0)$.

Proof. Suppose U satisfies the equation (1) under Convention 1 for given m and n. Let z_i be what y_i becomes when every x_j, $(j = 1, 2, \cdots, m)$, is replaced by x_{j+k}; then

$$\vdash B_k U x_0\, x_1\, x_2 \cdots x_{m+k} = U (x_0\, x_1\, x_2 \cdots x_k)\, x_{k+1}\, x_{k+2} \cdots x_{m+k} \qquad \text{(II B 1 Satz 3)},$$
$$= x_0\, x_1\, x_2 \cdots x_k\, z_1\, z_2 \cdots z_n. \qquad \text{(Hp.)}.$$

THEOREM 5. *The combinators $(B_p B)$, C_{p+1}, W_{p+1}, K_{p+1} are respectively of order $p+2$, $p+2$, $p+1$, $p+1$, and degree $p+1$, $p+2$, $p+2$, p, but not of any lower order or degree* $(p \geqq 0)$.

Proof. The combinators B, C, W, and K, are, by Rules B, C, W, and K, respectively of order $2, 2, 1, 1$, and degree $1, 2, 2, 0$. That the combinators $(B_p B)$, C_{p+1}, W_{p+1} and K_{p+1} have the orders in indicated in the theorem then follows by Theorem 4.

The combinators mentioned cannot have orders and degrees less than those stated. For let U be one of these combinators and suppose

$$\vdash U x_0 \, x_1 \, x_2 \cdots x_m = x_0 \, y_1 \, y_2 \cdots y_n.$$

Then since U is in the normal form (II D) the expression on the left must reduce to that on the right. But such a reduction cannot take place if $m <$ the value stated in the theorem.

THEOREM 6. *If U and V are regular combinators of orders m and p and degrees n and q respectively, and if r and s are integers such that*

$$r \geqq m, \qquad r \geqq m+p-n, \qquad r+n+q = s+m+p;$$

then $(U \cdot V)$ is of order r and degree s.

Proof. Let $\vdash U x_0 \, x_1 \, x_2 \cdots x_m = x_0 \, y_1 \, y_2 \cdots y_n$, where $y_1, y_2, \cdots, y_n$ are combinations of $x_1, x_2, \cdots, x_m$. If we then define

$$y_{n+j} \equiv x_{m+j}, \qquad (j = 1, 2, 3, \cdots),$$

then, for an arbitrary $r \geqq m$,

$$\vdash U x_0 \, x_1 \, x_2 \cdots x_r = x_0 \, y_1 \, y_2 \cdots y_t, \qquad t = r-m+n.$$

Again, let $\vdash V x_0 \, x_1 \, x_2 \cdots x_p = x_0 \, z_1 \, z_2 \cdots z_q$, where $z_1, z_2, \cdots, z_q$ are combinations of $x_1, x_2, \cdots, x_p$. Let $z_1', z_2', \cdots, z_q'$ be what $z_1, z_2, \cdots, z_q$ respectively become when every x_i is replaced by y_i, and let $z_{q+k}' \equiv y_{p+k}$; then for arbitrary $t \geqq p$

$$\vdash V x_0 \, y_1 \, y_2 \cdots y_t = x_0 \, z_1' \, z_2' \cdots z_s', \qquad s = t-p+q.$$

From the last two formulas, if $r \geqq m$ and $t = r-m+n \geqq p$ (i. e. $r \geqq m-n+p$),

$$\begin{aligned} \vdash (U \cdot V) \, x_0 \, x_1 \, x_2 \cdots x_r &= U(V x_0) \, x_1 \, x_2 \cdots x_r \quad \text{(II B 4 Satz 1)}, \\ &= V x_0 \, y_1 \, y_2 \cdots y_t \\ &= x_0 \, z_1' \, z_2' \cdots z_s'. \end{aligned}$$

Hence $(U \cdot V)$ is of order r and degree s provided $r \geqq$ the values stated in the theorem, and s satisfies the stated equation.

THEOREM 7. *If U and V are regular combinators such that*

(a) *the minimum order and degree of U are respectively m, n;*

(b) *the minimum order and degree of V are respectively p, q;*

(c) *U, V, and $(U \cdot V)$, are all as they stand in the normal form;*

then the minimum order, r, and degree s, of $(U \cdot V)$ satisfy the same conditions as in Theorem 6, viz.

$$r \geqq m, \qquad r \geqq m - n + p, \qquad r + n + q = s + m + p.$$

Proof. Suppose

$$\vdash (U \cdot V)\, x_0\, x_1\, x_2 \cdots x_r = x_0\, z_1\, z_2 \cdots z_s, \tag{1}$$

then by the hypothesis of normality the expression on the left must reduce to that on the right (II D 6 Satz 6). In the course of this reduction the terms of U must disappear first, then those of V, so that we have an equation of the form

$$\begin{aligned} \vdash (U \cdot V)\, x_0\, x_1\, x_2 \cdots x_r &= U (V x_0)\, x_1\, x_2 \cdots x_r \\ &= V x_0\, y_1\, y_2 \cdots y_t, \end{aligned} \tag{2}$$

where $y_1, y_2, \cdots, y_t$ are combinations of $x_1, x_2, \cdots, x_r$. In this reduction V plays no essential role; the reduction would go through this far just the same if were left out, which is equivalent to saying that U has the order r and degree t. Then by Hp. (a) $r \geqq m$; furthermore, since the degree corresponding to any order is unique, $t = r - m + n$.

In the further reduction the expression on the right of (2) must reduce to that on the right of (1). A precisely similar reduction must go through if variables $x_1, x_2, \cdots, x_t$ be substituted for $y_1, y_2, \cdots, y_t$, since the latter expressions are left intact through out the reduction. Then V is of order t and degree s; hence, as in the previous case, $t \geqq p$, $s = t - p + q$ whence, by definition of t, $r \geqq m - n + p$, $r + n + q = s + m + p$, q. e. d.

Remark. The foregoing theorems give a method whereby the minimum order and degree (and hence by Theorem I all possible orders and degrees) of any regular combinator may be determined without the use of variables. In fact let U be a regular combinator. We may suppose without loss of generality that U is in the normal form (II D 6 Satz 1). Then U is of the form

$$(U_1 \cdot U_2 \cdots U_N),$$

where each U_i is of one of the forms $(B_p B)$,[10] C_{p+1}, W_{p+1}, K_{p+1}, the order of these terms being such that U is in the normal form. Then the minimum order and degree of each U_i is determined by Theorem 5. Furthermore let

[10] See definition at beginning of II D. A term of the form $(B_p\, B_q)$ can be factored into q terms of the form $(B_p\, B)$ by II B 4 Satz 7.

$$V_j \equiv (U_1 \cdot U_2 \cdots U_j),$$

so that

$$\vdash V_{j+1} = V_j \cdot U_{j+1}.$$

We have already found the minimum order and degree of V_1. Suppose we have done this for V_j; then V_{j+1} will have an order and degree as given by Theorem 6, and the smallest pair of values of r and s satisfying the conditions of Theorem 6 will be the minimum order and degree of V_{j+1} by Theorem 7. Hence we can find the minimum order and degree of V_{j+1}, hence, by induction, of V_N, and so of U.

The significance of Theorem 7 and of the last part of Theorem 5 is that U cannot have any order or degree lower than any which can be inferred from Theorems 5 (first part) and 6.

THEOREM 8. *Let $\mathfrak{P}(m, n)$ be for each positive or zero integral value of m and n a property of regular combinators such that:*

(a) *if U has the property $\mathfrak{P}(m, n)$ then it has the property $\mathfrak{P}(m+k, n+k)$ for all $k \geqq 0$;*

(b) *$(B_p B)$, C_{p+1}, W_{p+1}, K_{p+1} have respectively the properties $\mathfrak{P}(p+2, p+1)$, $\mathfrak{P}(p+2, p+2)$, $\mathfrak{P}(p+1, p+2)$, $\mathfrak{P}(p+1, p)$;*

(c) *if U and V have respectively the properties $\mathfrak{P}(m, n)$, $\mathfrak{P}(p, q)$, then $(U \cdot V)$ has the property $\mathfrak{P}(r, s)$, provided,*

$$r \geqq m, \qquad r \geqq m-n+p, \qquad r+n+q = s+m+p.$$

Then every regular combinator of order m and degree n has the property $\mathfrak{P}(m, n)$.

Proof. By the foregoing remark, if U has the order m and degree n, then we must be able to demonstrate this fact by the use of Theorems 1, 5, and 6. If throughout this demonstration we replace the three theorems quoted respectively by the three hypotheses of this theorem, we shall then have a demonstration that U has the property $\mathfrak{P}(m, n)$, q. e. d.

4. The formalizing combinator. First properties of formal implication.

DEFINITION 1.

$$\begin{aligned} \Phi_0 &\equiv I, \\ \Phi_1 &\equiv W_3 \cdot C_2 \cdot B_2 B_1 \cdot B, \\ \Phi_{n+1} &\equiv \Phi_1 \cdot \Phi_n, \qquad\qquad n = 1, 2, 3, \cdots. \end{aligned}$$

DISCUSSION. The combinator here defined I have called the formalizing combinator, because by means of it it is possible to define the relation of formal implication for functions of one or more variables in terms of

ordinary implication. Thus if P is ordinary implication, it follows from Theorem 1 (below) that $(\Phi_n P)$ is that function of two functions of n variables, whose value for the given functions $f(x_1, x_2, \cdots, x_n)$ and $g(x_1, x_2, \cdots, x_n)$ is the function $f(x_1, \cdots, x_n) \rightarrow g(x_1, \cdots, x_n)$. Thus formal implication for functions of n variables is $(B\Pi_n(\Phi_n P))$. The properties of this formalizing combinator here deduced will be useful in the discussion of formal implication in the sequel; although the theorems are somewhat more general than is necessary for the immediate purpose.

THEOREM 1. *If $X, Y, Z, u_1, u_2, \cdots, u_n$ are entities; then*

$$\vdash \Phi_n XYZu_1 u_2 \cdots u_n = X(Yu_1 u_2 \cdots u_n)(Zu_1 u_2 \cdots u_n).$$

Proof. For $n = 0$, trivial. For $n = 1$, the theorem follows from the considerations of II C and II D. In fact the expression $(\Phi_1 XYZu_1)$, reduces (in the sense of II C 1 Festsetzung 3) to $X(Yu_1)(Zu_1)$. For $n > 1$ the theorem may be proved by induction.

THEOREM 2.

$$\vdash \Phi_{m+n} = \Phi_m \cdot \Phi_n, \qquad m, n = 0, 1, 2, 3. \cdots.$$

Proof. Follows from Def. 1 and the associativity of the dot product as in proof of II B 4 Satz 5. For $m = 0$ or $n = 0$, the theorem follows from II B 4 Satz 4.

DEFINITION 2.

$$\Psi \equiv W_1 \cdot C_2 \cdot B_2 B \cdot B.$$

THEOREM 3. *If X, Y, Z, U, V, are entities, then*

(a) $\vdash \Psi UXYZ = U(XY)(XZ)$.

(b) $\vdash (C_1 \cdot B\Psi) UVXYZ = UX(VY)(VZ)$.

(c) $\vdash \Psi XI = X$.

(d) $\vdash \Psi(\Psi XY)Z = \Psi X(Y \cdot Z)$.

(e) $\vdash B_{m+2}X(\Psi YZ) = \Psi(B_{m+2}XY)Z, \quad m = 0, 1, 2, \cdots.$

Proof. Formulas (a) and (b) follow at once because the left hand sides reduce to the right hand sides.

To prove (c), we notice first the left hand side is, by Rule C, $C_1 \Psi IX$; hence it is sufficient to prove

$$\vdash C_1 \Psi I = I.$$

This however follows from II E 3 Satz 4, since the two combinators correspond to the same sequence (Folge). In fact

$$\begin{aligned}\vdash C_1 \Psi I x_0 x_1 x_2 &= \Psi \\ &= x_0 (I x_1)(I x_2) \qquad \text{(Formula (a))}, \\ &= x_0 x_1 x_2 .\end{aligned}$$

To prove (d) it is similarly sufficient to show

$$\vdash B_2 \Psi \Psi = C_1 (B_2 \Psi) B,$$

for if we write XYZ after either side of this equation, the two expressions reduce to those on either side of (d). That the equation just written is true follows from II E 3 Satz 4, for the combinators appearing on either side both correspond to the sequence

$$x_0 (x_1 (x_2 x_3)) (x_1 (x_2 x_4)) x_5 x_6 \cdots .$$

To prove (e) it is sufficient (for similar reasons) to show

$$\vdash C_1 B_{m+4} \Psi = B \Psi \cdot B_{m+2} .$$

This, however, follows from § 3 Theorem 3 and II D 2 Satz 2, since Ψ is of order 3 and degree 2.

THEOREM 4. *If U is a regular combinator of order p and degree q, and X, Y, Z, are entities; then*

$$\vdash U(\Phi_{n+q} XYZ) = \Phi_{n+p} X (UY)(UZ).$$

Proof. First let $n = 0$. Then we have

$$\begin{aligned}&\vdash U(\Phi_q XYZ) = B_3 U \Phi_q XYZ \qquad \text{(II B 1 Satz 3)}. \\ &\vdash \Phi_p X(UY)(UZ) = (C_1 \cdot B\Psi) \Phi_p UXYZ \qquad \text{(Theorem 3 (b))}.\end{aligned}$$

It is thus sufficient to show that

$$\vdash B_3 U \Phi_q = (C_1 \cdot B \Psi) \Phi_p U.$$

This last, however, follows from II E 3 Satz 4; for as I shall show forthwith the combinators $(B_3 U \Phi_q)$ and $((C_1 \cdot B\Psi) \Phi_p U)$ correspond to the same sequence of variables. In fact we have, by § 3 Convention 1,

$$\vdash U x_0 x_1 x_2 \cdots x_p = x_0 y_1 y_2 \cdots y_q ,$$

where $y_1, y_2, \cdots, y_q$ are combinations of $x_1, x_2, \cdots, x_p$. Let z_i be what y_i becomes when every x_j is replaced by x_{j+3}. Then

$$\begin{aligned}
\vdash B_3\, U\, \Phi_q\, x_1 x_2 \cdots x_{p+3} &= U(\Phi_q\, x_1 x_2 x_3) x_4 x_5 \cdots x_{p+3} && \text{(II B 1 Satz 3)}\\
&= \Phi_q\, x_1 x_2 x_3 z_1 z_2 \cdots z_q\\
&= x_1 (x_2 z_1 z_2 \cdots z_q)(x_3 z_1 z_2 \cdots z_q), && \text{(Theorem 1).}\\
\vdash (C_1 \cdot B\Psi)\Phi_p x_1 x_2 \cdots x_{p+3} &= \Phi_p\, x_1 (U x_2)(U x_3) x_4 x_5 \cdots x_{p+3} && \text{(Theorem 3(b))}\\
&= x_1 (U x_2 x_4 x_5 \cdots x_{p+3})(U x_3 x_4 x_5 \cdots x_{p+3})\\
& && \text{(Theorem 1)}\\
&= x_1 (x_2 z_1 z_2 \cdots z_q)(x_3 z_1 z_2 \cdots z_q).
\end{aligned}$$

Thus the theorem is proved for $n = 0$.

To prove the theorem for $n > 0$, it is merely necessary to substitute $\Phi_n X$ for X in the above proof. The theorem follows at once since, by Theorem 2 and II B 4 Satz 1,

$$\vdash \Phi_{p+n} X = \Phi_p (\Phi_n X) \quad \text{and} \quad \vdash \Phi_{q+n} X = \Phi_q (\Phi_n X).$$

DEFINITION 3.

$$P_n \equiv \Phi_n P, \qquad n = 0, 1, 2, \cdots.$$

THEOREM 5.

$$\vdash P_{m+n} = \Phi_m P_n, \qquad m, n = 0, 1, 2, \cdots.$$

Proof. Follows at once from Theorem 2 and II B 4 Satz 1.

THEOREM 6. *If* $X, Y, u_1, u_2, \cdots, u_n$ *are entities; then*

$$\vdash P_{m+n} X Y u_1 u_2 \cdots u_n = P_m (X u_1 u_2 \cdots u_n)(Y u_1 u_2 \cdots u_n).$$

Proof. Follows from Theorems 1 and 5.

THEOREM 7. *If* X *and* Y *are arbitrary entities and* U *is any regular combinator of order* p *and degree* q; *then*

$$\vdash U(P_{n+q} X Y) = P_{n+p} (UX)(UY), \qquad n = 0, 1, 2, \cdots;$$

in particular

$$\vdash B_k (P_{n+1} X Y) = P_{n+k+1} (B_k X)(B_k Y) \qquad n, k = 0, 1, 2, \cdots.$$

Proof. The general case follows from Theorems 4 and 5. The particular one then follows from since B_k is of order $k+1$ and degree 1.

DEFINITION 4.

$$\mathsf{IP}_n \equiv B_2 \Pi_n P_n, \qquad n = 0, 1, 2, \cdots.$$

The entity IP_n has then for its interpretation the relation of formal implication for functions of n variables.

THEOREM 8. *If* X *and* Y *are entities such that*

$$\vdash \mathsf{IP}_{m+n} X Y;$$

then for arbitrary entities $u_1, u_2, \cdots, u_n$,

$$\vdash \mathsf{IP}_m (X u_1 u_2 \cdots u_n)(Y u_1 u_2 \cdots u_n).$$

Proof. Suppose

$$\begin{aligned} &\vdash \mathsf{IP}_{m+n} XY \\ = &\vdash \Pi_{m+n} (P_{m+n} XY). && \text{(Def. 4; II B 1 Satz 3).}\end{aligned}$$

Then

$$\begin{aligned} &\vdash \Pi_m (P_{m+n} XY u_1 u_2 \cdots u_n) && \text{(§ 2 Theorem 4),} \\ = &\vdash \Pi_m (P_m (X u_1 u_2 \cdots u_n)(Y u_1 u_2 \cdots u_n)) && \text{(Theorem 6),} \\ = &\vdash \mathsf{IP}_m (X u_1 u_2 \cdots u_n)(Y u_1 u_2 \cdots u_n), && \text{q. e. d.}\end{aligned}$$

THEOREM 9. *If* X *and* Y *are entities such that*

(1) $$\vdash \mathsf{IP}_{n+p} XY,$$

and if U *is a regular combinator of order* q *and degree* p, *and such furthermore that for any entity* z *for which* $\vdash \Pi_{n+p} Z$, *it follows that* $\vdash \Pi_{n+q}(UZ)$; *then* $\vdash \mathsf{IP}_{n+q}(UX)(UY)$.

Proof. By Hp. (1) and Def. 4,

$$\begin{aligned} &\vdash \Pi_{n+p} (P_{n+p} XY). \\ \therefore\ &\vdash \Pi_{n+q} (U(P_{n+p} XY)) && \text{(Hp. regarding } U) \\ = &\vdash \Pi_{n+q} (P_{n+q} (UX)(UY)) && \text{(Theorem 7)} \\ = &\vdash \mathsf{IP}_{n+q} (UX)(UY) \quad \text{q. e. d.} && \text{(Def. 4).}\end{aligned}$$

5. **The axioms (ΠZ) and their first consequences.** In this section we shall introduce new axioms as follows:

$$\begin{aligned} &\text{Ax. } (\Pi B). && \vdash \mathsf{IP}_1\, \Pi_1\, (\Pi_2 \cdot B). \\ &\text{Ax. } (\Pi C). && \vdash \mathsf{IP}_1\, \Pi_2\, (\Pi_2 \cdot C). \\ &\text{Ax. } (\Pi W). && \vdash \mathsf{IP}_1\, \Pi_2\, (\Pi_1 \cdot W). \\ &\text{Ax. } (\Pi K). && \vdash \mathsf{IP}_1\, \Pi_0\, (\Pi_1 \cdot K). \\ &\text{Ax. } (\Pi P). && \vdash \mathsf{IP}_2\, \mathsf{IP}_1\, (\Psi\, P_0\, \Pi_1).\end{aligned}$$

To obtain the interpretation of these axioms we may proceed as follows: replace IP_k at the beginning by its definition (§ 4 Def. 4); then remove the prefix Π_k and place variables $x_1, x_2, \cdots, x_k$ after the resulting expressions and reduce, with due regard for the theorems of § 4 and the interpretating P. The expressions so obtained must be true for all values of $x_1, x_2, \cdots, x_k$, which may be indicated in the usual manner by writing a prefix $(x_1, x_2, \cdots, x_k)$ before the expression. The interpretations so obtained are:

$$\begin{array}{lll}
(\Pi B). & (x_1) & (\Pi_1 x_1 \rightarrow \Pi_2 (B x_1)). \\
(\Pi C). & (x_1) & (\Pi_2 x_1 \rightarrow \Pi_2 (C x_1)). \\
(\Pi W). & (x_1) & (\Pi_2 x_1 \rightarrow \Pi_1 (W x_1)). \\
(\Pi K). & (x_1) & (\quad x_1 \rightarrow \Pi_1 (K x_1)). \\
(\Pi P). & (x_1, x_2) & (\mathsf{IP}_1 x_1 x_2 \rightarrow (\Pi_1 x_1 \rightarrow \Pi_1 x_2)).
\end{array}$$

These may be further translated as follows:

$$\begin{array}{lll}
(\Pi B). & (f) & [(x) f x \rightarrow (g, x) f(g x)]. \\
(\Pi C). & (f) & [(x, y) f(x, y) \rightarrow (x, y) f(y, x)]. \\
(\Pi W). & (f) & [(x, y) f(x, y) \rightarrow (x) f(x, x)]. \\
(\Pi K). & (p) & [p \rightarrow (x) K p x]. \\
(\Pi P). & (f, g) & [(x)(f x \rightarrow g x) \rightarrow ((x) f x \rightarrow (x) g x)].
\end{array}$$

Convention 1. An entity Z shall be said to have the property $\mathfrak{P}_1(p, q)$ if the following two formulas both hold:

(A) $\qquad \vdash \mathsf{IP}_1\, \Pi_p (\Pi_q \cdot Z).$

(B) $\qquad \vdash C\, B_{q+1} Z = B Z \cdot B_p.$

Theorem 1. *The combinators B, C, W, and K have respectively the properties $\mathfrak{P}_1(1,2)$, $\mathfrak{P}_1(2,2)$, $\mathfrak{P}_1(2,1)$, $\mathfrak{P}_1(0,1)$.*

Proof. Follows at once from the axioms (ΠB), (ΠC), (ΠW), (ΠK), and axioms B, C, W, and K.

Theorem 2. *If Z is any entity satisfying formula* (A) *of the property $\mathfrak{P}_1(p, q)$, and if X is any entity such that $\vdash \Pi_p X$; then $\vdash \Pi_q (Z X)$.*

Proof. From (A) and § 4 Theorem 8[11] follows

$$\begin{aligned} &\vdash P_0 (\Pi_p X)((\Pi_q \cdot Z) X) \\ = &\vdash P_0 (\Pi_p X)(\Pi_q (Z X)), \qquad \text{(II B 4 Satz 1)}. \end{aligned}$$

The theorem then follows by Rule P.

Corollary. *If X is an entity such that $\vdash \Pi_1 X$, then $\vdash \Pi_2 (B X)$.*

Proof. Follows from Theorems 1 and 2.

Theorem 3. *If Z is an entity having the property $\mathfrak{P}_1(p, q)$, then Z also has the property $\mathfrak{P}_1(p+n, q+n)$, $n = 0, 1, 2, \cdots$.*

Proof. The part of the theorem relating to (B) follows from II D 2

[11] For the case that $m = 0$, $n = 1$, $u_1 \equiv x$. It is of course an immediate consequence of the definitions that

$$\vdash \mathsf{IP}_0 = P_0 = P.$$

Satz 2 (or, if preferred, from § 3 Theorems 1 and 3). It remains to show that (A) is true when p and q are replaced by $p+n$ and $q+n$ respectively.

From formula (A) of $\mathfrak{P}_1(p,q)$ follows

$$\vdash \mathsf{IP}_2(B\Pi_p)\,(B(\Pi_q\cdot Z)) \qquad \text{(Theorem 2 Cor.; § 4 Theorem 9, } n=0\text{)}.$$
$$\therefore \vdash \mathsf{IP}_1(B\Pi_p(B_p\Pi_n))\,(B(\Pi_q\cdot Z)\,(B_p\Pi_n)) \qquad \text{(§ 4 Theorem 8)}.$$

But

$$\vdash B\Pi_p\,(B_p\Pi_n) = \Pi_p\cdot B_p\Pi_n = \Pi_{p+n}$$
$$\text{(II B 4 Def. 1; § 2 Theorem 3)}.$$
$$\vdash B(\Pi_q\cdot Z)\,(B_p\Pi_n) = \Pi_q\cdot Z\cdot B_p\Pi_n \qquad \text{(II B 4 Def. 1 and Satz 3)},$$
$$= \Pi_q\cdot B_q\Pi_n\cdot Z \qquad \text{(Formula (B); II D 2 Satz 1)},$$
$$= \Pi_{q+n}\cdot Z \qquad \text{(§ 2 Theorem 3)}.$$

Inserting[12] the last two formulas in the two preceeding them, we have

$$\vdash \mathsf{IP}_1\,\Pi_{p+n}\,(\Pi_{q+n}\cdot Z), \quad \text{q. e. d.}$$

THEOREM 4. *If Z is an entity having the property $\mathfrak{P}_1(p,q)$, and if X is an entity such that $\vdash \Pi_{p+n}X$; then $\vdash \Pi_{q+n}(ZX)$.*

Proof. Follows from Theorems 2 and 3.

THEOREM 5. *If X and Y are entities such that $\vdash \mathsf{IP}_1XY$; then $\vdash P_0(\Pi_1X)\,(\Pi_1Y)$.*

Proof. From Ax. (ΠP) and § 4 Theorem 8 follows

$$\vdash P_0(\mathsf{IP}_1XY)\,(\Psi P_0\Pi_1XY)$$
$$= \vdash P_0(\mathsf{IP}_1XY)\,(P_0(\Pi_1X)\,(\Pi_1Y)) \qquad \text{(§ 4 Theorem 3a)},$$

whence the theorem follows by Rule P.

THEOREM 6. *If Z is an entity having the property $\mathfrak{P}_1(p,q)$, and if X is an entity such that $\vdash \Pi_{p+n+1}X$; $n\geqq 0$; then $\vdash \Pi_{q+n+1}(BZX)$.*

Proof. By Theorem 3 Z has the property $\mathfrak{P}_1(p+n,q+n)$. But from (A) for $\mathfrak{P}_1(p+n,q+n)$ follows

$$\vdash \mathsf{IP}_2(B\Pi_{p+n})\,(B(\Pi_{q+n}\cdot Z)) \qquad \text{(Theorem 2 Cor.; § 4 Theorem 9)}$$
$$= \vdash \mathsf{IP}_2(B\Pi_{p+n})\,(B\Pi_{q+n}\cdot BZ) \qquad \text{(II B 4 Satz 2)}.$$

Hence

$$\vdash \mathsf{IP}_1(B\Pi_{p+n}X)\,(B\Pi_{q+n}(BZX)) \qquad \text{(§ 4 Theorem 8; II B 4 Satz 1)}.$$
$$\therefore \vdash P_0(\Pi_1(B\Pi_{p+n}X))\,\big(\Pi_1(B\Pi_{q+n}(BZX))\big) \qquad \text{(Theorem 5)},$$
$$= \vdash P_0(\Pi_{p+n+1}X)\,(\Pi_{q+n+1}(BZX)) \qquad \text{(II B 4 Satz 1; § 2 Theorem 3)}.$$

The theorem then follows by Rule P.

[12] This insertion is of the kind which can be justified by the use of the properties of equality alone. (Cf. I D Satz 7).

$$
\begin{array}{lll}
(\Pi B). & (x_1) & (\Pi_1 x_1 \rightarrow \Pi_2 (B x_1)). \\
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(\Pi P). & (x_1, x_2) & (\mathrm{IP}_1 x_1 x_2 \rightarrow (\Pi_1 x_1 \rightarrow \Pi_1 x_2)).
\end{array}
$$

These may be further translated as follows:

$$
\begin{array}{lll}
(\Pi B). & (f) & [(x) fx \rightarrow (g, x) f(gx)]. \\
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(\Pi W). & (f) & [(x, y) f(x, y) \rightarrow (x) f(x, x)]. \\
(\Pi K). & (p) & [p \rightarrow (x) Kpx]. \\
(\Pi P). & (f, g) & [(x) (fx \rightarrow gx) \rightarrow ((x) fx \rightarrow (x) gx)].
\end{array}
$$

Convention 1. An entity Z shall be said to have the property $\mathfrak{P}_1(p, q)$ if the following two formulas both hold:

(A) $\vdash \mathrm{IP}_1\, \Pi_p (\Pi_q \cdot Z)$.

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Theorem 1. *The combinators B, C, W, and K have respectively the properties* $\mathfrak{P}_1(1,2)$, $\mathfrak{P}_1(2,2)$, $\mathfrak{P}_1(2,1)$, $\mathfrak{P}_1(0,1)$.

Proof. Follows at once from the axioms (ΠB), (ΠC), (ΠW), (ΠK), and axioms B, C, W, and K.

Theorem 2. *If Z is any entity satisfying formula* (A) *of the property $\mathfrak{P}_1(p, q)$, and if X is any entity such that $\vdash \Pi_p X$; then $\vdash \Pi_q (Z X)$.*

Proof. From (A) and § 4 Theorem 8[11] follows

$$
\begin{aligned}
& \vdash P_0 (\Pi_p X) ((\Pi_q \cdot Z) X) \\
= & \vdash P_0 (\Pi_p X) (\Pi_q (Z X)), \qquad \text{(II B 4 Satz 1)}.
\end{aligned}
$$

The theorem then follows by Rule P.

Corollary. *If X is an entity such that $\vdash \Pi_1 X$, then $\vdash \Pi_2 (B X)$.*

Proof. Follows from Theorems 1 and 2.

Theorem 3. *If Z is an entity having the property $\mathfrak{P}_1(p, q)$, then Z also has the property $\mathfrak{P}_1(p+n, q+n)$, $n = 0, 1, 2, \cdots$.*

Proof. The part of the theorem relating to (B) follows from II D 2

[11] For the case that $m = 0$, $n = 1$, $u_1 \equiv x$. It is of course an immediate consequence of the definitions that

$$\vdash \mathrm{IP}_0 = P_0 = P.$$

Satz 2 (or, if preferred, from § 3 Theorems 1 and 3). It remains to show that (A) is true when p and q are replaced by $p+n$ and $q+n$ respectively.

From formula (A) of $\mathfrak{P}_1(p, q)$ follows

$$\vdash \mathsf{IP}_2(B\Pi_p)\,(B(\Pi_q\cdot Z)) \qquad \text{(Theorem 2 Cor.; § 4 Theorem 9, } n=0\text{)}.$$

$$\therefore \vdash \mathsf{IP}_1(B\Pi_p(B_p\Pi_n))\,(B(\Pi_q\cdot Z)\,(B_p\Pi_n)) \qquad \text{(§ 4 Theorem 8)}.$$

But

$$\vdash B\Pi_p\,(B_p\Pi_n) = \Pi_p\cdot B_p\Pi_n = \Pi_{p+n}$$

(II B 4 Def. 1; § 2 Theorem 3).

$$\begin{aligned}\vdash B(\Pi_q\cdot Z)\,(B_p\Pi_n) &= \Pi_q\cdot Z\cdot B_p\Pi_n && \text{(II B 4 Def. 1 and Satz 3)},\\ &= \Pi_q\cdot B_q\Pi_n\cdot Z && \text{(Formula (B); II D 2 Satz 1)},\\ &= \Pi_{q+n}\cdot Z && \text{(§ 2 Theorem 3)}.\end{aligned}$$

Inserting[12] the last two formulas in the two preceeding them, we have

$$\vdash \mathsf{IP}_1\,\Pi_{p+n}\,(\Pi_{q+n}\cdot Z), \quad \text{q. e. d.}$$

THEOREM 4. *If Z is an entity having the property $\mathfrak{P}_1(p, q)$, and if X is an entity such that $\vdash \Pi_{p+n}X$; then $\vdash \Pi_{q+n}(ZX)$.*

Proof. Follows from Theorems 2 and 3.

THEOREM 5. *If X and Y are entities such that $\vdash \mathsf{IP}_1\,XY$; then $\vdash P_0(\Pi_1 X)\,(\Pi_1 Y)$.*

Proof. From Ax. (ΠP) and § 4 Theorem 8 follows

$$\begin{aligned}&\vdash P_0(\mathsf{IP}_1\,XY)\,(\Psi P_0\,\Pi_1\,XY)\\ &= \vdash P_0(\mathsf{IP}_1\,XY)\,(P_0(\Pi_1 X)\,(\Pi_1 Y)) \quad \text{(§ 4 Theorem 3a)},\end{aligned}$$

whence the theorem follows by Rule P.

THEOREM 6. *If Z is an entity having the property $\mathfrak{P}_1(p, q)$, and if X is an entity such that $\vdash \Pi_{p+n+1}X$; $n \geqq 0$; then $\vdash \Pi_{q+n+1}(BZX)$.*

Proof. By Theorem 3 Z has the property $\mathfrak{P}_1(p+n, q+n)$. But from (A) for $\mathfrak{P}_1(p+n, q+n)$ follows

$$\begin{aligned}&\vdash \mathsf{IP}_2(B\Pi_{p+n})\,(B(\Pi_{q+n}\cdot Z)) && \text{(Theorem 2 Cor.; § 4 Theorem 9)}\\ &= \vdash \mathsf{IP}_2(B\Pi_{p+n})\,(B\Pi_{q+n}\cdot BZ) && \text{(II B 4 Satz 2)}.\end{aligned}$$

Hence

$$\begin{aligned}&\vdash \mathsf{IP}_1(B\Pi_{p+n}X)\,(B\Pi_{q+n}(BZX)) && \text{(§ 4 Theorem 8; II B 4 Satz 1)}.\\ \therefore\ &\vdash P_0(\Pi_1(B\Pi_{p+n}X))\,\big(\Pi_1(B\Pi_{q+n}(BZX))\big) && \text{(Theorem 5)},\\ &= \vdash P_0(\Pi_{p+n+1}X)\,(\Pi_{q+n+1}(BZX)) && \text{(II B 4 Satz 1; § 2 Theorem 3)}.\end{aligned}$$

The theorem then follows by Rule P.

[12] This insertion is of the kind which can be justified by the use of the properties of equality alone. (Cf. I D Satz 7).

THEOREM 7. *If X is an entity such that $\vdash \Pi_m X$; then*

$$\begin{array}{lll}
\text{a)} & \textit{if } m \geqq 1, & \vdash \Pi_{m+1}(BX). \\
\text{b)} & \textit{if } m \geqq 2, & \vdash \Pi_{m+1}(BBX). \\
\text{c)} & \textit{if } m \geqq 2, & \vdash \Pi_m \ (C_1 X). \\
\text{d)} & \textit{if } m \geqq 3, & \vdash \Pi_m \ (C_2 X). \\
\text{e)} & \textit{if } m \geqq 2, & \vdash \Pi_{m-1}(W_1 X). \\
\text{f)} & \textit{if } m \geqq 3, & \vdash \Pi_{m-1}(W_2 X). \\
\text{g)} & \textit{if } m \geqq 0, & \vdash \Pi_{m+1}(K_1 X). \\
\text{h)} & \textit{if } m \geqq 1, & \vdash \Pi_{m+1}(K_2 X).
\end{array}$$

Proof. Follows from Theorems 1, 4, 6.

THEOREM 8. *If X is an entity such that $\vdash \Pi_m X$, $m \geqq 2$; then $\vdash \Pi_{m+n}(\Phi_n X)$.*

Proof. It is sufficient to prove the theorem for the case $n = 1$; for if the theorem is proved for $n = 1$, it will follow for general n by induction, since

$$\vdash \Phi_{n+1} X = \Phi_1(\Phi_n X).$$

For $n = 1$: in the first place,

$$\vdash \Phi_1 = C_2 \cdot W_2 \cdot B \cdot C_2 \cdot BB,$$

since, if the regular combinator on the right be transformed into the normal form by the process of II D, we shall have Φ_1. Now suppose

$$\begin{array}{ll}
\vdash \Pi_m X, & m \geqq 2. \\
\text{Then} & \\
\vdash \Pi_{m+1}(BBX) & \text{(Theorem 7b).} \\
\therefore \vdash \Pi_{m+1}(C_2(BBX)) = \vdash \Pi_{m+1}((C_2 \cdot BB)X) & \text{(Theorem 7d; II B 4 Satz 1).} \\
\therefore \vdash \Pi_{m+2}((B \cdot C_2 \cdot BB)X) & \text{(Theorem 7a; II B 4 Satz 1).} \\
\therefore \vdash \Pi_{m+1}((W_2 \cdot B \cdot C_2 \cdot BB)X) & \text{(Theorem 7f; II B 4 Satz 1).} \\
\therefore \vdash \Pi_{m+1}((C_2 \cdot W_2 \cdot B \cdot C_2 \cdot BB)X) & \text{(Theorem 7d; II B 4 Satz 1),} \\
= \vdash \Pi_{m+1}(\Phi_1 X), \text{ q. e. d.} &
\end{array}$$

THEOREM 9. *If X is an entity such that $\vdash \Pi_m X$, $m \geqq 2$; then $\vdash \Pi_{m+1}(\Psi X)$.*

Proof. As in Theorem 8, $\vdash \Psi = W_1 \cdot C_2 \cdot B \cdot BB$. The proof of the theorem now follows as in Theorem 8.

6. **Further properties of formal implication.** For the further development of the theory we shall need certain generalizations of the Ax. (ΠP). These generalizations might be derived from the formula

$$\vdash \mathsf{IP}_2 \, \mathsf{IP}_{n+p}(\Psi \, \mathsf{IP}_n(B_n \Pi_p));$$

whose interpretation for $n = p = 1$ is

$$(f, g)\,\{(x, y)\,[f(x, y) \to g(x, y)] \to (x)\,[(y)\,f(x, y) \to (y)\,g(x, y)]\}.$$

The formula in question I am, however, unable to prove on the basis of the primitive frame as presented up to now. Nevertheless it is possible to establish the principle which the formula represents as a theorem. This is accomplished in Theorem 2, which is all of the present section that is used in the next.

The remaining theorems of this section contain a further investigation of this subject on its own account. In Theorems 3 and 4 some special cases of the above formula, and some more general formulas, are proved on the basis of the present primitive frame. If we be allowed to assume that formal implication is transitive, then it is shown in Theorem 5, that not only can the above formula be proved for any given values of n and p, but that an even more general formula can also be established. The most general formula is that stated in Theorem 5 (a). It reduces to the above formula for $m = k = 0$.

If we denote, for the moment, the formula in Theorem 5 a by $[m, n, p, k]$, then an outline of its mode of development is as follows: $[0, 0, 1, 0]$ is Ax. (ΠP); $[m, 0, 1, 0]$ is proved in Theorem 1; $[m, n, 1, 0]$ in Theorem 3; $[m, n, 1, k]$ in Theorem 4. The induction on p is what requires the transitive property of $\mathbf{IP}_{m+2}$.

The interpretation of the formula $[1, 1, 1, 1]$ is as follows

$$\begin{aligned}(f, g, x)\,\{(y, z)\,[(u)\,f(x, y, z, u) \to (u)\,g\,(x, y, z, u)]\\ \to (y)\,[(z, u)\,f(x, y, z, u) \to (z, u)\,g\,(x, y, z, u)]\}.\end{aligned}$$

The reader may find it advantageous to work out the interpretations of other formulas.

THEOREM 1.

$$\vdash \mathbf{IP}_{m+2}(B_{m+2}\,\Pi_1\,P_{m+1})(\Psi P_m(B_m\,\Pi_1)), \qquad m = 0, 1, 2, \cdots$$

Proof. For $m = 0$, this is Ax. (ΠP). Let $m > 0$. Let us define

$$(1) \qquad F_k \equiv P_{k+2}(B_{k+2}\,\Pi_1\,P_{k+1})(\Psi P_k(B_k\,\Pi_1)), \qquad k = 0, 1, 2, \cdots,$$

so that the formula to be proved becomes

$$\vdash \Pi_{m+2}\,F_m,$$

while Ax. (ΠP) is $\vdash \Pi_2 F_0$. I shall show that

$$(2) \qquad \vdash F_m = \Phi_m F_0.$$

The theorem will then follow from Ax. (ΠP) and § 5 Theorem 8.

By § 4 Theorem 5 and (1)

$$\vdash F_k = \Phi_{k+2} P_0 (B_{k+2} \Pi_1 (\Phi_k P_1)) (\Psi (\Phi_k P_0) (B_k \Pi_1)).$$

Let S be the normal combinator such that

$$\vdash S x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 = x_0 x_5 (x_1 x_6 (x_2 x_7)) (x_3 (x_2 x_5) (x_4 x_6)).$$

By II D 6 Sätze 4 and 7 this defines S essentially uniquely. Let

$$T_k \equiv S \Phi_{k+2} B_{k+2} \Phi_k \Psi B_k, \qquad k = 0, 1, 2, \cdots.$$

Then,

$$(3) \qquad \vdash F_m = T_m P_0 \Pi_1 P_1.$$

$$\vdash F_0 = T_0 P_0 \Pi_1 P_1.$$

$$(4) \qquad \vdash \Phi_m F_0 = \Phi_m (T_0 P_0 \Pi_1 P_1) = B_3 \Phi_m T_0 P_0 \Pi_1 P_1.$$

The combinators T_m and $B_3 \Phi_m T_0$ both correspond to the sequence of variables

$$x_0 \left| x_1 (x_2 (x_3 x_5 x_6 \cdots x_{m+4}) (x_4 x_5 x_6 \cdots x_{m+4})) \right| \\ \left| x_0 (x_1 (x_3 x_5 x_6 \cdots x_{m+4})) (x_2 (x_4 x_5 x_6 \cdots x_{m+4})) \right|,$$

as may be seen by carrying out the somewhat laborious reduction. Hence

$$(5) \qquad \vdash T_m = B_3 \Phi_m T_0, \qquad \text{(II E 3 Satz 4)}.$$

From (3), (4), (5), follows (2). The theorem is then proved.

THEOREM 2. *If X and Y are entities such that $\vdash \mathsf{IP}_m X Y$; then*

$$\vdash \mathsf{IP}_{m-n} (B_{m-n} \Pi_n X) (B_{m-n} \Pi_n Y), \qquad n = 0, 1, 2, \cdots, m.$$

Proof. If $n = 0$ the theorem is trivial. It therefore suffices to prove it for $n \geqq 1$, $m \geqq 1$. If $m = 1$, $n = 1$, the theorem follows by § 5 Theorem 5.

Suppose now a method is available whereby we may prove the theorem for given entities X and Y for which $m \leqq k$. Then for $m = k+1$ it may be proved as follows, X and Y being given entities.

$$\vdash \mathsf{IP}_{k+2}(B_{k+2}\Pi_1 P_{k+1})\,(\Psi P_k(B_k\Pi_1)) \qquad \text{(Theorem 1).}$$

$$\therefore\ \vdash \mathsf{IP}_k(B_{k+2}\Pi_1 P_{k+1}XY)\,(\Psi P_k(B_k\Pi_1)XY), \qquad \text{(§ 4 Theorem 8),}$$

$$= \vdash \mathsf{IP}_k(B_k\Pi_1(P_{k+1}XY))\,(P_k(B_k\Pi_1 X)\,(B_k\Pi_1 Y))$$

$$\text{(II B 1 Satz 2; § 4 Theorem 3a).}$$

Now apply the theorem for $m = n = k$ to this last expression. We have

$$\vdash P_0\Big(\Pi_k(B_k\Pi_1(P_{k+1}XY))\Big)\,\Big(\Pi_k(P_k(B_k\Pi_1 X)\,(B_{k+1}\Pi_1 Y))\Big).$$

Now if Z, U, and V are arbitrary entities,

$$\vdash \Pi_k(B_k\Pi_1 Z) = (\Pi_k\cdot B_k\Pi_1)\,Z = \Pi_{k+1}Z, \qquad \text{(§ 2 Theorem 3).}$$

$$\vdash \Pi_k(P_k UV) = \mathsf{IP}_k UV, \qquad \text{(§ 4 Def. 4).}$$

Consequently, applying these last two equations to the formula next preceeding them, we have

$$\vdash P_0(\mathsf{IP}_{k+1}XY)\,(\mathsf{IP}_k(B_k\Pi_1 X)\,(B_k\Pi_1 Y)).$$

Hence if X and Y are entities satisfying the hypothesis of this theorem for $m = k+1$, it follows from Rule P that

$$\vdash \mathsf{IP}_k(B_k\Pi_1 X)\,(B_k\Pi_1 Y),$$

which is the formula to be proved for $n = 1$.

To obtain the result for general n apply the theorem with m and n replaced by k and $n-1$ respectively to the last formula. Then

$$\vdash \mathsf{IP}_{k-n+1}(B_{k-n+1}\Pi_{n-1}(B_k\Pi_1 X))\,(B_{k-n+1}\Pi_{n-1}(B_k\Pi_1 Y)).$$

But

$$\begin{aligned}\vdash B_{k-n+1}\Pi_{n-1}(B_k\Pi_1 X) &= (B_{k-n+1}\Pi_{n-1}\cdot B_k\Pi_1)\,X \quad &&\text{(II B 4 Satz 1),}\\ &= B_{k-n+1}(\Pi_{n-1}\cdot B_{n-1}\Pi_1)\,X &&\text{(II B 4 Satz 6),}\\ &= B_{k-n+1}\Pi_n X, &&\text{(§ 2 Theorem 3),}\end{aligned}$$

and $\vdash B_{k-n+1}\Pi_{n-1}(B_k\Pi_1 Y) = B_{k-n+1}\Pi_n Y$, (similar proof).

$$\therefore\ \vdash \mathsf{IP}_{k-n+1}(B_{k-n+1}\Pi_n X)\,(B_{k-n+1}\Pi_n Y),$$

which is the conclusion of the theorem for $m = k+1$, $n = n$.

COROLLARY. *If X and Y are entities such that* $\vdash \mathsf{IP}_m XY$; *then* $\vdash P_0(\Pi_m X)\,(\Pi_m Y)$.

Proof. This is the case $n = m$.

THEOREM 3. *The following formulas hold for* $n \geqq 0$, $m \geqq 0$

$$\vdash \mathsf{IP}_{m+2}(B_{m+2}\,\Pi_{n+1}\,P_{m+n+1})\,(\Psi\,(B_{m+2}\,\Pi_n\,P_{m+n})\,(B_{m+n}\,\Pi_1)).$$
$$\vdash \mathsf{IP}_2\,\mathsf{IP}_{n+1}\,(\Psi\,\mathsf{IP}_n\,(B_n\,\Pi_1)).$$

Proof. Since the second of these formulas is derived from the first by putting $m = 0$, it will suffice to prove the first. For this purpose replace m in Theorem 1 by $m+n$, then we have

$$\vdash \mathsf{IP}_{m+n+2}(B_{m+n+2}\,\Pi_1\,P_{m+n+1})\,(\Psi\,P_{m+n}\,(B_{m+n}\,\Pi_1)).$$

Now apply Theorem 2, with the m of that theorem replaced by the present $m+n+2$. Then

(1) $\vdash \mathsf{IP}_{m+2}\,(B_{m+2}\,\Pi_n\,(B_{m+n+2}\,\Pi_1\,P_{m+n+1}))\,(B_{m+2}\,\Pi_n\,(\Psi\,P_{m+n}\,(B_{m+n}\,\Pi_1)))$.

But

$$\vdash B_{m+2}\,\Pi_n\,(B_{m+n+2}\,\Pi_1\,P_{m+n+1}) = (B_{m+2}\,\Pi_n \cdot B_{m+n+2}\,\Pi_1)\,P_{m+n+1}$$
(II B 4 Satz 1),
$$= B_{m+2}\,(\Pi_n \cdot B_n\,\Pi_1)\,P_{m+n+1}$$
(II B 4 Satz 6),

(2) $\qquad = B_{m+2}\,\Pi_{n+1}\,P_{m+n+1}$ (§ 2 Theorem 3).

(3) $\vdash B_{m+2}\,\Pi_n\,(\Psi\,P_{m+n}\,(B_{m+n}\,\Pi_1)) = \Psi\,(B_{m+2}\,\Pi_n\,P_{m+n})\,(B_{m+n}\,\Pi_1)$ (§ 4 Theorem 3e).

Inserting (2) and (3) in (1) we have the formula to be proved.

THEOREM 4. *The following formulas hold for* $n > 0$, $k \geqq 0$, $m \geqq 0$:

(a) $\vdash \mathsf{IP}_{m+2}\,(\Psi\,(B_{m+2}\,\Pi_{n+1}\,P_{m+n+1})\,(B_{m+n+1}\,\Pi_k))$
$\qquad (\Psi\,(B_{m+2}\,\Pi_n\,P_{m+n})\,(B_{m+n}\,\Pi_{k+1}))$.

(b) $\vdash \mathsf{IP}_2\,(\Psi\,\mathsf{IP}_{n+1}\,(B_{n+1}\,\Pi_k))\,(\Psi\,P_n\,(B_n\,\Pi_{k+1}))$.

Proof. As in Theorem 4, the second formula is obtained from the first by putting $m = 0$. Hence it suffices to prove the first.

Introduce temporarily the abbreviation

$$X_n \equiv B_{m+2}\,\Pi_n\,P_{m+n},$$

then (a) becomes

$$\vdash \mathsf{IP}_{m+2}\,(\Psi\,X_{n+1}\,(B_{m+n+1}\,\Pi_k))\,(\Psi\,X_n\,(B_{m+n}\,\Pi_{k+1})).$$

To prove this we have

$$\vdash \mathsf{IP}_{m+2}\,X_{n+1}\,(\Psi\,X_n\,(B_{m+n}\,\Pi_1)) \qquad \text{(Theorem 3)}.$$

Now Ψ is of order 3 and degree 2; hence from the last formula follows

$$\vdash \mathsf{IP}_{m+3}(\Psi X_{n+1})\left(\Psi(\Psi X_n(B_{m+n}\Pi_1))\right), \quad (\S\ 5 \text{ Theorem } 9;\ \S\ 4 \text{ Theorem } 9).$$

$$\therefore \vdash \mathsf{IP}_{m+2}(\Psi X_{n+1}(B_{m+n+1}\Pi_k))\left(\Psi(\Psi X_n(B_{m+n}\Pi_1))(B_{m+n+1}\Pi_k)\right),$$
(§ 4 Theorem 8),

$$= \vdash \mathsf{IP}_{m+2}(\Psi X_{n+1}(B_{m+n+1}\Pi_k))(\Psi X_n(B_{m+n}\Pi_1\cdot B_{m+n+1}\Pi_k)),$$
(§ 4 Theorem 3 d).

From this the formula to be proved follows by I D Satz 7, since

$$\vdash B_{m+n}\Pi_1\cdot B_{m+n+1}\Pi_k = B_{m+n}(\Pi_1\cdot B\Pi_k) \quad (\text{II B 4 Satz 6}),$$
$$= B_{m+n}\Pi_{k+1} \quad (\S\ 2 \text{ Theorem } 3).$$

THEOREM 5. *If the use of the transitivity of* IP_{m+2} $(m \geqq 0)$ *be allowed, then the following can be proved for* $n \geqq 0,\ k \geqq 0,\ p \geqq 1$:

$$\text{(a)} \quad \vdash \mathsf{IP}_{m+2}(\Psi(B_{m+2}\Pi_{n+p}P_{m+n+p})(B_{m+n+p}\Pi_k))$$
$$(\Psi(B_{m+2}\Pi_n P_{m+n})(B_{m+n}\Pi_{p+k}))$$

in particular for $m = 0$

$$\text{(b)} \quad \vdash \mathsf{IP}_2(\Psi\mathsf{IP}_{n+p}(B_{n+p}\Pi_k))(\Psi\mathsf{IP}_n(B_n\Pi_{p+k})).$$

Proof. Introduce temporarily the definition

$$Z_n^k \equiv \Psi(B_{m+2}\Pi_n P_{m+n})(B_{m+n}\Pi_k).$$

Then the formula (a) becomes

$$\vdash \mathsf{IP}_{m+2}Z_{n+p}^k Z_n^{k+p}.$$

For the case $p = 1$ this was proved in Theorem 4. Suppose it true for $p = j$, then

$$\vdash \mathsf{IP}_{m+2}Z_{n+j+1}^k Z_{n+1}^{k+j},$$

also

$$\vdash \mathsf{IP}_{m+2}Z_{n+1}^{k+j} \quad Z_n^{k+j+1} \quad (\text{Theorem } 4).$$

$$\therefore \vdash \mathsf{IP}_{m+2}Z_{n+j+1}^k Z_n^{k+j+1} \quad (\text{by the assumed transitive property}).$$

This is the formula to be proved for $p = j+1$. Hence the formula may be proved for any p.

COROLLARY. *Under the same circumstances the following may be proved for* $n \geqq 0,\ p \geqq 1$:

$$\text{(c)} \quad \vdash \mathsf{IP}_2\mathsf{IP}_{n+p}(\Psi\mathsf{IP}_n(B_n\Pi_p)).$$

$$\text{(d)} \quad \vdash \mathsf{IP}_2\mathsf{IP}_p(\Psi P_0\Pi_p).$$

Proof. (c) is the special case of (b) where $k = 0$.[13] (d) is the special case of (c) where $n = 0$.

7. Proof of the principle of substitution.

THEOREM 1. *If X is an entity such that $\vdash \Pi_p X$; then*

$$\vdash \Pi_{p+m}(B_m X), \qquad m = 0, 1, 2, \cdots.$$

Proof. For $m = 0$, trivial. For $m = 1$, proved in § 5 Theorem 7a. For $m > 1$, it follows by induction. For suppose we have established

$$\vdash \Pi_{p+k}(B_k X),$$

then

$$\vdash \Pi_{p+k+1}(B(B_k X)) = \Pi_{p+k+2}(B_{k+1} X)$$

(§ 5 Theorem 7a; II B 1 Satz 5).

THEOREM 2. *If Z is an entity such that $\vdash \mathsf{IP}_1 \Pi_p (\Pi_q \cdot Z)$; then*

$$\vdash \mathsf{IP}_1 \Pi_{p+m}(\Pi_{q+m} \cdot B_m Z).$$

Proof. By hypothesis $\vdash \mathsf{IP}_1 \Pi_p (\Pi_q \cdot Z)$. To this formula we apply § 4 Theorem 9, with $U \equiv B_m$. The hypotheses regarding U in that theorem are fulfilled by Theorem 1 (since B_m is of order $m+1$ and degree 1). Hence

$$\vdash \mathsf{IP}_{m+1}(B_m \Pi_p)(B_m \Pi_q \cdot B_m Z) \qquad \text{(II B 4 Satz 6).}$$
$$\therefore \vdash \mathsf{IP}_1(B\Pi_m(B_m \Pi_p))(B\Pi_m(B_m \Pi_q \cdot B_m Z)) \qquad \text{(§ 6 Theorem 2),}$$
$$= \vdash \mathsf{IP}_1(\Pi_m \cdot B_m \Pi_p)(\Pi_m \cdot B_m \Pi_q \cdot B_m Z) \qquad \text{(II B 4 Def. 1 and Satz 3),}$$
$$= \vdash \mathsf{IP}_1 \Pi_{p+m}(\Pi_{q+m} \cdot Z), \quad \text{q. e. d,} \qquad \text{(§ 2 Theorem 3).}$$

COROLLARY. *If an entity Z has the property $\mathfrak{P}_1(p, q)$; then $B_m Z$ has the property $\mathfrak{P}_1(p+m, q+m)$ $(m = 0, 1, 2, \cdots)$.*

Proof. That part of the theorem which relates to formula (A) follows from what has just been proved. The corresponding property for formula (B) follows from § 3 Theorems 3 and 4.

CONVENTION 1. A regular combinator shall be said to have the property $\mathfrak{P}_2(p, q)$, when for every entity X and integer $k \geqq 0$ such that $\vdash \Pi_{p+k} X$, it follows that $\vdash \Pi_{q+k}(ZX)$.

THEOREM 3. *If a regular combinator has the property $\mathfrak{P}_1(p, q)$; then it has the property $\mathfrak{P}_2(p, q)$.*

Proof. See § 5 Theorem 4.

THEOREM 4. *The combinators $(B_p B)$, C_{p+1}, W_{p+1}, K_{p+1}, have respectively the properties $\mathfrak{P}_2(p+1, p+2)$, $\mathfrak{P}_2(p+2, p+2)$, $\mathfrak{P}_2(p+2, p+1)$, and $\mathfrak{P}_2(p, p+1)$.*

[13] This follows by definition of Π_0, II B 2 Satz 1, and § 4 Theorem 3c.

Proof. Follows from § 5 Theorem 1, and Theorem 2 Corollary and Theorem 3 of this section.

THEOREM 5. *If U and V are regular combinators such that*

(a) *U has the property $\mathfrak{P}_2(n, m)$,*

(*b*) *V has the property $\mathfrak{P}_2(q, p)$;*

then $(U \cdot V)$ has the property $\mathfrak{P}_2(s, r)$, provided $r \geqq m$, $r \geqq m-n+p$, $r+n+q = s+m+p$.

Proof. Let X be an entity and $k \geqq 0$, $s \geqq 0$, integers such that

$$\vdash \Pi_{s+k} X. \tag{1}$$

Then, if $s \geqq q$, it follows from Hp. (b) that

$$\vdash \Pi_{t+k}(VX), \qquad t = s+p-q;$$

whence, if $t \geqq n$, we have from Hp. (a),

$$\begin{aligned} &\vdash \Pi_{r+k}(U(VX)), && r = t+m-n; \\ = &\vdash \Pi_{r+k}((U \cdot V)X), && \text{(II B 4 Satz 1).} \end{aligned} \tag{2}$$

Thus the formula (2) follows from formula (1) provided that $s \geqq q$, $t = s+p-q \geqq n$, $r = t+m-n$. These latter conditions are, however, equivalent to those stated in the theorem.

THEOREM 6. *Every regular combinator of order m and degree n has the property $\mathfrak{P}_2(n, m)$.*

Proof. Follows from § 3 Theorem 8. For if we identify the $\mathfrak{P}(m, n)$ of that theorem with the present $\mathfrak{P}_2(n, m)$, then the hypotheses (b) and (c) are fulfilled by Theorems 4 and 5 just proved, while the hypothesis (a) follows immediately from the definition of $\mathfrak{P}_2(n, m)$.

THEOREM 7. *If U and V are regular combinators having the properties $\mathfrak{P}_1(n, m)$ and $\mathfrak{P}_1(q, p)$ respectively, and if*

$$r \geqq m, \qquad r \geqq m-n+p, \qquad r+n+q = s+m+p;$$

then $(U \cdot V)$ may be shown to have the property $\mathfrak{P}_1(s, r)$ provided the use of the transitive property of IP_2 be allowed.

Proof. Under the given conditions $(U \cdot V)$ satisfies formula (B) of $\mathfrak{P}_1(s, r)$ by § 3 Theorems 3 and 6. It remains to show that formula (A) is satisfied.

Let $t = s+p-q$. Then if $s \geqq q$,

$$\vdash \mathrm{IP}_2 \Pi_s (\Pi_t \cdot V), \qquad \text{(Hp.; § 5 Theorem 3).} \tag{1}$$

Let $r = t + m - n$. Then if $t \geqq n$,

$$\vdash \mathrm{IP}_2\, \Pi_t(\Pi_r \cdot U), \qquad \text{(same reason).}$$

$$\therefore \vdash \mathrm{IP}_3\,(B\Pi_t)\,(B(\Pi_r \cdot U)), \qquad \text{(§ 4 Theorem 9; § 5 Theorem 7a).}$$

$$\therefore \vdash \mathrm{IP}_2\,(B\Pi_t V)\,(B(\Pi_r \cdot U)\, V) \qquad \text{(§ 4 Theorem 8),}$$

(2) $$= \vdash \mathrm{IP}_2\,(\Pi_t \cdot V)\,(\Pi_r \cdot U \cdot V), \qquad \text{(II B 4 Def. 1).}$$

From (1) and (2) and the assumed transitive property the formula to be proved follows. The restrictions on r and s, viz., $s \geqq q$, $t = s + p - q \geqq n$, $r = t + m - n$, are the equivalent to those stated in the theorem.

THEOREM 8. *If U as any regular combinator of order m and degree n; then the formula*

$$\vdash \mathrm{IP}_2\, \Pi_n\,(\Pi_m \cdot U)$$

can be proved, provided the transitive property of IP_2 *may be used.*

Proof. If we identify the $\mathfrak{P}(m, n)$ in § 3 Theorem 8 with the present $\mathfrak{P}_1(n, m)$, then the hypotheses (a), (b) and (c) of those theorems follow from Theorem 2 Cor., Theorem 8, and § 5 Theorems 1 and 3. The theorem then follows by definition of $\mathfrak{P}_1(n, m)$.

THE PENNSYLVANIA STATE COLLEGE,
March 24, 1930.

NOTES ON DIFFERENTIAL GEOMETRY.[1]

By A. P. Mellish.[2]

I. On ovals and curves of constant width.

1. By an *oval* we mean a *convex closed curve.* For the sake of simplicity the following discussion will be restricted to curves whose curvature is continuous and never vanishes, although our results can be extended, with suitable modifications, to the general case. An extensive use will be made of vector-calculus notation, the bold-faced types designating vectors. Thus the position-vector $\mathbf{r}$ of a variable point on the curve will have continuous second derivative with respect to s, length of arc. The symbols $\mathbf{t}$ and $\mathbf{n}$ will designate respectively the unit tangent and unit normal vectors whose orientation coincides with that of the OX, OY axes. We shall consider simultaneously *a pair of opposite points* (P, P'), at which the tangents are parallel. The distance between these tangents and the chord $\overrightarrow{PP'}$ will be called respectively *the width of the curve,* and *the diameter at* P, and will be designated by μ and $\mathbf{d}$. The width μ, as well as the curvature $\varkappa$, and the radius of curvature $\varrho = 1/\varkappa$, will be assumed >0. Finally if a point P' is opposite to P, the geometric quantities which correspond to P' will be designated by the same letters as those for P, with addition of a prime ($'$). We assume

$$\mathbf{t}' = -\mathbf{t}, \qquad \mathbf{n}' = -\mathbf{n}. \tag{1}$$

With this notation we have

$$\mathbf{r}' = \mathbf{r} + \mu\mathbf{n} + \lambda\mathbf{t}. \tag{2}$$

On differentiating with respect to s and using the well known formulas

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \varkappa\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\varkappa\mathbf{t}, \tag{3}$$

we have

$$\mathbf{t}'\frac{ds'}{ds} = \mathbf{t}\left(1 - \varkappa\mu + \frac{d\lambda}{ds}\right) + \mathbf{n}\left(\varkappa\lambda + \frac{d\mu}{ds}\right), \tag{4}$$

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[1] Received June 6, 1930.

[2] We publish here some fragments found among the papers of a young Canadian mathematician of great promise, Arthur Preston Mellish (Born June 10, 1905; died February 7, 1930). His death is a source of profound regret to all who had personal contact with him, and especially to his colleagues at Brown University. [J. Tamarkin.]

whence, in view of (1),

$$(5)\qquad -\frac{ds'}{ds} = 1-\varkappa\mu+\frac{d\lambda}{ds}, \qquad 0 = \varkappa\lambda+\frac{d\mu}{ds}.$$

Let φ be the angle made by $\mathfrak{t}$ and an arbitrary fixed direction. Then

$$(6)\qquad ds = \varrho\, d\varphi = \frac{d\varphi}{\varkappa}, \qquad ds' = \varrho'\, d\varphi = \frac{d\varphi}{\varkappa'},$$

and relations (5) reduce to

$$(7)\qquad ds+ds' = \mu\, d\varphi - d\lambda, \qquad \lambda\, d\varphi + d\mu = 0,$$

or, which is the same, to

$$(8)\qquad \varrho+\varrho' = \mu-\frac{d\lambda}{d\varphi}, \qquad \frac{d\mu}{d\varphi} = -\lambda.$$

Elimination of λ yields a linear differential equation of the second order for μ,

$$(9)\qquad \frac{d^2\mu}{d\varphi^2}+\mu = f(\varphi) \text{ where } f(\varphi) = \varrho+\varrho'.$$

The general solution of this equation is

$$(10)\quad \mu(\varphi) = \sin\varphi\left[\int_0^{\varphi} f(t)\cos t\, dt + C_1\right] - \cos\varphi\left[\int_0^{\varphi} f(t)\sin t\, dt + C_2\right].$$

To determine the arbitrary constants C_1 and C_2 we observe that the functions $\mu(\varphi)$ and $\lambda(\varphi)$, as well as $f(\varphi)$ are periodic in φ, of period π, so that we must have

$$(11)\qquad \mu(\varphi+\pi) = \mu(\varphi), \qquad \lambda(\varphi+\pi) = \lambda(\varphi).$$

In view of (8), (9), and of the periodicity of $f(\varphi)$, conditions (11) are equivalent to

$$(12)\qquad \mu(\pi) = \mu(0), \quad \frac{d\mu}{d\varphi}\Big/_{\varphi=\pi} = \frac{d\mu}{d\varphi}\Big/_{\varphi=0},$$

whence

$$C_1 = -\frac{1}{2}\int_0^{\pi} f(t)\cos t\, dt, \quad C_2 = -\frac{1}{2}\int_0^{\pi} f(t)\sin t\, dt.$$

On setting

$$(13)\qquad \begin{aligned} U(\varphi) &= \int_0^{\varphi} f(t)\cos t\, dt - \frac{1}{2}\int_0^{\pi} f(t)\cos t\, dt, \\ V(\varphi) &= \int_0^{\varphi} f\sin t\, dt - \frac{1}{2}\int_0^{\pi} f\sin t\, dt, \end{aligned}$$

we have the final result

$$\begin{aligned}\mu(\varphi) &= U(\varphi)\sin\varphi - V(\varphi)\cos\varphi,\\ \lambda(\varphi) &= -U(\varphi)\cos\varphi - V(\varphi)\sin\varphi.\end{aligned} \tag{14}$$

Let L be the total length of the curve. Then, by (7),

$$L = \int_{\varphi=0}^{\varphi=\pi}(ds+ds') = \int_0^\pi \mu\, d\varphi; \quad \int_0^\pi \lambda\, d\varphi = 0. \tag{15}$$

Another important relation follows from (14),

$$|\boldsymbol{d}|^2 = \mu^2+\lambda^2 = [U(\varphi)]^2+[V(\varphi)]^2. \tag{16}$$

2. So far we have dealt with general ovals. Let us see now what conclusions can be derived from the formulas above concerning the curves of constant width, that is the ovals for which the distance between any pair of parallel tangents is the same constant a. We shall prove the following

THEOREM. *The statements;*

(i) *a curve is of constant width;*

(ii) *a curve is of constant diameter;*

(iii) *all the normals of a curve (an oval) are double;*

(iv) *the sum of radii of curvature at opposite points of a curve (an oval) is constant;*

are equivalent, in the sense that, whenever one of statements (i–iv) *holds true, all other statements also hold.*

(v) *All curves of the same (constant) width a have the same length L given by*

$$L = \pi a. \tag{17}$$

Proof. Let $\mu = a$ be constant. Then, by (8), (9), (16),

$$\lambda = 0, \qquad \varrho+\varrho' = a, \qquad |\boldsymbol{d}|^2 = a^2$$

which implies, respectively, (iii), (iv), (ii).

Statement (iii) implies $\lambda = 0$; hence, by (8), $\mu = a$ so that (i), and consequently, (ii), (iv) hold.

Statement (ii) implies

$$|\boldsymbol{d}|^2 = \lambda^2+\mu^2 = a^2.$$

On differentiating and using (8) and (7), we have

$$0 = \lambda\, d\lambda+\mu\, d\mu = \lambda(d\lambda-\mu\, d\varphi) = -\lambda(ds+ds').$$

Since $ds+ds' \neq 0$ this gives $\lambda = 0$, that is (iii).

Finally, if

$$\varrho + \varrho' = a \text{ (constant)}$$

then, by (13) and (14), $\mu = a$, which implies (i).

Statement (v) of the theorem follows immediately from (15).

Remark 1. The following properties of the curves of constant width are easily verified:

(vi) *The evolute of a curve of constant width has one and only one tangent parallel to any given direction* (such curves may be termed as "*curves of zero width*").

(vii) *The medial* (that is the locus of mid-points of the diameters) *of a curve of constant width is a curve of zero width.*

(viii) *Any oval which is an involute of a curve of zero width or else, which is parallel to a curve of constant width* (*including those of zero width*) *is a curve of constant width.*

Remark 2. It follows at once from (8) and the periodicity of μ that any oval has at least two double normals, the width being an extremum (maximum and minimum) at the corresponding points of the curve.[3]

II. On ovaloids and surfaces of constant width.

1. By an *ovaloid* we mean a *convex closed surface.* It will be assumed that the principal curvatures of the surface are continuous and do not vanish. The notation of the preceding note will be used, mutatis mutandis (by replacing "tangent" by "tangent plane"). If $\boldsymbol{n}$ is the unit normal vector at P,[4] then the unit normal vector $\boldsymbol{n}'$ at P' will be chosen in such a way that

$$\boldsymbol{n}' = -\boldsymbol{n}. \tag{1}$$

The lines of curvature at P will be taken as the coördinate lines $u =$ const., $v =$ const., the corresponding principal radii of curvature and principal curvatures being respectively ϱ_b, ϱ_a[5] and $\varkappa_b$, $\varkappa_a$. The differentiations with respect to u, v will be designated by the subscripts 1, 2.

With this notation we have

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[3] The results of this note are mostly not new. However, the elegant method of deriving them is new and due to Mellish. The idea of applying this method to the discussion of general ovals was suggested to Mellish by Professor D. J. Struik. [J. T.]

[4] As to general facts and notation used the reader is referred to Weatherburn's Differential Geometry of three dimensions, Cambridge, 1927, especially Chapters 3–5. This treatise will be referred to by (W.)

[5] Instead of Weatherburn's β, α.

$$(2)\qquad E = \mathbf{r}_1^2 \neq 0, \quad F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0, \quad G = \mathbf{r}_2^2 \neq 0,$$

$$(3)\qquad L = \mathbf{r}_{11} \cdot \mathbf{n} \neq 0, \quad M = \mathbf{r}_{12} \cdot \mathbf{n} = 0, \quad N = \mathbf{r}_{22} \cdot \mathbf{n} \neq 0,$$

$$(4)\qquad \varkappa_a = \frac{1}{\varrho_a} = \frac{L}{E}, \quad \varkappa_b = \frac{1}{\varrho_b} = \frac{N}{G},$$

$$(5)\qquad \mathbf{n}_1 = -\varkappa_a \mathbf{r}_1, \quad \mathbf{n}_2 = -\varkappa_b \mathbf{r}_2,$$

$$(6)\qquad \begin{aligned} \mathbf{r}_{11} &= \frac{E_1}{2E}\mathbf{r}_1 - \frac{E_2}{2G}\mathbf{r}_2 + L\mathbf{n}, \\ \mathbf{r}_{12} &= \frac{E_2}{2E}\mathbf{r}_1 + \frac{G_1}{2G}\mathbf{r}_2, \\ \mathbf{r}_{22} &= -\frac{G_1}{2E}\mathbf{r}_1 + \frac{G_2}{2G}\mathbf{r}_2 + N\mathbf{n}, \end{aligned}$$

together with the corresponding equations at the point P'.

The following theorem will be important for our discussion and is interesting in itself.

THEOREM 1. *The radius of curvature R of the orthogonal section of any cylinder circumscribed about any surface is given by the formula*

$$(7)\qquad R = \varrho_a \cos^2\theta + \varrho_b \sin^2\theta,$$

where ϱ_a, ϱ_b are the principal radii of curvature at the corresponding point P of the surface and θ is the angle between the direction $v = const.$ and the generators of the cylinder.

Proof. Let $\mathbf{T}$ and $\mathbf{N}$ be the unit tangent and normal vectors of the orthogonal section (C) of the circumscribed cylinder at P, dS the element of the arc of (C), and ds_a, ds_b the elements of the arc of the lines $v =$ const., $u =$ const. Let the corresponding unit tangent vectors be $\mathbf{a} = \frac{\mathbf{r}_1}{\sqrt{E}}$, $\mathbf{b} = \frac{\mathbf{r}_2}{\sqrt{G}}$. We have then

$$(8)\qquad \mathbf{N} = \mathbf{n},$$

$$(9)\qquad \mathbf{T} = \mathbf{a}\cos\theta + \mathbf{b}\sin\theta,$$

$$(10)\qquad dS = ds_a\cos\theta + ds_b\sin\theta.$$

Differentiating (8) and denoting by $K = \frac{1}{R}$ the curvature of (C),

$$(11)\qquad -K\mathbf{T}\,dS = -\varkappa_a \mathbf{a}\,ds_a - \varkappa_b \mathbf{b}\,ds_b.$$

Comparison with (9) and (10) gives now

$$KdS = \frac{ds_a}{\varrho_a \cos\theta} = \frac{ds_b}{\varrho_b \sin\theta} = \frac{ds_a \cos\theta + ds_b \sin\theta}{\varrho_a \cos^2\theta + \varrho_b \sin^2\theta}$$
$$= \frac{dS}{\varrho_a \cos^2\theta + \varrho_b \sin^2\theta}$$

which is the desired result.[6]

2. Let us now differentiate the vectors $\boldsymbol{n}'$ and $\boldsymbol{r}'$ with respect to u, v. We get first from (1)

$$\text{(12)}\quad \begin{aligned} \frac{\partial \boldsymbol{n}'}{\partial u} &= \boldsymbol{n}_1' \frac{\partial u'}{\partial u} + \boldsymbol{n}_2' \frac{\partial v'}{\partial u} = -\varkappa_a' \boldsymbol{r}_1' \frac{\partial u'}{\partial u} - \varkappa_b' \boldsymbol{r}_2' \frac{\partial v'}{\partial u} = -\boldsymbol{n}_1 = \varkappa_a \boldsymbol{r}_1, \\ \frac{\partial \boldsymbol{r}'}{\partial v} &= \boldsymbol{n}_1' \frac{\partial u'}{\partial v} + \boldsymbol{n}_2' \frac{\partial v'}{\partial v} = -\varkappa_a' \boldsymbol{r}_1' \frac{\partial u'}{\partial v} - \varkappa_b' \boldsymbol{r}_2' \frac{\partial v'}{\partial v} = -\boldsymbol{n}_2 = \varkappa_b \boldsymbol{r}_2. \end{aligned}$$

On setting

$$\text{(13)}\qquad \boldsymbol{r}' = \boldsymbol{r} + x\boldsymbol{r}_1 + y\boldsymbol{r}_2 + z\boldsymbol{n},$$

we have, the tangent plane at P' being parallel to that at P,

$$\text{(14)}\qquad \boldsymbol{r}_1' \cdot \boldsymbol{n} = \boldsymbol{r}_2' \cdot \boldsymbol{n} = 0,$$

and, by (6),

$$\text{(15)}\quad \begin{aligned} \frac{\partial \boldsymbol{r}'}{\partial u} &= \boldsymbol{r}_1' \frac{\partial u'}{\partial u} + \boldsymbol{r}_2' \frac{\partial v'}{\partial u} \\ &= \left\{1 + x_1 + \frac{E_1 x + E_2 y}{2E} - \varkappa_a z\right\}\boldsymbol{r}_1 + \left\{\frac{-E_2 x + G_1 y}{2G} + y_1\right\}\boldsymbol{r}_2 \\ &\qquad + \{Lx + z_1\}\boldsymbol{n}, \\ \frac{\partial \boldsymbol{r}'}{\partial v} &= \boldsymbol{r}_1' \frac{\partial u'}{\partial v} + \boldsymbol{r}_2' \frac{\partial v'}{\partial v} \\ &= \left\{\frac{E_2 x - G_1 y}{2E} + x_2\right\}\boldsymbol{r}_1 + \left\{1 + y_2 + \frac{G_1 x + G_2 y}{2G} - \varkappa_b z\right\}\boldsymbol{r}_2 \\ &\qquad + \{Ny + z_2\}\boldsymbol{n}. \end{aligned}$$

Hence

$$\text{(16)}\qquad Lx + z_1 = 0, \qquad Ny + z_2 = 0.$$

We are prepared now to investigate the properties of the surfaces of constant width.

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[6] This theorem was proved first by Blaschke, Kreis und Kugel, Leipzig, 1916, pp. 117–118. See also R. Mehmke, Wien. Sitzungsberichte, 1917, pp. 1317–1321. Mellish's proof is different from that of Blaschke and was obtained independently. [J. T.]

THEOREM 2. *The principal directions at the opposite points of a surface of constant width are parallel.*

Proof. If our surface is of constant width, then

$$z = \mu = \text{const.}, \quad z_1 = 0, \quad z_2 = 0, \tag{17}$$

and, from (16), it follows at once

$$x = 0, \quad y = 0. \tag{18}$$

Formulas (15) reduce then to

$$\begin{aligned} \mathbf{r}_1' \frac{\partial u'}{\partial u} + \mathbf{r}_2' \frac{\partial v'}{\partial u} &= (1 - \varkappa_a \mu)\, \mathbf{r}_1, \\ \mathbf{r}_1' \frac{\partial u'}{\partial v} + \mathbf{r}_2' \frac{\partial v'}{\partial v} &= (1 - \varkappa_b \mu)\, \mathbf{r}_2. \end{aligned} \tag{19}$$

Since the coefficients $\frac{\partial u'}{\partial u}$, $\frac{\partial v'}{\partial u}$ in (19) cannot vanish simultaneously, the parameters u', v' can be chosen in such a way that, at P', $\frac{\partial u'}{\partial u} \neq 0$. If now $\frac{\partial v'}{\partial u} \neq 0$, on comparing formulas (12) and (19) we get

$$\varkappa_a' = \varkappa_b' = \frac{-\varkappa_a}{1 - \varkappa_a \mu}.$$

This implies that P' is an umbilic, and since the position of P' is not restricted, all the points of our surface will be umbilics, which is possible when and only when the surface is a sphere (the case of a plane obviously being excluded).[7] Thus it is seen that if our surface is not a sphere, then necessarily $\frac{\partial v'}{\partial u} = 0$, whence $\mathbf{r}_1'$ is parallel to $\mathbf{r}_1$. The second of equations (19) shows then that $\frac{\partial u'}{\partial v} = 0$ and that $\mathbf{r}_2'$ is parallel to $\mathbf{r}_2$. The case of a sphere being trivial, Theorem 2 is completely proved.

COROLLARY. *For a surface of constant width the parameters (u, v), (u', v') can be chosen in such a way that the vectors $\mathbf{r}_1'$, $\mathbf{r}_2'$ be parallel and opposite to $\mathbf{r}_1$, $\mathbf{r}_2$ (and also parallel to the principal directions at P, P'),*

$$\mathbf{r}_1 \cdot \mathbf{r}_1' = -\sqrt{EE'}, \quad \mathbf{r}_2 \cdot \mathbf{r}_2' = -\sqrt{GG'}, \quad \mathbf{r}_1 \cdot \mathbf{r}_2' = \mathbf{r}_2 \cdot \mathbf{r}_1' = 0, \tag{20}$$

while

$$\sqrt{\frac{E'}{E}} \frac{\partial u'}{\partial u} = \frac{\varrho_a'}{\varrho_a}, \quad \sqrt{\frac{G'}{G}} \frac{\partial v'}{\partial v} = \frac{\varrho_b'}{\varrho_b}, \quad \frac{\partial v'}{\partial u} = \frac{\partial u'}{\partial v} = 0. \tag{21}$$

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[7] See e. g. Blaschke, Differentialgeometrie, vol. 1, 3rd edit., Berlin 1930, p. 97. [J. T.]

3. Other important properties of surfaces of constant width are given in Theorems 3, 4 below.

THEOREM 3. *A surface of constant width μ has the following properties:*

(i) *all its normals are double;*

(ii) *the sum of principal radii of curvature that correspond to parallel principal directions at any pair of opposite points is constant,*

$$\varrho_a + \varrho_a' = \varrho_b + \varrho_b' = \mu; \tag{22}$$

(iii) *the surface is of constant diameter,*

$$|\boldsymbol{d}| = \mu. \tag{23}$$

Conversely, any surface which has property (i) *is of constant width, while if it has property* (ii) *it will be of constant width provided the principal directions at the opposite points are parallel.*

Proof. Property (i) and its converse are proved already, since relations (17) and (18) are obviously equivalent. To prove (ii) we observe that if our surface is of constant width, then equations (19), (21) show

$$-\varrho_a'/\varrho_a = 1 - \mu/\varrho_a, \quad \text{or} \quad \varrho_a + \varrho_a' = \mu,$$
$$-\varrho_b'/\varrho_b = 1 - \mu/\varrho_b, \quad \text{or} \quad \varrho_b + \varrho_b' = \mu.$$

Conversely, if (22) are satisfied, and the principal directions at opposite points are parallel, then the angle θ of Theorem 1 will be the same at the opposite points P, P', which always are on the same circumscribed cylinder. If now R and R' are the radii of curvature of the orthogonal section of the cylinder at the points P, P' respectively, then, by Theorem 1,

$$R = \varrho_a \cos^2\theta + \varrho_b \sin^2\theta, \qquad R' = \varrho_a' \cos^2\theta + \varrho_b' \sin^2\theta,$$

whence $R + R' = \mu$. Hence the projection of our surface on any plane is bounded by a curve of constant width (Theorem of the preceding note) which implies that the surface itself is of constant width.

THEOREM 4. *The surface integral of the mean curvature $J = \frac{1}{2}(\varkappa_a + \varkappa_b)$ of a surface of constant width is equal to the double perimeter of the orthogonal section of any circumscribed cylinder.*

Proof. From (21) it is seen at once

$$\frac{ds_a'}{ds_a} = \frac{\sqrt{E'}\,du'}{\sqrt{E}\,du} = \frac{\varrho_a'}{\varrho_a}, \qquad \frac{ds_b'}{ds_b} = \frac{\sqrt{G'}}{\sqrt{G}}\,\frac{dv'}{dv} = \frac{\varrho_b'}{\varrho_b},$$

so that, by (22)

$$ds_a + ds_a' = \mu\varkappa_a\,ds_a = \mu\varkappa_a'\,ds_a',$$
$$ds_b + ds_b' = \mu\varkappa_b\,ds_b = \mu\varkappa_b'\,ds_b'.$$

Hence

$$ds_a\, ds_b + ds'_a\, ds'_b = \mu^2 \varkappa_a \varkappa_b\, ds_a\, ds_b - ds_a\, ds'_b - ds_b\, ds'_a,$$

$$2\, ds_a\, ds_b = \mu(\varkappa_a + \varkappa_b)\, ds_a\, ds_b - ds_a\, ds'_b - ds_b\, ds'_a.$$

On the other hand, the area of the surface is given by

$$S = \frac{1}{2}\int_S (ds_a\, ds_b + ds'_a\, ds'_b) = \frac{1}{2}\int_S 2\, ds_a\, ds_b.$$

This gives

$$\mu^2 \int_S \varkappa_a \varkappa_b\, dS = \mu \int_S (\varkappa_a + \varkappa_b)\, dS = 2\mu \int_S J\, dS,$$

which in view of the known formula[8] $\int_S \varkappa_a \varkappa_b\, dS = 4\pi$ gives the desired result

$$\int_S J\, dS = 2\pi\mu.$$ [9]

III. On Bertrand curves.

We give here a simple proof of the following

THEOREM. *If the curve (C_1) is of constant curvature and the curve (C_2) is of constant torsion, and if the points of these curves are in such a correspondence that the tangents at the corresponding points are parallel, then the locus (C) of the points which divide in a constant ratio the segments joining the correspondent points is a Bertrand curve.*[10]

Proof. We shall use the subscripts 1,2 to designate the geometric quantities corresponding to the curves (C_1), (C_2) while the same letters without subscripts will refer to the curve (C). Then the equation of (C) is

$$(1) \qquad \boldsymbol{r} = \frac{p\boldsymbol{r}_1 + q\boldsymbol{r}_2}{p+q}, \qquad p, q \text{ constant},$$

while by hypothesis,

$$(2) \qquad \boldsymbol{t}_1 = \text{constant}, \qquad \boldsymbol{t}_2 = \text{constant}, \qquad \boldsymbol{t}_1 = \boldsymbol{t}_2.$$

On differentiating (1) we have

$$(3) \qquad \boldsymbol{t}\, ds = \frac{1}{p+q}(p\boldsymbol{t}_1\, ds_1 + q\boldsymbol{t}_2\, ds_2) = \frac{p\, ds_1 + q\, ds_2}{p+q}\, \boldsymbol{t}_1$$

[8] Blaschke, Differentialgeometrie, p. 165. [J. T.]

[9] This theorem was proved, by an entirely different method, by Minkowski, Moskow Matem. Sbornik 25 (1904), pp. 505–508; Ges. Werke, vol. 2, pp. 277–279. [J. T.]

[10] This theorem is not new, see e. g. Tajima, On Bertrand curves, Tôhoku Math. Journ. 18 (1920), pp. 128–133. Mellish was aware of this paper and simplified considerably Tajima's proof by using vector calculus notation. [J. T.]

which shows that $\boldsymbol{t}$ is parallel to $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ and always can be chosen so that

$$(4) \qquad \boldsymbol{t} = \boldsymbol{t}_1 = \boldsymbol{t}_2,$$

$$(5) \qquad (p+q)\,ds = p\,ds_1 + q\,ds_2.$$

Differentiation of (4) gives

$$(6) \qquad \varkappa\,\boldsymbol{n}\,ds = \varkappa_1\,\boldsymbol{n}_1\,ds_1 = \varkappa_2\,\boldsymbol{n}_2\,ds_2,$$

and if we assume that $\varkappa$, $\varkappa_1$, $\varkappa_2$ are positive, then

$$(7) \qquad \boldsymbol{n} = \boldsymbol{n}_1 = \boldsymbol{n}_2,$$

$$(8) \qquad \varkappa\,ds = \varkappa_1\,ds_1 = \varkappa_2\,ds_2.$$

From (4) and (7),

$$(9) \qquad \boldsymbol{b} = \boldsymbol{b}_1 = \boldsymbol{b}_2,$$

and, differentiating,

$$(10) \qquad -\tau\,\boldsymbol{n}\,ds = -\tau_2\,\boldsymbol{n}_2\,ds_2, \qquad \tau\,ds = \tau_2\,ds_2.$$

Elimination of ds, ds_1, ds_2 gives

$$\varkappa\frac{p}{\varkappa_1} + \tau\frac{q}{\tau_2} = p+q$$

which is the desired result since p, q, $\varkappa_1$, τ_2 are constant.

If instead of $\boldsymbol{t}_1 = \boldsymbol{t}_2$ we were given the condition $\boldsymbol{b}_1 = \boldsymbol{b}_2$, the same result would follow in the same manner.

ÜBER FUNKTIONEN VON FUNKTIONALOPERATOREN.*

Von J. v. Neumann, Berlin.

Einleitung.

1. Den Gegenstand der vorliegenden Arbeit bilden gewisse Sätze über die funktionelle Abhängigkeit vertauschbarer (beschränkter) Funktionaloperatoren. Dieselben hängen mit Fragen zusammen, welche bereits in einer früheren Arbeit des Verfassers angeschnitten wurden[1]), deren vollständige Diskussion dort aber (aus räumlichen Rücksichten, da die genannte Arbeit auch anderen Gegenständen gewidmet war) nicht erfolgte. Hier soll eine erschöpfende Untersuchung und Erledigung erfolgen, im Verein mit einer (dadurch nahegelegten) vollständigen Entwicklung des Funktionsbegriffes für Funktionaloperatoren. Es handelt sich um den Beweis des folgenden Satzes: Zwei (oder sogar beliebig viele) vertauschbare[2]) Hermitesche Operatoren können immer als Funktionen eines einzigen dargestellt werden. In A, a. a. O. Anm. [1]), wurde nämlich nur gezeigt, daß sie in einem gewissen Sinne Limites von Polynomen desselben sind, da der allgemeine Begriff einer Funktion eines Operators dort nicht entwickelt wurde. Der oben genannte Satz soll hier übrigens in einer noch etwas verschärften und präzisierten Form bewiesen werden, auf die wir noch zu sprechen kommen.

2. Unter den für die folgenden Betrachtungen wesentlichen Begriffsbildungen stehen an erster Stelle diejenigen des Hilbertschen Raumes und seiner linearen Operatoren. Es ist wohl nicht nötig, dieselben des Näheren zu entwickeln, aber da wir bei ihrer Untersuchung eine geometrische, besonders für unsere Zwecke zugeschnittene, Terminologie verwenden werden, sei gesagt, daß wir uns an die diesbezüglichen Ausführungen und Begriffsbildungen von E und A anschließen[3]). Wie dort, nennen wir den (abstrakten) Hilbertschen Raum $\mathfrak{H}$, den Ring seiner beschränkten Operatoren $\boldsymbol{B}$. Ferner wird von den Ringen in $\boldsymbol{B}$ immer wieder die Rede sein: das sind diejenigen Teilmengen von $\boldsymbol{B}$, welche mit allen Elementen A, B auch aA, A^*[4]), $A+B$, AB enthalten, und außerdem eine gewisse Abgeschlossenheits-Eigenschaft besitzen[5]). Jede Teilmenge $\boldsymbol{M}$ von $\boldsymbol{B}$ ist in gewissen Ringen enthalten, u. a. in einem kleinsten: das ist der von ihr erzeugte Ring $\boldsymbol{R}(\boldsymbol{M})$[6]). Besteht insbesondere $\boldsymbol{M}$ aus einem einzigen Element A, so können wir aus A den Ring $\boldsymbol{R}(A)$ erzeugen. Dieser besteht

* Received October 20, 1930.

übrigens aus allen Polynomen $p(A)$ von A ($p(x)$ durchläuft die Polynome mit $p(0)=0$), sowie den Limites aller sogenannten stark doppelkonvergenten Folgen derselben[7]) — wenigstens, wenn A Hermitesch oder zumindest normal ist[8]).

Eine Menge $\boldsymbol{M}$ (in $\boldsymbol{B}$) ist Abelsch, wenn alle ihre Elemente miteinander und mit ihren $*$ vertauschbar sind; bei Mengen, die mit A auch A^* enthalten, z. B. Ringen, genügt also die Vertauschbarkeit aller Elemente[9]). A. a. O. Anm. [9]) wurde übrigens gezeigt, daß $\boldsymbol{M}$ dann und nur dann Abelsch ist, wenn es der Ring $\boldsymbol{R}(\boldsymbol{M})$ ist, und daß ein Ring dann und nur dann Abelsch ist, wenn er aus lauter normalen Operatoren besteht.

3. In A wurde nun bewiesen[10]): wenn A alle normalen Operatoren durchläuft, so durchläuft $\boldsymbol{R}(A)$ alle Abelschen Ringe, ja es ist zulässig, A auf die Hermiteschen zu beschränken (vgl. Anm. [8])). D. h. die Elemente des Ringes sind im weiter oben skizzierten Sinne Limites von Polynomen von A. Wir wünschen sie aber direkt als Funktionen von A darzustellen. Daher werden wir in dieser Arbeit den allgemeinen Begriff einer Funktion eines Hermiteschen (oder normalen) Operators einführen (und die für diesen Zweck zulässige Funktionenklasse genau abgrenzen), und dann beweisen: $\boldsymbol{R}(A)$ ist die Menge der $f(A)$, wobei $f(x)$ alle zulässigen Funktionen mit $f(0)=0$ durchläuft — übrigens darf man $f(x)$ auf die Funktionen der ersten Baireschen Klasse beschränken.

Da jede Abelsche Menge Teil eines Abelschen Ringes ist (z. B. dessen, den sie aufspannt), sind ihre Elemente Funktionen eines A — und jede Menge von vertauschbaren Hermiteschen Operatoren ist (da in ihr $B^*=B$ gilt) nach Definition Abelsch, so daß auch für sie das obige gilt. Wir werden die verschiedenen Formulierungs- und Verschärfungsmöglichkeiten dieses Satzes weiter unten noch näher analysieren.

4. Wie aus dem bisher Gesagten hervorgeht, ist unsere Hauptaufgabe die Einführung des allgemeinen Funktionsbegriffes für Operatoren. Hierfür sind zwei Wege vorhanden: einmal kann man es durch sukzessive Grenzübergänge für die Funktionen aller Baireschen Klassen tun, andererseits aber durch eine Definition mit Lebesgue-Stieltjesschen Integralen auf einen Schlag für die (allgemeinere) volle Funktionenklasse, für welche es überhaupt möglich ist. Die beiden Methoden sind übrigens mit zwei Wegen zur Einführung des allgemeinen Integralbegriffes, denen von Young bzw. Lebesgue[11]), aufs engste verwandt. Wir werden uns der zweiten bedienen, die allgemeinere Resultate zu erzielen gestattet, obwohl auch die erste genügen würde, um den in 3. genannten Satz zu sichern.

Ehe wir übrigens an unseren eigentlichen Gegenstand herantreten, müssen wir einen Hilfssatz über Lebesguesche Integrale beweisen, der, nebst einigen Folgerungen, vielleicht auch an und für sich nicht ganz ohne

Interesse ist und, soweit es dem Verfasser bekannt ist, in der Literatur bisher nicht vorkam.

Die Einteilung der Arbeit ist die folgende: Im Kapitel I werden die soeben erwähnten Hilfssätze über Lebesguesche Integrale bewiesen und dann das Lebesgue-Stieltjessche Integral, mitsamt seinen für unsere Zwecke wesentlichen Eigenschaften, eingeführt (da hierfür ein kurzer Exkurs in aller Strenge ausreicht, und wir es, etwas über die übliche Definitionsweise hinausgehend, auch für unstetige Differentiale brauchen, soll diese Darlegung erfolgen, obwohl sie nichts wesentlich Neues bietet). Im Kapitel II definieren wir die Funktionen von Operatoren und stellen ihre Haupteigenschaften auf; im Kapitel III wird der in 3. erwähnte Satz nebst einigen Folgerungen bewiesen. Im Kapitel IV verallgemeinern wir den Funktionsbegriff (der bis dahin nur für ein Argument, das Hermitesch sein muß, vorliegt) auf endlich oder abzählbar unendlich viele Argumente, die vertauschbare normale Operatoren sein dürfen.

I. Hilfssätze über Lebesguesche und Lebesgue-Stieltjessche Integrale.

1. Sei $\varphi(a)$ eine in einem Intervalle $0 \leqq a \leqq \alpha$ (α eine beliebige Zahl $\geqq 0$) definierte, monoton nicht-fallende, nie negative, überall nach rechts halbstetige Funktion — wir sagen kurz: eine M-Funktion.

Sei $f(x)$ eine in einer meßbaren Menge $\mathfrak{M}$ von endlichem Maße definierte, meßbare, nie negative und beschränkte Funktion. Wir definieren die Maßfunktion $\varphi(a)$ von $f(x)$ folgendermaßen:

$\varphi(a) =$ Maß der Menge aller x (in $\mathfrak{M}$) mit $f(x) \leqq a$. $\varphi(a)$ ist offenbar eine M-Funktion, als α wählen wir dabei die obere Grenze des Wertevorrats von $f(x)$.

Da jede M-Funktion $\varphi(a)$ auch die soeben genannten Eigenschaften der $f(x)$ besitzt, können wir zu $\varphi(a)$ ihre Maßfunktion, die M-Funktion $\psi(b)$ bilden. Zwischen $\varphi(a)$ und $\psi(b)$ besteht offenbar die folgende (auch zur Definition verwendbare) Relation:

$\psi(b) =$ Obere Grenze der Menge aller a ($\geqq 0, \leqq \alpha$) mit $\varphi(a) \leqq b$.

Das Definitionsintervall von $\psi(b)$ ist $0 \leqq b \leqq \varphi(\alpha)$. Aus der vorangehenden Relation folgert man mühelos: Die Maßfunktion von $\psi(b)$ ist wieder $\varphi(a)$, d. h. die Beziehung zwischen $\varphi(a)$ und $\psi(b)$ ist wechselseitig. Ferner ist $\varphi(\psi(b)) = b$, wenn b nicht am linken Ende oder im Inneren eines Konstanzintervalles von ψ liegt (d. h. am unteren Ende oder im Inneren eines Intervalles, das an einer Unstetigkeitsstelle von φ durch dessen „Sprung" ausgemacht wird); und ebenso $\psi(\varphi(a)) = a$ mit der entsprechenden Ausnahme für a. Somit sind $\varphi(a)$, $\psi(b)$ invers zueinander — soweit so etwas überhaupt möglich ist.

2. Sei $f(x)$ wie am Anfang von 1., $\varphi(a)$ ihre Maßfunktion, $\psi(b)$ die Maßfunktion von $\varphi(a)$. Wenn $f(x)$ eine M-Funktion ist, so ist, wie wir sahen, $f \equiv \psi$ (und nur dann, da $\psi(b)$ ein M ist); aber auch bei beliebigem $f(x)$ besteht eine recht innige Beziehung zwischen ihm und $\psi(b)$. Es gilt nämlich, wie wir zeigen wollen:

Satz 1. *Sei $g(u)$ eine beliebige Funktion*[12]. *Dann gilt die Gleichung*

$$\int_{\mathfrak{M}} g(f(x))\,dx = \int_0^{\text{Maß von }\mathfrak{M}} g(\psi(b))\,db$$

($0 \leqq b \leqq$ Maß von $\mathfrak{M}$ ist das Definitionsintervall von $\psi(b)$, da das Maß von $\mathfrak{M}$ das Maximum von $\varphi(a)$ ist) in dem Sinne, daß die linke Seite immer sinnvoll ist, wenn es die rechte ist (dies kommt auf die Lebesgue-Summierbarkeit der betreffenden Integranden heraus) und beide einander dann gleich sind.

Beweis: Es genügt zu zeigen: wenn alle Mengen der b mit $g(\psi(b)) \leqq m$ meßbar sind, so sind es auch alle Mengen der x mit $g(f(x)) \leqq m$, und sie haben bzw. die gleichen Maße. Dies steht jedenfalls fest, wenn folgendes für alle Mengen $\mathfrak{P}$ gilt: wenn die Menge der b mit $\psi(b)$ aus $\mathfrak{P}$ meßbar ist, so ist es auch diejenige der x mit $f(x)$ aus $\mathfrak{P}$, und beide haben dasselbe Maß. (Man wähle alsdann $\mathfrak{P}$ als Menge der u mit $g(u) \leqq m$.) Nennen wir die b-Menge $\mathfrak{P}_b$, die x-Menge $\mathfrak{P}_x$. Verwenden wir nunmehr $\mathfrak{P}_b$ als Ausgangspunkt. Es hat allenfalls die folgende Eigenschaft: sei I ein Konstanzintervall von $\psi(b)$[13], dann ist I entweder Teil von $\mathfrak{P}_b$ oder fremd zu $\mathfrak{P}_b$. Die Zahl der (paarweise fremden) Konstanzintervalle von $\psi(b)$ ist endlich oder abzählbar, sie mögen $I_1, I_2, \cdots$ heißen. Wir betrachten $\mathfrak{P}_b$ nun ganz unabhängig von $\mathfrak{P}$, als beliebige Menge, nur muß jedes I_n Teil von $\mathfrak{P}_b$ oder fremd zu $\mathfrak{P}_b$ sein (und $\mathfrak{P}_b$ Teil des Intervalles 0, Maß von $\mathfrak{M}$). $\mathfrak{P}_x$ ist die Menge aller x (von $\mathfrak{M}$), für die $f(x) = \psi(b)$ für ein geeignetes b von $\mathfrak{P}_b$ ist. Es gilt also zu zeigen: wenn $\mathfrak{P}_b$ meßbar ist, so ist es auch $\mathfrak{P}_x$, und beide Mengen haben dasselbe Maß.

Wir wollen zeigen: es genügt dies für diejenigen $\mathfrak{P}_b$ zu beweisen, die Intervalle $0 \leqq b < \beta$ oder $0 \leqq b \leqq \beta$ sind (und zu den I_n im oben genannten Verhältnis stehen). Von diesen überträgt es sich nämlich zunächst auf ihre Differenzen, d. h. auf alle Intervalle (mit beliebiger Endenzurechnung, wieder das obige Verhältnis zu den I_n) — also insbesondere auf die offenen Intervalle sowie auf die I_n. Von diesen überträgt es sich wiederum auf die Summen endlich oder abzählbar vieler solcher: also erstens auf alle Summen irgendwelcher I_n, zweitens auf alle offenen Mengen (d. h. die in $0 \leqq b \leqq$ Maß von $\mathfrak{M}$ relativ offenen, natürlich mit dem obigen Verhältnis zu den I_n). Insbesondere gilt sie für das volle Intervall $0 \leqq b \leqq$ Maß von $\mathfrak{M}$.

Nun genügt es für alle unsere $\mathfrak{P}_b$ Maß $\mathfrak{P}_b \geqq$ Äußeres Maß $\mathfrak{P}_x$ zu beweisen (ohne die Meßbarkeit von $\mathfrak{P}_x$!); denn durch Betrachtung der Komplementärmenge von $\mathfrak{P}_b$ folgt (wegen des über das volle Intervall $0 \leqq b \leqq$ Maß von $\mathfrak{M}$ gesagten) Maß $\mathfrak{P}_b \leqq$ Inneres Maß $\mathfrak{P}_x$, also Maß $\mathfrak{P}_b =$ Äußeres Maß $\mathfrak{P}_x =$ Inneres Maß $\mathfrak{P}_x$.

Sei nun Q_b die Vereinigungsmenge aller zu $\mathfrak{P}_b$ fremden I_n, dann gilt Maß $Q_b =$ Maß Q_x, also genügt es, Maß $(Q_b + \mathfrak{P}_b) \geqq$ Äußeres Maß $(Q_x + \mathfrak{P}_x)$ nachzuweisen. $Q_b + \mathfrak{P}_b$ umfaßt alle I_n, wenn wir es wieder mit $\mathfrak{P}_b$ bezeichnen, so finden wir: es genügt die obige Relation für diejenigen $\mathfrak{P}_b$ zu betrachten, die alle I_n umfassen.

Da $\mathfrak{P}_b$ Teil eines offenen $\mathfrak{P}'_b$ ist, dessen Maß das seine nur um $\leqq \varepsilon$ übertrifft (dies gelingt für jedes $\varepsilon > 0$), genügt es, die alle I_n umfassenden offenen $\mathfrak{P}'_b$ zu betrachten: gilt unsere Relation für diese, so gilt sie auch für $\mathfrak{P}_b$ selbst. Für offene $\mathfrak{P}'_b$ (die obigen haben ja zu allen I_n das früher verlangte Verhältnis[14])) gilt diese Relation aber, wie wir wissen, sogar mit dem $=$-Zeichen.

Daher genügt es in der Tat, die Intervalle $\mathfrak{P}_b : 0 \leqq b < \beta$ bzw. $0 \leqq b \leqq \beta$ zu betrachten. Aus dem Verhalten von $\mathfrak{P}_b$ zu den I_n folgt nun, daß es wirklich die b-Menge zu einem $\mathfrak{P}$ ist (b gehört zu $\mathfrak{P}_b$, wenn $\psi(b)$ zu $\mathfrak{P}$ gehört), und zwar ist (wegen der Monotonie von $\psi(b)$) $\mathfrak{P}$ ein Anfangsintervall des Intervalles $0, \infty$, ebenso wie $\mathfrak{P}_b$. D. h. $\mathfrak{P}$ ist eine Menge $0 \leqq u < a$ oder $0 \leqq u \leqq a$. Da $\psi(b)$ wie $f(x)$ nie negativ sind, können wir $0 \leqq u$ weglassen; da sich unsere Behauptung ($\mathfrak{P}_b$, $\mathfrak{P}_x$ beide meßbar und vom selben Maß) von einer aufsteigenden Folge von $\mathfrak{P}$-Mengen sofort auf deren Vereinigungsmenge überträgt, genügt es, $u \leqq a$ zu betrachten. Somit ist $\mathfrak{P}_b$ durch $\psi(b) \leqq a$, $\mathfrak{P}_x$ durch $f(x) \leqq a$ definiert. Daß beide Mengen meßbar sind, ist klar, und da $\psi(b)$, $f(x)$ dieselbe Maßfunktion $\varphi(a)$ haben, haben beide dasselbe Maß: $\varphi(a)$.

Damit sind wir am Ziele.

3. Aus dem in Satz 1. formulierten Tatbestande wollen wir einige Folgerungen ziehen. Zu diesem Zweck führen wir die folgende Begriffsbildung ein: wenn die Funktion $f(x)$ (definiert für alle reellen x, mit komplexen Werten) so beschaffen ist, daß $f(\Phi(a))$ für alle monoton nichtfallenden und nach rechts halbstetigen $\Phi(a)$ eine meßbare Funktion ist, so sagen wir, es ist Φ-meßbar[15]). Offenbar sind alle Polynome Φ-meßbar; und wenn $f_1, f_2, \cdots$ eine Folge Φ-meßbarer Funktionen ist, die gegen ein f konvergieren (d. h. für jedes x gilt $f_n(x) \to f(x)$ für $n \to \infty$), so ist auch f Φ-meßbar (wegen der analogen Eigenschaft der gewöhnlichen Meßbarkeit). Somit sind alle Funktionen aller Baireschen Klassen Φ-meßbar. Ferner ist es klar, daß mit $f(x), g(x)$ auch $f(x) \pm g(x)$, $af(x)$, $f(x)g(x)$ Φ-meßbar sind (wieder wegen der analogen Eigenschaft der

gewöhnlichen Meßbarkeit). Es ist aber keineswegs trivial, daß folgendes gilt:

SATZ 2. *Wenn* $g(u)$ *Φ-meßbar ist und* $f(x)$ *meßbar bzw. Φ-meßbar (aber reell!), so ist auch* $g(f(x))$ *meßbar bzw. Φ-meßbar.*

Beweis: Es genügt die Meßbarkeit zu betrachten, der Fall der Φ-Meßbarkeit wird durch Ersetzen von $f(x)$ durch $f(\Phi(a))$ darauf zurückgeführt. Ferner dürfen wir $g(u)$ als beschränkt annehmen, da wir es sonst etwa durch Tghyp $(g(u))$ ersetzen können. Der Wertevorrat von $f(x)$ darf als im Intervalle $0 < u < 2$ gelegen vorausgesetzt werden, denn sonst ersetzen wir $f(x)$ durch $1 + \text{Tghyp}\,(f(x))$ und $g(u)$ durch $\bar{g}(u) = g$ (Arctghyp $(u-1)$) (für $0 < u < 2$, sonst etwa $\bar{g}(u) = 0$). Schließlich möge auch der Definitionsbereich von $f(x)$ $0 < x < 2$ sein, da wir sonst f(Arctghyp $(x-1)$) betrachten.

Also: $g(u)$ ist beschränkt, $f(x)$ nie negativ, beschränkt, und in einer Menge von endlichem Maße definiert. Sei nun $\Phi(a)$ die Maßfunktion der Maßfunktion von $f(x)$, dann ist $g(\Phi(a))$ meßbar, also summierbar. Nach Satz 1. ist daher auch $g(f(x))$ summierbar, also meßbar.

Eine zweite, für später wichtige Tatsache ist diese:

SATZ 3. *Sei* $F(u)$ *eine Φ-meßbare Funktion,* $f_1(x), f_2(x), \cdots$ *irgendwelche meßbare Funktionen. Dann gibt es eine Funktion* $G(u)$, *die zur dritten Baireschen Klasse gehört, und für welche die Menge* ϑ *aller* u *mit* $F(u) \neq G(u)$ *die folgende Eigenschaft hat: die Menge aller* x, *für die irgendein* $f_n(x)$ *zu* ϑ *gehört, ist eine Lebesguesche Nullmenge*[16]).

Beweis: Indem wir $f_n(x)$ durch f_n(Arctghyp x) ersetzen (die Abbildung $x \to$ Arctghyp x bildet $-1 < x < 1$ auf alle x ab, und führt dabei Lebesguesche Nullmengen in ebensolche über), erreichen wir, daß sein Definitionsbereich $-1 < x < 1$ wird; und indem wir es sodann durch $f_n\left(\frac{2}{\beta-\alpha}\left(x - \frac{\alpha+\beta}{2}\right)\right)$ ersetzen ($\alpha < \beta$, sonst beide beliebig), machen wir $\alpha < x < \beta$ zum Definitionsbereich. Dies sei daher von vornherein vorausgesetzt, und zwar habe $f_n(x)$ den Definitionsbereich $\frac{1}{n+1} < x < \frac{1}{n}$. Alle diese Funktionen $f_n(x)$ $\left(n = 1, 2, \cdots, \frac{1}{n+1} < x < \frac{1}{n}\right)$ können wir nun zu einer einzigen $\bar{f}(x)$ zusammenfassen, die dann in $0 < x < 1$ definiert ist (für $x = \frac{1}{2}, \frac{1}{3}, \cdots$ sei sie etwa $= 0$) und meßbar. Es handelt sich also darum, $G(u)$ so zu wählen, daß die x mit $\bar{f}(x)$ aus ϑ eine Lebesguesche 0-Menge bilden. Sei nun $\psi(b)$ die Maßfunktion der Maßfunktion von $\bar{f}(x)$, nach Satz 1. ist die x-Menge der $\bar{f}(x)$ aus ϑ gewiß eine Lebesguesche Nullmenge, wenn die b-Menge der $\psi(b)$ aus ϑ eine ist (man wähle $g(u) = 1$ für die u von ϑ, sonst $= 0$). Also dürfen wir $\bar{f}(x)$ durch $\psi(b)$ ersetzen.

D. h.: es genügt den Fall zu betrachten, wo nur ein einziges $f_1(x)$ auftritt, und dieses eine M-Funktion ist (vgl. Anm. [16])). Wir nennen es $\varphi(x)$ (statt $\psi(b)$).

Zunächst ist $F(u)$ Φ-meßbar, also $F(\varphi(x))$ meßbar. Auf Grund des Vitalischen Satzes (vgl. Anm. [16])) existiert also eine Funktion $\overline{G}(x)$ aus der zweiten Baireschen Klasse, so daß bis auf eine Lebesguesche Nullmenge immer $F(\varphi(x)) = \overline{G}(x)$ gilt. Hätte nun $\overline{G}(x)$ die Form $G(\varphi(x))$, mit einem $G(u)$ aus der dritten Baireschen Klasse, so wäre damit der Beweis fertig. Er ist es aber auch, wenn es uns gelingt, $\overline{G}(x)$ durch Abänderungen in einer Lebesgueschen Nullmenge in ein solches $\overline{G}(x) = G(\varphi(x))$ zu verwandeln. Dies soll daher geschehen.

Daß $\overline{\overline{G}}(x)$ die Form $G(\varphi(x))$ (für irgendein $G(u)$) hat, besagt bloß, daß es in allen Konstanzintervallen von $\varphi(x)$ konstant ist. $F(\varphi(x))$ ist so, daher ist es $\overline{G}(x)$ bis auf eine Lebesguesche Nullmenge. Indem wir also $\overline{G}(x)$ in den Konstanzintervallen von $\varphi(x)$ konstant machen, verändern wir es bloß in einer Lebesgueschen Nullmenge. So entsteht ein $\overline{\overline{G}}(x) = G(\varphi(x))$; es ist nur noch zu zeigen, daß $G(u)$ von der dritten Baireschen Klasse ist. Übrigens können wir, wenn $\psi(u)$ die Maßfunktion von $\varphi(x)$ ist, $G(u) = \overline{\overline{G}}(\psi(u))$ setzen, da $\psi(\varphi(x)) = x$ ist, falls x nicht in einem Konstanzintervall von $\varphi(x)$ liegt, und wenn es in einem solchen liegt, so liegen $\psi(\varphi(x))$ und x im nämlichen, und $\overline{\overline{G}}(x)$ ist dort konstant. Also ist $\overline{\overline{G}}(x) = G(\varphi(x))$.

Zunächst ist $\overline{\overline{G}}(x)$ von der zweiten Baireschen Klasse: denn es entsteht aus einer solchen Funktion, $\overline{G}(x)$, indem diese in endlich oder abzählbar vielen Intervallen (den Konstanzintervallen von $\varphi(x)$) abgeändert, und zwar konstant gemacht, wird. Dies ist offenbar für die zweite Bairesche Klasse allgemein richtig, wenn es bei einer endlichen Zahl von Intervallen für die erste Bairesche Klasse stimmt[17]). Eine Funktion der ersten Baireschen Klasse ist nun ein Limes einer überall konvergenten Folge stetiger Funktionen, wird sie auf die genannte Weise geändert, so ist sie Limes der Folge der ebenso abgeänderten stetigen Funktionen. Diese sind aber auch nach dieser Änderung alle überall stetig — mit Ausnahme endlich vieler Stellen (der Enden der Änderungsintervalle). Man erkennt leicht, daß die Limites solcher Funktionen auch als Limites stetiger Funktionen darstellbar sind[18]). Als Letztes müssen wir noch zeigen: da $\overline{\overline{G}}(x)$ zur zweiten Baireschen Klasse gehört, gehört $G(u) = \overline{\overline{G}}(\psi(u))$ zur dritten. Nun entsteht $\overline{\overline{G}}(x)$ durch zwei sukzessive Grenzübergänge aus gewissen stetigen Funktionen $f_{m,n}(x)$[19]), also $G(u) = \overline{\overline{G}}(\psi(u))$ ebenso aus den $f_{m,n}(\psi(u))$, welche (wie $\psi(u)$) nach rechts halbstetig sind. Da jede nach rechts halbstetige Funktion zur ersten Baireschen Klasse gehört[20]), gehört $G(u)$ zur dritten.

Damit schließen wir unsere aufs Lebesguesche Maß und Integral bezüglichen Betrachtungen ab.

4. Das Lebesgue-Stieltjessche Integral $\int f(x)\, d\,\varphi(x)$ soll nun definiert werden, und zwar zunächst für den Fall, wo $\varphi(x)$ eine M-Funktion ist. Dann ist der Definitionsbereich von $\varphi(x)$ ein Intervall $0 \leqq x \leqq \alpha$, daher sollen auch die Integrationsgrenzen diese Zahlen sein — wir halten sie zunächst fest. Die Maßfunktion von $\varphi(x)$ sei $\psi(y)$, dann definieren wir (an Lebesgue anlehnend):

$$\int f(x)\, d\,\varphi(x) = \int f(\psi(y))\, dy,$$

wo die rechte Seite ein gewöhnliches Lebesguesches Integral ist. Sie ist sicher sinnvoll, wenn $f(x)$ Φ-meßbar und beschränkt ist — darum wollen wir dies im folgenden stets über $f(x)$ voraussetzen.

Das so definierte Integral hat die folgenden Eigenschaften:

a) $\int a f(x) \cdot d\varphi(x) = a \int f(x) \cdot d\varphi(x)$ (a eine komplexe Zahl).

b) $\int (f(x) + g(x)) \cdot d\,\varphi(x) = \int f(x) \cdot d\varphi(x) + \int g(x) \cdot d\varphi(x)$.

c) $\int f(x) \cdot d\, a\, \varphi(x) = a \int f(x) \cdot d\varphi(x)$ $(a > 0)$.

d) $\int f(x) \cdot d(\varphi(x) + \chi(x)) = \int f(x) \cdot d\varphi(x) + \int f(x) \cdot d\chi(x)$.

e) Gelte $f(x) \geqq 0$ in einer Menge $\mathfrak{M}$, wobei die Menge der y für die $\psi(y)$ nicht zu $\mathfrak{M}$ gehört, eine Lebesguesche Nullmenge ist[21]). Dann ist $\int f(x) \cdot d\varphi(x) \geqq 0$.

f) Sei $f(x) = g(x)$, in einer Menge $\mathfrak{M}$ wie in e), dann ist

$$\int f(x) \cdot d\varphi(x) = \int g(x) \cdot d\psi(x).$$

g) Sei $f_n(x) \to f(x)$ für $n \to \infty$, in einer Menge $\mathfrak{M}$ wie in e), ferner seien die $f_n(x)$ gleichmäßig beschränkt. Dann gilt

$$\int f_n(x) \cdot d\varphi(x) \to \int f(x) \cdot d\varphi(x).$$

Da a), b) sofort aus den entsprechenden Eigenschaften des Lebesgueschen Integrals folgen, und auch e), f), g) durch die Definition sofort in bekannte Eigenschaften desselben übergehen, sind nur c), d) zu prüfen. Aber nach f) und nach Satz 3 genügt es, diese für $f(x)$, $g(x)$ aus der dritten Baireschen Klasse zu beweisen, und wegen g) sogar für die der nullten, d. h. für stetige $f(x)$, $g(x)$. Dann aber haben wir es offenbar mit gewöhnlichen

Riemann-Stieltjesschen Integralen zu tun $\left(\text{denn} \int f(\psi(y))\, dy\right.$ ist, da $f(\psi(y))$ nach rechts halbstetig ist, ein Riemannsches $\left.\text{Integral}\right)$, und für diese gelten c), d) bekanntlich.

Ferner erkennt man: wenn eine monoton wachsende stetige, also ein-eindeutige, Abbildung $\xi = \omega(x)$, $x = \omega^{-1}(\xi)$ von $0 \leqq x \leqq \alpha$ auf $0 \leqq \xi \leqq \beta$ gegeben ist, so ist

$$\int f(\xi)\, d\varphi(\xi) = \int f(\omega(x))\, d\varphi(\omega(x)).$$

(Daß $f(\omega(x))$ Φ-meßbar ist, folgt offenbar aus der Φ-Meßbarkeit von $f(\xi)$). Es ist nämlich, wie man sofort erkennt, $\omega^{-1}(\psi(y))$ die Maßfunktion von $\varphi(\omega(x))$ ($\psi(y)$ sei diejenige von $\varphi(x)$), also $f(\omega[\omega^{-1}(\psi(y))]) = f(\psi(y))$.

Dies ermöglicht uns nun, die Definition von $\int f(x)\, d\varphi(x)$ auf den Fall zu übertragen, wo $\varphi(x)$ für alle reellen Zahlen definiert ist, und entsprechend von $-\infty$ bis $+\infty$ integriert wird (im übrigen sei wieder $f(x)$ Φ-meßbar und beschränkt, $\varphi(x)$ monoton nichtfallend, nach rechts halbstetig, und beschränkt). Sei nämlich $\xi = \varrho(x)$, $x = \varrho^{-1}(\xi)$ eine monoton wachsende stetige, also ein-eindeutige, Abbildung von $0 < x < \alpha$ auf $-\infty < \xi < \infty$. Wir setzen dann

$$\int f(\xi)\, d\varphi(\xi) = \int f(\varrho(x))\, d\varphi(\varrho(x)).$$

Von der Wahl von $\varrho(x)$ ist die definierende rechte Seite unabhängig: um dies für zwei $\varrho_1(x)$, $\varrho_2(x)$ zu erkennen, genügt es, im obigen Resultat $\omega(x) = \varrho_2(\varrho_1^{-1}(x))$ zu setzen. Auch für diesen Integralbegriff gelten a)—g), wie man ohne weiteres sieht. (Bei e)—g) ist die Forderung über $\mathfrak{M}$ in der Formulierung in Anm. [21]) zu stellen, da wir die „Maßfunktion" für das allgemeine $\varphi(x)$ nicht definiert haben.) Ferner sind auch bei diesem $\int$ die Variablentransformationen $\xi = \varrho(x)$, $x = \varrho^{-1}(\xi)$ unwirksam, wie aus der Definition folgt, nur müssen alle x auf alle ξ abgebildet werden. Mit Hilfe von $\int$, genauer $\int_{-\infty}^{\infty}$, können wir nun auch alle $\int_{\mathfrak{M}}$ definieren [22]): wenn $f(x)$ gegeben ist, so sei $f_{\mathfrak{M}}(x) = f(x)$ in $\mathfrak{M}$ und $= 0$ in der Komplementärmenge, und

$$\int_{\mathfrak{M}} f(x) \cdot d\varphi(x) = \int f_{\mathfrak{M}}(x) \cdot d\varphi(x).$$

Aus der Gültigkeit von a) —g) für $\int$ folgt sofort diejenige für $\int_{\mathfrak{M}}$, ferner folgt aus der Definition von $\int_{\mathfrak{M}}$ und aus b), g) sofort:

h) Seien $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ paarweise elementfremde Φ-meßbare Mengen (in endlicher oder abzählbarer Zahl), dann ist ($\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$):

$$\int\limits_{\mathfrak{M}_1} f(x)\cdot d\varphi(x) + \int\limits_{\mathfrak{M}_2} f(x)\cdot d\varphi(x) + \cdots = \int\limits_{\mathfrak{M}} f(x)\cdot d\varphi(x).$$

Schließlich ist $\int\limits_{\overline{\mathfrak{M}}}$ mit dem eingangs definierten $\int$ im Intervalle $0, \alpha$ identisch, wenn $\overline{\mathfrak{M}}$ die Menge $0 < x < \alpha$ ist. Sei nämlich $f(x)$ außerhalb von $0 < x < \alpha$ als 0 definiert und $\varphi(x)$ als $\varphi(0)$ bzw. als $\varphi(\alpha)$, und betrachten wir $\int f(x-\varepsilon)\, d\varphi(x-\varepsilon)$ für $0 < x < \alpha + 2\varepsilon$ (ε irgendeine feste Zahl > 0). Sei $\psi(y)$ die Maßfunktion von $\varphi(x)$ (in $0, \alpha$), dann ist die Maßfunktion von $\varphi(x+\varepsilon)$ (in $0, \alpha+2\varepsilon$) gleich $\psi(y)+\varepsilon$ für $0 \leqq y < \varphi(\alpha)$ und $= \alpha + 2\varepsilon$ (nicht, wie $\psi(y)+\varepsilon$, $= \alpha+\varepsilon$) für $y = \varphi(\alpha)$ ($0 \leqq \alpha \leqq \varphi(\alpha)$ ist ja der Definitionsbereich). Hieraus folgt $\int\limits_{0,\,\alpha+2\varepsilon} f(x-\varepsilon)\cdot d\varphi(x-\varepsilon)$ $= \int\limits_{0,\,\alpha} f(x)\, d\varphi(x)$. Nun ist aber die linke Seite gleich $\int f(x)\, d\varphi(x)$ (von $-\infty$ bis $+\infty$), es genügt dazu das $0, \alpha+2\varepsilon$ auf $-\infty, +\infty$ abbildende $\varrho(x)$ in $\varepsilon < x < \alpha+\varepsilon$ gleich $x-\varepsilon$ zu wählen, was möglich ist. Und dieses Integral ist, da $f(x)$ außerhalb $0 < x < \alpha$ verschwindet, gleich $\int\limits_{\overline{\mathfrak{M}}} f(x)\cdot d\varphi(x)$, wenn $\overline{\mathfrak{M}}$ diese Menge ist.

Es ist noch erwähnenswert, daß die Variablentransformationsformel auf Grund der Definition die folgende Form annimmt:

$$\int\limits_{\overline{\mathfrak{M}}} f(\xi)\cdot d\varphi(\xi) = \int\limits_{\varrho^{-1}(\overline{\mathfrak{M}})} f(\varrho(x))\cdot d\varphi(\varrho(x)),$$

wo $\varrho^{-1}(\overline{\mathfrak{M}})$ das $x = \varrho^{-1}(\xi)$-Bild von $\overline{\mathfrak{M}}$ ist. Ferner können wir in e) —g) (Anm. [21]) statt der Komplementärmenge von $\mathfrak{M}$ den in $\overline{\mathfrak{M}}$ liegenden Teil derselben nehmen: denn außerhalb von $\overline{\mathfrak{M}}$ ist ja ohnehin $f_{\overline{\mathfrak{M}}}(x)$ $= g_{\overline{\mathfrak{M}}}(x) = f_{n,\overline{\mathfrak{M}}}(x) = 0$, also alles erfüllt.

5. Wir gehen nun dazu über, $\int\limits_{\overline{\mathfrak{M}}} f(x)\, d\varphi(x)$ für solche $\varphi(x)$ zu definieren, die (im ganzen Intervall $-\infty, +\infty$) von beschränkter Schwankung sind. ($f(x)$, $\overline{\mathfrak{M}}$ seien wie früher: Φ-meßbar und beschränkt, bzw. Φ-meßbar.) Ein solches $\varphi(x)$ kann bekanntlich immer auf diese Form gebracht werden:

$$\varphi(x) = \psi_1(x) - \psi_2(x) + i\,(\psi_3(x) - \psi_4(x)),$$

wo $\psi_1, \psi_2, \psi_3, \psi_4$ M-Funktionen (d. h. reell, nie negativ [$\varphi(x)$ darf komplex sein!], monoton nichtfallend und nach rechts halbstetig) sind —

d. h. diese Gleichung gilt mit Ausnahme der (endlich oder abzählbar vielen) Stellen x, wo $\varphi(x)$ nicht nach rechts halbstetig ist[23]). Wir definieren nun:

$$\int_{\mathfrak{M}} f(x)\, d\varphi(x) = \int_{\mathfrak{M}} f(x)\, d\psi_1(x) - \int_{\mathfrak{M}} f(x)\, d\psi_2(x) + \\ + i\left(\int_{\mathfrak{M}} f(x)\, d\psi_3(x) - \int_{\mathfrak{M}} f(x)\, d\psi_4(x)\right).$$

Daß dies von der speziellen Wahl von $\psi_1, \psi_2, \psi_3, \psi_4$ unabhängig ist, erkennt man, indem man beachtet, daß $\int_{\mathfrak{M}} f(x)\, d\chi_1(x) - \int_{\mathfrak{M}} f(x)\, d\chi_2(x)$ (χ_1, χ_2 M-Funktionen) nur von $\chi_1(x) - \chi_2(x)$ abhängt, was unmittelbar aus d) folgt. Daß es für M-Funktionen $\varphi(x)$ die alte Definition ist, folgt daraus, daß dann $\psi_1 \equiv \varphi$, $\psi_2 \equiv \psi_3 \equiv \psi_4 \equiv 0$ gesetzt werden kann.

a)—h) können auch auf diesen Integralbegriff übertragen werden, wie man aus der Definition sofort abliest, ebenso die Variablentransformationsformel am Ende von **4**. Immerhin ist noch hinzuzufügen: Ad c) Dies gilt für alle komplexen a. Es genügt ja, es für M-Funktionen $\varphi(x)$ zu beweisen, und bei diesen, da es für positive a ohnehin gilt (und wegen d)) für $a = \pm 1, \pm i$. Dies sind Potenzen von i, also genügt $a = i$, und das liest man an der Definition ab. Ad e)—g), die in Anm. [21]) für $\mathfrak{M}$ formulierte Bedingung. Hier sind die $\psi_1, \psi_2, \psi_3, \psi_4$ einzusetzen, und nicht etwa φ selbst! Immerhin ist zu beachten, daß bei der Wahl der $\psi_1, \psi_2, \psi_3, \psi_4$ nach Anm. [23]) diese nur dort unstetig sind, wo φ es ist. Ad e) Mit $\geqq 0$, $\leqq 0$ gilt dies nur für M-Funktionen $\varphi(x)$, mit $= 0$ aber offenbar allgemein. Wenn $f(x)$, $\varphi(x)$ reell sind, ist das Integral offenbar auch reell.

Zum Schluß beweisen wir noch die folgende wichtige Relation:

i) Seien $f(x)$, $g(x)$ Φ-meßbar und beschränkt, $\varphi(x)$ von beschränkter Schwankung. Dann ist auch $\int^{x} g(y) \cdot d\varphi(y)$ (als Funktion von x betrachtet, $\int^{x}$ sei das Integral über die Menge der $y \leqq x$) von beschränkter Schwankung, und es gilt:

$$\int f(x) \cdot d\left[\int^{x} g(y) \cdot d\varphi(y)\right] = \int f(x)\, g(x) \cdot d\varphi(x).$$

Da sich jedes $f(x)$ und jedes $g(x)$ mit den Faktoren ± 1, $\pm i$ additiv aus nichtnegativen (und dabei Φ-meßbaren und beschränkten) zusammensetzen läßt, und ebenso jedes $\varphi(x)$ aus nichtnegativen monoton nichtfallenden und beschränkten (wegen a)—d)), dürfen wir uns von vornherein

auf solche beschränken. Dann ist aber $\int^{x} g(y)\cdot d\varphi(y)$ monoton nichtfallend und beschränkt $\left(\geqq 0,\ \leqq \int g(y)\cdot d\varphi(y) \text{ nach e), h)}\right)$, also von beschränkter Schwankung — d. h. die erste Behauptung bewiesen. Es bleibt noch der Beweis der Integralformel zu führen. Nach f) und Satz 3 brauchen wir nur solche $f(x)$, $g(x)$ zu betrachten, die zur dritten Baireschen Klasse gehören, und wegen der in g) zum Ausdruck kommenden „Stetigkeit" beider Seiten in $f(x)$, können wir $f(x)$ sogar auf die nullte Bairesche Klasse beschränken: d. h. es als stetig annehmen. Aber für stetiges $f(x)$ ist die linke Seite, $\int f(x)\, d\eta(x)$ $\left(\eta(x) = \int^{x} g(y)\cdot d\varphi(y)\right)$, ein gewöhnliches Riemann-Stieltjessches Integral, also in $\eta(x)$ „stetig". Und nach g) sind sowohl $\eta(x)$, als auch die rechte Seite, $\int f(x)\, g(x)\cdot d\varphi(x)$, in $g(x)$ „stetig", so daß wir auch $g(x)$ als zur nullten Baireschen Klasse gehörig, d. h. stetig, ansehen dürfen. Dann aber haben wir es auf beiden Seiten nur noch mit Riemann-Stieltjesschen Integralen zu tun, so daß beide Seiten auch in $\varphi(x)$ „stetig" sind. Und da es sicher eine überall gegen $\varphi(x)$ konvergierende Folge überall stetiger und differentiierbarer (nichtnegativer monoton nichtfallender) Funktionen gibt, können wir $\varphi(x)$ als solche voraussetzen. Dann können wir die Riemann-Stieltjesschen Integrale als Riemannsche Integrale schreiben:

$$\int f(x)\cdot d\left[\int^{x} g(y)\cdot d\varphi(y)\right] = \int f(x)\cdot \frac{d}{dx}\left[\int^{x} g(y)\cdot \frac{d}{dy}\varphi(y)\cdot dy\right]\cdot dx$$

$$= \int f(x)\, g(x)\cdot \frac{d}{dx}\varphi(x)\cdot dx,$$

$$\int f(x)\, g(x)\cdot d\varphi(x) = \int f(x)\, g(x)\cdot \frac{d}{dx}\varphi(x)\cdot dx.$$

Aber dies setzt die Gültigkeit der behaupteten Relation in Evidenz.

II. Allgemeine Funktionen von Operatoren.

1. Sei A ein beschränkter Hermitescher Operator, somit einer aus $\boldsymbol{B}$. Wir schreiben A in der Hilbertschen Spektralform [24]):

$$(Af, g) = \int \lambda\, d(E(\lambda) f, g)$$

(λ ist die Integrationsvariable), wo $E(\lambda)$ die zu A gehörige Zerlegung der Einheit ist. (Die Bezeichnungsweise ist die in E und A, a. a. O. Anm. [24]), auseinandergesetzte. Unter $f, g, \cdots, \varphi, \psi, \cdots$ verstehen wir von nun an

stets Funktionen des Hilbertschen Raumes, allgemeine [im Reellen definierte] Funktionen nennen wir $F, G, \cdots$). Das Integral ist ein Riemann-Stieltjessches[25]), und gewiß (trotz des unbeschränkten λ!) sinnvoll, da $E(\lambda)$ (also auch $(E(\lambda)f, g)$) außerhalb eines endlichen Intervalles konstant ist[26]), so daß es genügt, über dieses zu integrieren.

Betrachten wir $E(\lambda)$ als Funktion von λ. Im Sinne der für Projektions-Operatoren definierten Relation $\leqq$ ist es monoton nichtfallend, nach rechts halbstetig[27]), und unterhalb einer geeigneten Schranke $= 0$, oberhalb einer geeigneten Schranke $= 1$. Die Vereinigungsmenge seiner (offen genommenen) Konstanzintervalle heiße $\mathfrak{T}$, die Komplementärmenge von $\mathfrak{T}$ heiße $\mathfrak{S}$. $\mathfrak{T}$ ist offen, $\mathfrak{S}$ abgeschlossen, $\mathfrak{S}$ ist nach dem vorhin Gesagten beschränkt. Für alle f, g ist $(E(\lambda)f, g)$ in den Konstanzintervallen von $E(\lambda)$ konstant, d. h. $\mathfrak{T}$ durch die Konstanzintervalle dieser Funktion überdeckt. $\mathfrak{S}$ ist bekanntlich das „Spektrum“ von A. Ferner sei $\mathfrak{P}$ die Menge aller Unstetigkeitsstellen von $E(\lambda)$. Nur an diesen kann ein $(E(\lambda)f, g)$ unstetig sein (vgl. Anm. [27])). $\mathfrak{P}$ ist natürlich Teilmenge von $\mathfrak{S}$ und bekanntlich endlich oder abzählbar, es ist das „Punktspektrum“ von A.

Sei nun $F(x)$ eine beschränkte Φ-meßbare Funktion. Wir betrachten die Ausdrücke

$$\int F(\lambda)\, d(E(\lambda)f, g).$$

Da $\mathfrak{T}$ aus lauter Konstanzintervallen der Funktion hinter dem d-Zeichen besteht, sind diese Ausdrücke alle vom Verhalten von $F(x)$ in $\mathfrak{T}$ unabhängig (I. 4. und 5., Eigenschaft f), zusammen mit Anm. [21])). Daher können wir z. B. $F(x)$ in $\mathfrak{T} = 0$ setzen (da $\mathfrak{T}$ offen ist, bleibt $F(x)$ dabei Φ-meßbar), oder, was dasselbe ist, $\int$ durch $\int_{\mathfrak{S}}$ ersetzen. Infolgedessen braucht $F(x)$ gar nicht beschränkt zu sein es genügt, wenn es in $\mathfrak{S}$ beschränkt ist[28]).

Es gilt allgemein die folgende Abschätzung

$$\left| \int F(\lambda)\, d(E(\lambda)f, g) \right| \leqq \sqrt{\int |F(\lambda)| \cdot d(|E(\lambda)f|^2) \cdot \int |F(\lambda)| \cdot d(|E(\lambda)g|^2)}.$$

Zunächst genügt es, dieselbe für die $F(\lambda)$ der dritten Baireschen Klasse zu beweisen (Satz 3 und f)), und infolgedessen sogar für diejenigen der nullten (g)), d. h. für die stetigen. Dann handelt es sich um Riemann-Stieltjessche Integrale, die somit bzw. Limites von

$$\sum_1^N {}^n F(\lambda_n) \{(E(\lambda_n)f, g) - (E(\lambda_{n-1})f, g)\}$$
$$= \sum_1^N {}^n F(x_n) (\{E(\lambda_n) - E(\lambda_{n-1})\} f, g).$$

$$\sum_1^N {}^n \,|\,F(\lambda_n)\,|\,\{|\,E(\lambda_n)f\,|^2 - |\,E(\lambda_{n-1})f\,|^2\}$$

$$= \sum_1^N {}^n \,|\,F(x_n)\,|\,|\{E(\lambda_n) - E(\lambda_{n-1})\}f\,|^2 \ {}^{29)},$$

$$\sum_1^N {}^n \,|\,F(\lambda_n)\,|\,\{|\,E(\lambda_n)\,g\,|^2 - |\,E(\lambda_{n-1})\,g\,|^2\}$$

$$= \sum_1^N {}^n \,|\,F(x_n)\,|\,|\{E(\lambda_n) - E(\lambda_{n-1})\}\,g\,|^2 \ {}^{29)}$$

$(\lambda_0 < \lambda_1 < \cdots < \lambda_N, \quad \lambda_0 \to -\infty, \quad \lambda_N \to +\infty, \quad \underset{n=1,\cdots,N}{\mathrm{Max}} (\lambda_n - \lambda_{n-1}) \to 0)$ sind. Wegen

$$|\{E(\lambda_n) - E(\lambda_{n-1})\}f, g\,| \leq |\{E(\lambda_n) - E(\lambda_{n-1})\}f\,| \cdot |\{E(\lambda_n) - E(\lambda_{n-1})\}\,g\,| \ {}^{30)},$$

und der Schwarzschen Ungleichheit

$$\left|\sum_1^N {}^n \, a_n\, b_n\right| \leq \sqrt{\sum_1^N {}^n \,|\,a_n\,|^2 \ \sum_1^N {}^n \,|\,b_n\,|^2}$$

ist aber der erste Ausdruck absolut genommen $\leq$ als die Quadratwurzel des Produkts der beiden letzten, und daraus folgt die Behauptung.

Gilt nun allgemein $|\,F(x)\,| \leq C$ (wir wissen übrigens, daß es genügt, wenn dies in $\mathfrak{S}$ gilt), so ist

$$\int |\,F(x)\,| \cdot d\,|\,E(\lambda)f\,|^2 \leq \int C \cdot d\,|\,E(\lambda)f\,|^2 = C \cdot (|\,1f\,|^2 - |\,0f\,|^2) = C \cdot |\,f\,|^2,$$

$$\int |\,F(x)\,| \cdot d\,|\,E(\lambda)\,g\,|^2 \leq \int C \cdot d\,|\,E(\lambda)\,g\,|^2 = C \cdot (|\,1g\,|^2 - |\,0g\,|^2) = C \cdot |\,g\,|^2,$$

also

$$\left|\int F(x) \cdot d\,(E(\lambda)f, g)\right| \leq C \cdot |\,f\,|\,|\,g\,|.$$

Wenn wir bei festgehaltenem f $\int F(x) \cdot d\,(E(\lambda)f, g) = L(g)$ als Funktion von g ansehen, so haben wir darum:

$$L(a_1 g_1 + \cdots + a_n g_n) = \bar{a}_1 L(g_1) + \cdots + \bar{a}_n L(g_n), \quad |\,L(g)| \leq D \cdot |\,g\,|$$
$$(D = C \cdot |\,f\,|).$$

Nach einem Satze von F. Rieß[31]) existiert daher ein und nur ein f^*, so daß stets $(f^*, g) = L(g)$ gilt. Wir nennen es $A'f$ (es hängt ja von f ab), so daß

$$(A'f, g) = \int F(\lambda) \cdot d\,(E(\lambda)f, g)$$

definitorisch ist.

Wie man sofort erkennt, ist A' ein linearer Operator, und $|(A'f, g)| \leqq C \cdot |f|\,|g|$ hat die Folge $|A'f| \leqq C \cdot |f|$ (man setze $g = A'f$), so daß er auch beschränkt ist — d. h. A' gehört zu $\boldsymbol{B}$. Diesen Operator A' wollen wir nun $F(A)$ nennen.

2. Wir zählen die einfachsten Eigenschaften von $F(A)$ auf:

a) Für $F(x) \equiv 0$ bzw. $\equiv 1$ bzw. $\equiv x$ ist $F(A) = 0$ bzw. $= 1$ bzw. $= A$.

b) Für $F(x) \equiv \overline{G(x)}$ (konjugiert-komplex) ist $F(A) = G(A)^*$ [32]).

c) Für $F(x) \equiv aG(x)$ ist $F(A) = aG(A)$.

d) Für $F(x) \equiv G(x) + H(x)$ ist $F(A) = G(A) + H(A)$.

e) Für $F(x) \equiv G(x) \cdot H(x)$ ist $F(A) = G(A) \cdot H(A)$.

f) Für $F(x) \equiv G(H(x))$ ist $F(A) = G(H(A))$ [33]).

g) Die Gültigkeit von $F(x) = G(x)$ in einer Menge $\mathfrak{M}$ zieht $F(A) = G(A)$ nach sich, wenn $\mathfrak{M}$ die folgende Eigenschaft hat: Sei $\xi = \varrho(x)$, $x = \varrho^{-1}(\xi)$ eine monoton wachsende stetige, d. h. ein-eindeutige, Abbildung eines Intervalles $0 < x < \alpha$ auf $-\infty < \xi < \infty$. Wie $\varrho(x)$ gewählt wird, ist gleichgültig, denn das gleich zu definierende $\varrho(\Psi_f(y))$ hängt, wie man leicht erkennt, nur scheinbar davon ab. $\varrho(\Psi_f(y))$ ist nämlich im am Ende von I., 1. skizzierten Sinne aufzufassende Inverse von $|E(\xi)f|^2$. Wir bilden $\Phi_f(x) = |E(\varrho(x))f|^2$ und seine Maßfunktion $\Psi_f(y)$, ferner $\varrho(\Psi_f(y))$. Für jedes f muß nun die Menge aller y, für die $\varrho(\Psi_f(y))$ nicht zu $\mathfrak{M}$ gehört, eine Lebesguesche Nullmenge sein [34]).

h) Gelte $F_n(x) \to G(x)$ für $n \to \infty$, in einer Menge $\mathfrak{M}$ wie in g), ferner seien die $F_n(x)$ (wenigstens in einer Menge $\mathfrak{M}$ wie vorhin) gleichmäßig beschränkt. Dann gilt $F_n(A) \to G(A)$, und zwar im Sinne der „starken Doppelkonvergenz“ in $\boldsymbol{B}$ [35]), d. h. es ist $F_n(A)f \to G(A)f$, $F_n(A)^*f \to G(A)^*f$ („stark“ im Hilbertschen Raume).

Die Behauptungen a)—h) gehen dahin, daß sich eine Reihe von Eigenschaften von den reellen Zahlen x (für die allein unsere Funktionen unmittelbar definiert sind), auf alle Hermiteschen beschränkten Operatoren übertragen. Wir wollen sie nun beweisen.

a)—d) und g) folgen sofort aus der Definition der $F(A), \cdots$ (d. h. der $(F(A)f, g), \cdots$). Dabei ist bei g) zu beachten, daß die dortige Fassung der Bedingung für $\mathfrak{M}$ darum die richtige ist, weil die in den betreffenden Definitionen hinter dem d-Zeichen stehenden Funktionen von beschränkter Schwankung, $(E(\lambda)f, g)$, als Linearaggregate von vier $|E(\lambda)h|^2$ $\left(\text{mit } h = \dfrac{f \pm g}{2}, \dfrac{f \pm ig}{2}\right.$ und den bzw. Koeffizienten $\left.\pm 1, \pm i\right)$ aufzufassen sind (vgl. Anm. [25])). Und da die Integrale über das Intervall $-\infty < \lambda < \infty$ erstreckt sind, müssen sie dann noch durch $\lambda = \varrho(x)$, $x = \varrho^{-1}(\lambda)$ auf ein Intervall $0 < x < \alpha$ reduziert werden (vgl. I., 4.), was genau zu unserer $\mathfrak{M}$-Bedingung führt.

Noch eines ist erwähnenswert: Da im zu beweisenden $(F(A)f, g) = (G(A)f, g)$ beide Seiten in f stetig sind, genügt es, dies für f, g aus einer überall dichten Menge zu beweisen, welche (da der Hilbertsche Raum so parabel ist[36]) abzählbar gewählt werden kann. Dann bilden auch die für h (in g) heißt es f) in Frage kommenden Werte $\left(\text{die } \frac{f \pm g}{2}, \frac{f \pm ig}{2}\right)$ eine abzählbare Menge. Also: es genügt, die Forderungen für $\mathfrak{M}$ in g) für eine geeignete abzählbare Menge von f zu stellen.

Ebenso von selbst ergibt sich die Richtigkeit von h), wenn darin $F_n(A) \to F(A)$ bloß im Sinne der „schwachen" Konvergenz in $\boldsymbol{B}$ ausgesprochen wird (vgl. a. a. O. Anm. [35]), d. h. wenn nur $(F_n(A)f, g) \to (F(A)f, g)$ für $n \to \infty$ zu beweisen ist. Um auch die vorige Zusatzbemerkung über g) zu übertragen (d. i. die Hinreichendheit des Betrachtens einer geeigneten abzählbaren Menge von f in der Bedingung für $\mathfrak{M}$), genügt es, sich zu vergegenwärtigen, daß die $F_n(x)$ $(n = 1, 2, \cdots)$ gleichmäßig beschränkte Funktionen von x sind, also die $F_n(A)$ gleichmäßig beschränkte Operatoren (vgl. 1.), also die $(F_n(A)f, g)$ gleichmäßig stetige Funktionen von f, g.

In e) ist für alle f, g die Gleichheit von

$$(F(A)f, g) = \int F(\lambda)\, d(E(\lambda)f, g) = \int G(\lambda)\, H(\lambda)\, d(E(\lambda)f, g)$$

und von

$$(G(A)\, H(A)f, g) = \big(H(A)f, (G(A))^* g\big) = \int H(\lambda)\, d\big(E(\lambda)f, (G(A))^* g\big)$$
$$= \int H(\lambda)\, d(G(A)\, E(\lambda)f, g)$$

zu beweisen. Nun ist

$$(G(A)\, E(\lambda)f, g) = \int G(\lambda')\, d(E(\lambda')\, E(\lambda)f, g)$$
$$= \int G(\lambda')\, d\big(E(\text{Min}(\lambda', \lambda)f, g)\big).$$

Der Ausdruck hinter dem d-Zeichen ist für $\lambda' \geqq \lambda$ konstant $(\text{Min}(\lambda', \lambda) = \lambda$, λ ist fest, λ' die Variable!), so daß wir das Integral durch $\int^{\lambda}$ ersetzen dürfen (vgl. I. 4., e) und Anm. [21]), und in diesem Integrationsbereich ist $(\lambda' \leqq \lambda)$, ist $\text{Min}(\lambda', \lambda) = \lambda'$, also gilt

$$(G(A)\, E(\lambda)f, g) = \int^{\lambda} G(\lambda')\, d(E(\lambda')f, g).$$

Somit ist

$$\int H(\lambda)\, G(\lambda) \cdot d(E(\lambda)f, g) = \int H(\lambda) \cdot d\left[\int^{\lambda} G(\lambda') \cdot d(E(\lambda')f, g)\right]$$

zu beweisen, und das ist gerade die Aussage von I. 4., i).

Nun können wir auch h) voll beweisen. Betrachten wir nämlich neben der Folge $F_n(x)$ und $F(x)$ auch die Folge $\overline{F_n(x)}\, F_n(x)$ und $\overline{F(x)}\, F(x)$, auf beide ist h) im bisher bewiesenen Umfange anwendbar, d. h. $F_n(A)$ strebt „schwach" (in $\boldsymbol{B}$) gegen $F(A)$, und $F_n(A)^* F_n(A)$ ebenso gegen $F(A)^* F(A)$ (hier sind die soeben bewiesenen b), e) heranzuziehen!). D. h. es ist für alle f, g $(F_n(A)f, g) \to (F(A)f, g)$ und $(F_n(A)^* F_n(A)f, g) \to$ $\to (F(A)^* F(A)f, g)$, $(F_n(A)f, F_n(A)g) \to (F(A)f, F(A)g)$. Letzteres besagt für $f = g$ $|F_n(A)f| \to |F(A)f|$. Hieraus folgt aber bekanntlich $F_n(A)f \to F(A)f$ im Sinne der „starken" Topologie (des Hilbertschen Raumes)[37]). Die „Doppelkonvergenz" ergibt sich schließlich, wenn wir die $\overline{F_n(x)}$ und $\overline{F(x)}$ betrachten.

Es bleibt noch übrig, f) zu beweisen. Nach g) können wir die darin vorkommenden $G(x)$, $H(x)$ (und mit ihnen $F(x) = G(H(x))$) auf gewissen Mengen abändern, ohne daß das Resultat beeinflußt wird, falls diese Menge die folgenden Eigenschaften haben: $H(x)$ werde nicht geändert, $G(x)$ auf einer Menge ϑ. Damit $G(H(A))$ invariant bleibe, muß für jedes f die Menge der y, für die $\varrho(\psi_f(y))$ (man bilde es nach g) zu $H(A)$) zu ϑ gehört, eine Lebesguesche Nullmenge sein. Damit $F(A)$ invariant bleibe, muß für jedes f die Menge der y, für die $\varrho(\psi'_f(y))$ (man bilde es nach g) zu A) zur Änderungsmenge von $F(x)$, d. h. $H(\psi'_f(y))$ zu ϑ, gehört, eine Lebesguesche Nullmenge sein. Auf Grund der Bemerkung nach dem Beweise von g) genügt es dabei, jedesmal eine geeignete abzählbare Menge von f zu berücksichtigen. Daher ist Satz 3 anwendbar: wir dürfen $G(x)$ als zur dritten Baireschen Klasse gehörend annehmen, und wegen h) sogar als zur nullten gehörend — die ihrerseits als aus allen Polynomen bestehend angenommen werden kann. Also sei $G(x)$ ein Polynom.

Nun ist für $F(x) = a_0(H(x))^n + a_1(H(x))^{n-1} + \cdots + a_n$ nach a)—d)

$$F(A) = a_0(H(A))^n + a_1(H(A))^{n-1} + \cdots + a_n 1.$$

Und wenn wir $H(x) = x$ setzen, für $G(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$

$$G(B) = a_0 B^n + a_1 B^{n-1} + \cdots + a_n 1 \quad \text{(vgl. a)).}$$

Daher gilt f) für alle Polynome $G(x)$.

3. Wir sind nun in der Lage, den folgenden Satz über die Konvergenz von Operatoren-Funktionen-Folgen zu beweisen:

SATZ 4. *Sei $F_1(x)$, $F_2(x)$, $\cdots$ eine Folge Φ-meßbarer Funktionen und A ein Hermitescher Operator. Dann sind die folgenden Aussagen gleichbedeutend:*

α) Für jedes f von $\mathfrak{H}$ konvergieren die Punkte $F_1(A)f$, $F_2(A)f$, $\cdots$ von $\mathfrak{H}$ gegen einen Limes, und zwar im Sinne der „starken" Topologie von $\mathfrak{H}$ (vgl. a. a. O. Anm. [35])).

β) *α) gilt für die* $F_1(A)f$, $F_2(A)f, \cdots$, *und für die* $F_1(A)^*f$, $F_2(A)^*f, \cdots$.

γ) Alle $F_1(x)$, $F_2(x), \cdots$ *sind in einer Menge* $\mathfrak{M}$, *die die in* 2., g) *angegebene Eigenschaft hat, gleichmäßig beschränkt. Wenn* $m_1, m_2, \cdots$ *und* $n_1, n_2, \cdots$ *zwei gegen Unendlich strebende Folgen ganzer Zahlen sind und* f *ein beliebiges Element von* $\mathfrak{H}$, *so hat die Folge* $F_{m_1}(x) - F_{n_1}(x)$, $F_{m_2}(x) - F_{n_2}(x), \cdots$ *eine Teilfolge* $F_{m_{p_1}}(x) - F_{n_{p_1}}(x)$, $F_{m_{p_2}}(x) - F_{n_{p_2}}(x), \cdots$ $(p_1 < p_2 < \cdots)$, *für welche die Menge* $\mathfrak{M}$ *aller* x *für die sie gegen* 0 *konvergiert, für dieses* f *die in* 2., g) *angegebene Eigenschaft hat.*

Beweis: *β)* entsteht aus *α)* durch Hinzufügen derselben Forderung für $F_1(A)^*$, $F_2(A)^*, \cdots$, d. h. für $\overline{F_1(x)}$, $\overline{F_2(x)}, \cdots$. Da hierfür *γ)* ebenso lautet, genügt es *α)* und *γ)* zu vergleichen.

Betrachten wir zunächst den ersten Fall, wo *γ)* zutrifft. Wäre *α)* falsch, so gäbe es ein f, für welches die $F_n(A)f$ keinen starken Limes (in $\mathfrak{H}$) haben, d. h. (wegen der topologischen Vollständigkeit von $\mathfrak{H}$) das Cauchysche Konvergenzkriterium unerfüllt ist. Also: es gibt ein $\varepsilon > 0$, so daß für beliebig große m geeignete $n \geqq m$ existieren, für welche

$$|(F_m(A) - F_n(A))f| = |F_m(A)f - F_n(A)f| \geqq \varepsilon$$

ist. Daher lassen sich zwei gegen Unendlich strebende Folgen $m_1, m_2, \cdots$ und $n_1, n_2, \cdots$ finden, so daß stets $|(F_{m_\nu}(A) - F_{n_\nu}(A))f| \geqq \varepsilon$ $(\nu = 1, 2, \cdots)$ gilt. Insbesondere kann für die $p_1, p_2, \cdots$ aus *γ)* nicht $(F_{m_\nu}(A) - F_{n_\nu}(A))f \to 0$ sein, obwohl dies wegen 2., h) der Fall sein sollte[38]).

Betrachten wir nun den zweiten Fall, wo *α)* zutrifft. Die erste Hälfte von *γ)* beweisen wir so: Da die Folge $F_n(A)f$ für jedes f konvergiert, sind die Operatoren $F_n(A)$ gleichmäßig beschränkt (vgl. A, Seite 382, der dort für stark konvergente $F_n(A)$ skizzierte, Banach-Saksssche, Beweis gilt ungeändert auch hier), d. h. es gibt eine Zahl C, so daß für alle f und n $|F_n(A)f| \leqq C \cdot |f|$ gilt. Wenn wir also beweisen können, daß allgemein dies gilt: wenn für alle f $|G(A)f| \leqq C \cdot |f|$ ist, so gilt $|G(x)| \leqq C$ in einer Menge $\mathfrak{M}$ mit der in 2., g) angegebenen Eigenschaft — dann sind wir am Ziele (da die Durchschnittsmenge der zu $F_1(x)$, $F_2(x), \cdots$ derart gebildeten Mengen $\mathfrak{M}$ offenbar wieder die Eigenschaft aus 2., g) hat).

Für reelles $G(x)$, also Hermitesches $G(A)$, folgt aus $|G(A)f| \leqq C \cdot |f|$ (für alle f) nach bekannten Sätzen von Hilbert, daß das Spektrum von $G(A)$ ganz im Intervalle $-C \leqq x \leqq C$ liegt. Wenn also eine Funktion $H(x)$ in diesem Intervalle überall verschwindet, so ist nach II., 2., g) und Anm. [34]) $H(G(A)) = 0$, und daraus folgt durch Anwenden der Schlußbemerkung von II. auf $H(G(x))$, daß $H(G(x)) = 0$ in einer Menge $\mathfrak{M}$ vom erwähnten Charakter gilt. Setzen wir nun

$$H(x) = \begin{cases} 0 \text{ für } -C \leqq x \leqq C, \\ 1 \text{ sonst}, \end{cases}$$

so ist damit unsere Behauptung für reelle $G(x)$ bewiesen. Bei komplexem $G(x)$ haben wir $|G(A)f| \leqq C \cdot |f|$, für $H(x) = \overline{G(x)}$ $|H(A)f| = |G(A)^* f| \leqq C \cdot |f|$ (vgl. E., Anhang II., Seite 112), also für $K(x) = \Re\, G(x) = \frac{1}{2}(G(x) + H(x))$ $|K(A)f| \leqq C \cdot |f|$. Somit gilt $|\Re\, G(x)| \leqq C$ in einer $\mathfrak{M}$-Menge. Betrachten wir statt $G(x)$ $\theta G(x)$ (θ konstant, $|\theta| = 1$), so ergibt sich ebenso: $|\Re(\theta G(x))| \leqq C$ in einer $\mathfrak{M}$-Menge. Wählen wir eine auf $|\theta| = 1$ überall dichte θ-Folge, so gilt obiges sogar für alle θ dieser Folge gleichzeitig im Durchschnitt einer $\mathfrak{M}$-Mengenfolge, also in einer $\mathfrak{M}$-Menge. Wo aber dies gilt, muß $|\Re(\theta G(x))| \leqq C$ für alle θ mit $|\theta| = 1$ gelten — und dann ist $|G(x)| \leqq C$.

Es bleibt übrig, die zweite Hälfte von γ) zu beweisen. Wäre sie falsch, so gäbe es zwei Folgen $m_1, m_2, \cdots$ und $n_1, n_2, \cdots$ sowie ein f, so daß für keine Teilfolge der Folge $F_{m_1}(x) - F_{n_1}(x)$, $F_{m_2}(x) - F_{n_2}(x), \cdots$ die Konvergenz gegen 0 in einer solchen Menge stattfindet, der alle $\varrho(\Psi_f(y))$ angehören, bis auf eine Lebesguesche Nullmenge der y.

Nehmen wir an, daß für jedes $\varepsilon > 0$, $\eta > 0$ ein $\nu_0 = \nu_0(\varepsilon, \eta)$ existiert, so daß für jedes feste $\nu \geqq \nu_0$ die Menge $\mathfrak{M}_\nu(\varepsilon)$ aller x, für die $|F_{m_\nu}(x) - F_{n_\nu}(x)| \geqq \varepsilon$ ist, so beschaffen ist, daß alle $\varrho(\Psi_f(y))$ zu ihr gehören, bis auf eine y-Menge von Lebesgueschem Maße $\leqq \eta$. Dann sei $p_1 \geqq \nu_0(\frac{1}{2}, \frac{1}{2})$, $p_2 \geqq \nu_0(\frac{1}{4}, \frac{1}{4})$ und $> p_1$, $p_3 \geqq \nu_0(\frac{1}{8}, \frac{1}{8})$ und $> p_2, \cdots$. Das Zutreffen von $|F_{m_{p_\nu}}(x) - F_{n_{p_\nu}}(x)| < \frac{1}{2^\nu}$ bei allen $\nu \geqq \mu$ erfolgt für alle x der Durchschnittsmenge

$$\mathfrak{M}_{p_\mu}\left(\frac{1}{2^\mu}\right) \cdot \mathfrak{M}_{p_{\mu+1}}\left(\frac{1}{2^{\mu+1}}\right) \cdot \ldots,$$

welcher alle $\varrho(\Psi_f(y))$ angehören, bis auf eine y-Menge von Lebesgueschem Maße $\leqq \frac{1}{2^\mu} + \frac{1}{2^{\mu+1}} + \cdots = \frac{1}{2^{\mu-1}}$. Für diese x konvergiert aber die Folge $F_{m_{p_\nu}}(x) - F_{n_{p_\nu}}(x)$ offenbar gegen 0; nennen wir die Konvergenzmenge der x $\mathfrak{M}$, so sehen wir: alle $\varrho(\Psi_f(y))$ gehören zu $\mathfrak{M}$, bis auf eine y-Menge von Lebesgueschem Maße $\leqq \frac{1}{2^{\mu-1}}$. Dies gilt für alle μ, also hat diese y-Menge das Lebesguesche Maß 0, entgegen dem früher Festgestellten. Daher existieren zwei $\varepsilon > 0$, $\eta > 0$, so daß für unendlich viele ν die Menge $\mathfrak{M}_\nu(\varepsilon)$ die $\varrho(\Psi_f(y))$ nur bis auf eine y-Menge von Lebesgueschem Maße $\geqq \eta$ enthält.

Für diese ν gilt aber:

$$\begin{aligned} &|(F_{m_\nu}(A) - F_{n_\nu}(A))f|^2 \\ &\quad = \left((F_{m_\nu}(A) - F_{n_\nu}(A))^2 f, f\right) \\ &\quad = \int_{-\infty}^{\infty} (F_{m_\nu}(\xi) - F_{n_\nu}(\xi))^2\, d|E(\xi)f|^2 \end{aligned}$$

$$= \int_0^{\alpha} \left(F_{m_\nu}(\varrho(x)) - F_{n_\nu}(\varrho(x))\right)^2 d\,|E(\varrho(x))f|^2$$

$$= \int_0^{|f|^2} \left(F_{m_\nu}(\varrho(\psi_f(y))) - F_{n_\nu}(\varrho(\psi_f(y)))\right)^2 dy$$

$$\geqq \int_{\varrho(\psi_f(y)) \text{ in } \mathfrak{M}_\nu(\varepsilon)} \left(F_{m_\nu}(\varrho(\psi_f(y))) - F_{n_\nu}(\varrho(\psi_f(y)))\right)^2 dy$$

$$\geqq \int_{\varrho(\psi_f(y)) \text{ in } \mathfrak{M}_\nu(\varepsilon)} \varepsilon^2 \cdot dy \geqq \varepsilon^2 \eta .$$

Daher ist

$$\limsup_{\nu \to \infty} |F_{m_\nu}(A) f - F_{n_\nu}(A) f| \geqq \sqrt{\varepsilon^2 \eta} > 0,$$

was mit der Konvergenz der $F_n(A)f$, d. h. α) unvereinbar ist. —

Zu diesem Satze bemerken wir noch: α) bzw. β) besagen, daß für jedes f $F_n(A)f$ einen starken Limes (in $\mathfrak{H}$) hat, bzw. $F_n(A)f$ und $F_n(A)^* f$ je einen solchen haben. In die Aussage von γ) geht dieses f auch ein. Satz 4 sagte nun aus, daß die Gültigkeit von α) für alle f der von β) für alle f und der von γ) für alle f gleichwertig ist — der Beweis zeigt aber, daß α), β) auch dann Folgen von γ) sind, wenn wir sie alle nur für ein bestimmtes f nehmen. Die gleichmäßige Beschränktheit der $F_n(x)$ in einer Menge $\mathfrak{M}$ mit der Eigenschaft aus 2., g) für alle f sei hier durchweg vorausgesetzt.) Da nun die $F_n(A)$, $F_n(A)^*$ gleichmäßig beschränkt sind (weil es die $F_n(x)$ im wesentlichen sind), ist die f-Menge in $\mathfrak{H}$, für die α) bzw. β) gilt, eine abgeschlossene lineare Mannigfaltigkeit — daher gelten sie überall, wenn sie etwa für ein vollständiges normiertes Orthogonalsystem gelten. Wegen der Gleichwertigkeit gilt dies auch für γ), d. h. γ) wird nicht abgeschwächt, wenn seine zweite Hälfte anstatt für alle f nur für die (abzählbar vielen) Elemente irgendeines vollständigen Orthogonalsystems gilt.

Wir beweisen nun weiter:

SATZ 5. *Seien die $F_n(x)$ und A wie in Satz 4. Wenn die dort erwähnte Konvergenz stattfindet, so gibt es einen Operator B aus* **B**, *derart, daß die $F_n(A)f$ gegen Bf und die $F_n(A)^* f$ gegen $B^* f$ konvergieren — d. h. B ist starker Doppellimes der $F_n(A)$ (vgl. a. a. O. Anm.[35])). Dieses B ist wieder von der Form $F(A)$, wobei $F(x)$ auch beschränkt und Φ-meßbar ist.*

Damit übrigens $F(A)$ starker (bzw. starker Doppel-) Limes der $F_n(A)$ sei, ist folgendes notwendig und hinreichend: Alle $F_n(x)$ sind in einer Menge $\mathfrak{M}$, die die in 2., g) angegebene Eigenschaft hat, gleichmäßig beschränkt. Wenn $n_1, n_2, \cdots$ eine gegen Unendlich strebende Folge ist, und f ein beliebiges Element von $\mathfrak{H}$, so hat die Folge $F_{n_1}(x), F_{n_2}(x), \cdots$ eine Teilfolge $F_{n_{\nu_1}}(x)$,

$F_{n_{p_2}}(x), \cdots (p_1 < p_2 < \cdots)$, *für welche die Menge* $\mathfrak{M}$ *aller* x, *für die sie gegen* $F(x)$ *konvergiert, für dieses* f *die in* 2., g) *angegebene Eigenschaft hat.*

Beweis: Die zweite Hälfte folgt sofort aus Satz 4, wenn wir die dortige Folge $F_1(x), F_2(x), \cdots$ durch $F_1(x), F(x), F_2(x), F(x), \cdots$ ersetzen. Die erste Hälfte folgt aus der zweiten, wenn wir eine Funktion $F(x)$ mit den in der zweiten Hälfte genannten Eigenschaften aufweisen können. Dabei ist aber zu bemerken, daß es wieder ausreicht, sich auf ein vollständiges normiertes Orthogonalsystem, d. h. eine abzählbare f-Menge $f_1, f_2, \cdots$ zu beschränken — denn dies ist für die zweite Hälfte unseres Satzes ebenso zulässig, wie für Satz 4, γ), aus dem sie gewonnen wurde. Dabei ist unser Ausgangspunkt die Konvergenz der $F_n(A)$, also die in Satz 4, γ) genannten Eigenschaften der $F_n(x)$.

Sei zunächst f beliebig, aber fest, ferner $\varepsilon > 0$, $\eta > 0$. Es muß ein $\nu_0 = \nu_0(\varepsilon, \eta)$ geben, so daß für jedes gegebene System $m, n \geqq \nu_0$ die Menge $\mathfrak{M}_{m,n}(\varepsilon)$ aller x mit $|F_m(x) - F_n(x)| \geqq \varepsilon$ alle $\varrho(\Psi_f(y))$ enthält, mit Ausnahme einer y-Menge von Lebesgueschem Maße $\leqq \eta$. Im entgegengesetzten Falle gäbe es nämlich zwei gegen Unendlich strebende Folgen $m_1, m_2, \cdots$ und $n_1, n_2, \cdots$, so daß für jedes $\mathfrak{M}_{m_\nu, n_\nu}(\varepsilon)$ $(\nu = 1, 2, \cdots)$ die y mit nicht dazu gehörendem $\varrho(\Psi_f(y))$ ein Lebesguesches Maß $\geqq \eta$ haben. Aus der m_ν, n_ν-Folge wählen wir die in Satz 4, γ) beschriebene m_{p_ν}, n_{p_ν}-Teilfolge $(p_1 < p_2 < \cdots)$ aus. Da nun alle in Frage kommenden y in einer Menge endlichen Maßes (nämlich $0 \leqq y \leqq |f|^2$, der Wertevorrat von $\Phi_f(x) = |E(\varrho(x))f|^2$) enthalten sind, ist der Satz von Arzelà-Young[39]) anwendbar: die Menge der y, für die $\varrho(\Psi_f(y))$ für unendlich viele ν dem $\mathfrak{M}_{m_{p_\nu}, n_{p_\nu}}(\varepsilon)$ nicht angehört, hat auch ein Maß $\geqq \eta$. Für ein solches $x = \varrho(\Psi_f(y))$ ist also unendlich oft $|F_{m_{p_\nu}}(x) - F_{n_{p_\nu}}(x)| \geqq \varepsilon$, also gewiß nicht $F_{m_{p_\nu}}(x) - F_{n_{p_\nu}}(x) \to 0$. Nach Satz 4, γ) sollten aber diejenigen y, deren $x = \varrho(\Psi_f(y))$ eine nicht gegen 0 konvergente Folge $F_{m_{p_\nu}}(x) - F_{n_{p_\nu}}(x)$ erzeugen, eine Menge vom Lebesgueschen Maße 0 bilden. Dies ist ein Widerspruch.

Nun sei $n_1, n_2, \cdots$ eine gegen Unendlich strebende Folge. Wir wählen p_1 mit $n_{p_1} \geqq \nu_0\left(\frac{1}{2}, \frac{1}{2}\right)$; $p_2 > p_1$ mit $n_{p_2} \geqq \nu_0\left(\frac{1}{2}, \frac{1}{2}\right), \nu_0\left(\frac{1}{4}, \frac{1}{4}\right)$, $p_3 > p_2$ mit $n_{p_3} \geqq \nu_0\left(\frac{1}{4}, \frac{1}{4}\right), \nu_0\left(\frac{1}{8}, \frac{1}{8}\right), \cdots$. Daher gilt $|F_{n_{p_\nu}}(x) - F_{n_{p_{\nu+1}}}(x)| \leqq \frac{1}{2^\nu}$ für alle $\varrho(\Psi_f(y))$, mit Ausnahme einer y-Menge vom Lebesgueschen Maße $\leqq \frac{1}{2^\nu}$. Es gilt also für alle $\nu \geqq \mu$ (μ gegeben) zugleich, mit Ausnahme einer y-Menge vom Lebesgueschen Maße $\leqq \frac{1}{2^\mu} + \frac{1}{2^{\mu+1}} + \cdots = \frac{1}{2^{\mu-1}}$. Für diese y, bzw. ihre $x = \varrho(\Psi_f(y))$, ist aber dann für alle $\nu \geqq \mu$, $\nu < \varrho$

$$|F_{n_{p_\nu}}(x) - F_{n_{p_\varrho}}(x)| \leqq \frac{1}{2^\nu} + \frac{1}{2^{\nu+1}} + \cdots \frac{1}{2^{\varrho-1}} < \frac{1}{2^{\nu-1}}.$$

Also erfüllen die $F_{n_{p_\nu}}(x)$ das Cauchysche Konvergenzkriterium, d. h. die Folge $F_{n_{p_\nu}}(x)$ $(\nu = 1, 2, \cdots)$ konvergiert. Das Maß der Menge der y, für deren $x = \varrho(\Psi_f(y))$ sie divergiert, ist somit $\leqq \frac{1}{2^{\mu-1}}$ — und da dies für alle μ gilt, ist sie $= 0$.

Nunmehr wollen wir f variieren. Wir nehmen eine gegen Unendlich strebende Folge $n_1, n_2, \cdots$ und gehen wie folgt vor: Aus $F_{n_1}(x), F_{n_2}(x), \cdots$ wählen wir nach obigem eine Teilfolge $F_{r_1}(x), F_{r_2}(x), \cdots$ aus, die für alle $\varrho(\Psi_{f_1}(y))$ konvergiert, mit Ausnahme einer y-Menge vom Maße 0. Aus $F_{r_2}(x), F_{r_3}(x), \cdots$ wählen wir eine Teilfolge $F_{s_1}(x), F_{s_2}(x), \cdots$ aus, die für alle $\varrho(\Psi_{f_2}(y))$ konvergiert, mit Ausnahme einer y-Menge vom Maße 0. Aus $F_{s_2}(x), F_{s_3}(x), \cdots$ wählen wir eine Teilfolge $F_{t_1}(x), F_{t_2}(x), \cdots$ aus, die für alle $\varrho(\Psi_{f_3}(y))$ konvergiert, mit Ausnahme einer y-Menge vom Maße 0, $\cdots$. Die Folge $F_{r_1}(x), F_{s_1}(x), F_{t_1}(x), \cdots$ ist nun eine Teilfolge von $F_{n_1}(x), F_{n_2}(x), \cdots$, und für jede der Folgen $F_{r_1}(x), F_{r_2}(x), \cdots, F_{s_1}(x), F_{s_2}(x), \cdots, F_{t_1}(x), F_{t_2}(x), \cdots, \cdots$ eine Teilfolge nach Wegnahme endlich vieler Glieder. Die Menge aller x, für die sie konvergiert, heiße $\mathfrak{M}$, dann enthält wegen der letzten Bemerkung $\mathfrak{M}$ alle $\varrho(\Psi_{f_n}(y))$ mit Ausnahme einer y-Menge vom Maße 0, und zwar für jedes n. Statt $F_{r_1}(x), F_{s_1}(x), \cdots$ schreiben wir lieber $F_{n_{p_1}}(x), F_{n_{p_2}}(x), \cdots (p_1 < p_2 < \cdots)$, die Limesfunktion nennen wir $F(x)$ — diese ist vorläufig nur in $\mathfrak{M}$ definiert, außerhalb $\mathfrak{M}$ möge sie etwa 0 sein. Da die $F_{n_{p_\nu}}(x)$ gleichmäßig beschränkt sind, ist es auch $F(x)$; da sie Φ-meßbar sind, sind es auch $\mathfrak{M}$ und $F(x)$ (man sieht dies genau so ein, wie für die gewöhnliche Meßbarkeit).

Auf Grund unserer Konstruktion scheint $F(x)$ noch von der Folge $n_1, n_2, \cdots$ abzuhängen. Seien $m_1, m_2, \cdots$ und $n_1, n_2, \cdots$ zwei solche Folgen, und $m_{p_1}, m_{p_2}, \cdots$ bzw. $n_{q_1}, n_{q_2}, \cdots$ die nach obigem hergestellten Teilfolgen, derart, daß $F_{m_{p_\nu}}(x)$ für $\nu \to \infty$ gegen $F'(x)$ konvergiert und $F_{n_{q_\nu}}(x)$ für $\nu \to \infty$ gegen $F''(x)$ — beides für alle $x = \varrho(\Psi_{f_l}(y))$ mit Ausnahme einer y-Menge vom Maße 0, und zwar für jedes l. Nach Satz 4, γ), gibt es nun, wenn wir l für den Augenblick festhalten, eine (von l abhängige) Folge $u_1 < u_2 < \cdots$, so daß $F_{m_{p_{u_\nu}}}(x) - F_{n_{q_{u_\nu}}}(x) \to 0$ für alle $\varrho(\Psi_{f_l}(y))$ stattfindet, mit Ausnahme einer y-Menge vom Maße 0. Daher ist $F'(x) = F''(x)$, für alle $\Psi_f(y)$, mit Ausnahme einer y-Menge vom Maße 0, und dies gilt für alle l. D. h.: $F(x)$ ist im wesentlichen von der Folge $n_1, n_2, \cdots$ unabhängig, wenn wir es für eine solche Folge $n_1, n_2, \cdots$ wählen und festhalten, wird es doch auch für alle anderen Folgen, die in der zweiten

Hälfte unseres Satzes auseinandergesetzten Eigenschaften haben. Damit haben wir alles nachgeholt, was notwendig war, um auch die erste Hälfte unseres Satzes zu beweisen. —

Ehe wir diesen Teil abschließen, sei noch erwähnt, daß weder Satz 4 noch Satz 5 für die „schwache" Topologie von $\boldsymbol{B}$ (vgl. a. a. O. Anm. [35])) zutrifft, die einfachen diesbezüglichen Gegenbeispiele übergehen wir. Ferner bemerken wir, daß ein notwendiges und hinreichendes Kriterium für das Bestehen von $F(A) = G(A)$ leicht aus der zweiten Hälfte von Satz 5 ableitbar ist: es genügt, alle $F_n(x)$ gleich $G(x)$ zu setzen. Dann finden wir: $F(x) = G(x)$ muß für jedes f für alle $\varrho(\Psi_f(y))$ zutreffen, mit Ausnahme einer y-Menge vom Lebesgueschen Maße 0, d. h. die hinreichende Bedingung in 2, g) ist auch notwendig. Ferner genügt es, ebenso wie in Satz 5, dies anstatt für alle f, nur für ein vollständiges normiertes Orthogonalsystem zu verlangen. Schließlich sei noch Folgendes hervorgehoben: In der zweiten Hälfte von Satz 4, γ) sowie von Satz 5 wird die dort auftretende Teilfolge $F_{m_\nu}(x) - F_{n_\nu}(x)$ bzw. $F_{n_\nu}(x) - F(x)$ $(\nu = 1, 2, \cdots)$ in Abhängigkeit von f gewählt, und ihr $\mathfrak{M}$ hat die Eigenschaft aus 2, g) nur für dieses f. Wir können aber an beiden Stellen ebensogut verlangen, daß sie von f unabhängig wählbar sei — ihr festes $\mathfrak{M}$ hat dann die Eigenschaft aus 2, g) allgemein. Sei nämlich $f_1, f_2, \cdots$ ein vollständiges normiertes Orthogonalsystem, ein Hilbertsches Diagonalverfahren, genau wie beim Beweise von Satz 5, liefert dann eine Teilfolge, die (d. h. deren $\mathfrak{M}$) das Gewünschte für alle f aus $f_1, f_2, \cdots$ tut. Daß aber ein solches $\mathfrak{M}$ das Gewünschte (d. i. das Erfülltsein der Eigenschaft aus 2, g) für alle f überhaupt leistet, folgt z. B. aus dem obigen Kriterium für $F(A) = G(A)$, wo es auch gleichgültig war, ob es für das dortige (offenbar allgemeine) $\mathfrak{M}$ für alle f oder nur für die $f_1, f_2, \cdots$ verlangt wird.

III. Operatorenfunktionen und -ringe.

1. Die Sätze 4, 5 ermöglichen einen einfachen Beweis der in Einleitung, 3, aufgestellten Behauptung.

SATZ 6. *Sei A ein beschränkter Hermitescher Operator, $\boldsymbol{R}(A)$ sein Ring. Dann ist $\boldsymbol{R}(A)$ die Menge aller $F(A)$, wobei $F(x)$ alle Φ-meßbaren Funktionen mit $F(0) = 0$ durchläuft. (Ob die $F(x)$ als beschränkt, oder, wie in Anm.[28]) bloß als in jedem endlichen Intervall beschränkt, postuliert werden, ist gleichgültig.)*

Beweis: Zunächst gehören alle diese $F(A)$ zu $\boldsymbol{R}(A)$. Denn es genügt, wie wir mehrmals schlossen, die $F(x)$ der dritten Baireschen Klasse zu berücksichtigen, und zwar natürlich die mit $F(0) = 0$. Sodann genügt es, wegen der Geschlossenheitseigenschaften der Ringe (und II., 2., h), oder Satz 5.), sogar die der nullten Baireschen Klasse zu betrachten, als welche

hier die Polynome $p(x)$ mit $p(0)=0$ fungieren dürfen. Daß aber ein solches $p(A)$ zum Ringe von A gehört, wissen wir längst, z. B. aus A, Seite 404.

Um die Umkehrung zu beweisen, erinnern wir daran, daß nach der Bemerkung a. a. O. jeder Operator von $\boldsymbol{R}(A)$ Limes einer stark doppelkonvergenten Folge $p_1(A), p_2(A), \cdots$ (alle $p_n(x)$ Polynome, $p_n(0)=0$) ist. Aus Satz 5 folgt daher, daß er die Form $F(A)$ hat, und aus der dort gegebenen Charakterisierung von $F(x)$, daß dieses beschränkt und Φ-meßbar ist, sowie die Möglichkeit, es mit $F(0)=0$ zu wählen.

Damit ist alles bewiesen.

Zwei Bemerkungen sind naheliegend: Erstens, daß es genügt, sich auf die $F(x)$ der dritten Baireschen Klasse zu beschränken. Zweitens, daß die Beschränkung $F(0)=0$ fallen gelassen werden kann, falls eine Abänderung von $F(x)$ im Punkte $x=0$ $F(A)$ nicht berührt. D. h., wenn für jedes f $\varrho(\Psi_f(y))=0$ nur für eine Lebesguesche Nullmenge eintritt, d. h. wenn 0 nicht der Wert dieser Funktion in einem Konstanzintervall ist. Also: $\varrho^{-1}(0)$ nicht ein Konstanzintervallwert von $\Psi_f(y)$, oder: $\varrho^{-1}(0)$ keine Unstetigkeitstelle von $\Phi_f(x)=|E(\varrho(x))f|^2$, oder: 0 keine Unstetigkeitstelle von $|E(\xi)f|^2$. D. h.: wenn 0 nicht zum Punktspektrum $\mathfrak{P}$ von A gehört, oder: wenn $Af=0$ $f=0$ zur Folge hat.

Die Hermiteschen Operatoren des Ringes $\boldsymbol{R}(A)$ erhalten wir, wenn wir uns auf die reellwertigen $F(x)$ beschränken. Denn daß für reelles $F(x)$ $F(A)$ Hermitesch ist, ist klar; ist umgekehrt $F(A)$ Hermitesch, so ist es gleich $F(A)^*$, also gleich $\frac{F(A)+F(A)^*}{2}=G(A)$, wo $G(x)$ die reelle Funktion $\frac{F(x)+\overline{F(x)}}{2}$ ist.

Da eine Abelsche Menge $\boldsymbol{M}$ von Operatoren aus $\boldsymbol{B}$ (d. h. eine Menge, für deren irgend zwei Elemente A, B A mit B und B^* bei der Multiplikation vertauschbar ist — daher sind alle ihre Elemente normal) stets Teil eines Abelschen Ringes ist (vgl. A, Seite 389), also, nach dem Hauptresultat von A (Seite 401, Satz 10), eines Ringes $\boldsymbol{R}(A)$, gibt es gewiß einen Hermiteschen Operator A, von dem alle Elemente von $\boldsymbol{M}$ Funktionen sind. Besteht $\boldsymbol{M}$ aus Hermiteschen Operatoren, so ist es Abelsch, wenn diese alle untereinander bei der Multiplikation vertauschbar sind, und dann sind die obengenannten Funktionen sogar reell wählbar.

Noch eines ist erwähnenswert: Sei A ein Hermitescher Operator, dann betrachten wir diejenigen Operatoren B (aus $\boldsymbol{B}$), die mit jedem Operator vertauschbar sind, der mit A vertauschbar ist. In A, Seite 404, wurde bewiesen, daß diese B alle normal sind, und zwar sind es gerade die Operatoren $a1+B'$, wo a alle komplexen Zahlen und B' alle Elemente

von $\boldsymbol{R}(A)$ durchläuft. Nach Satz 6 sind es also alle $a1+F'(A) = F(A)$ $(F(x) = a+F'(x))$, wo a alle komplexen Zahlen und $F'(x)$ die Funktionen mit $F'(0) = 0$ durchläuft — also $F(x)$ alle Funktionen. Wenn wir B als Hermitesch voraussetzen, so genügt es wiederum, die reellen $F(x)$ zu betrachten.

2. Sei $\boldsymbol{R}$ ein Abelscher Ring, dieser kann dann stets als $\boldsymbol{R}(A)$ geschrieben werden (vgl. A, Seite 401, Satz 10), A Hermitesch — aber A ist hierdurch, wie man leicht erkennt, noch lange nicht festgelegt. Sei also $\boldsymbol{R} = \boldsymbol{R}(A) = \boldsymbol{R}(B)$, welcher Zusammenhang muß dann zwischen A und B (beide seien Hermitesch) bestehen?

Da jedes zum Ringe des andern gehört, ist jedes Funktion, und zwar reelle Funktion, des anderen: $B = F(A)$, $A = G(B)$, und dabei $F(0) = 0$, $G(0) = 0$. Dies ist übrigens offenbar auch hinreichend, damit A zum Ringe von B und B zum Ringe von A gehöre, d. h. damit $\boldsymbol{R}(A) = \boldsymbol{R}(B)$ sei. Wir betrachten nun die (natürlich in jedem endlichen Intervalle beschränkte und Φ-meßbare) Funktion $F(x)$ (mit $F(0) = 0$) als gegeben, und fragen: wie muß sie beschaffen sein, damit ein (ebenfalls in jedem endlichen Intervalle beschränktes und Φ-meßbares) $G(x)$ (mit $G(0) = 0$) existiert, so daß für den durch $B = F(A)$ definierten Operator B $A = G(B)$ gilt?

Diese Forderung besagt $G(F(A)) = A$, also, nach der Schlußbemerkung von II., 3., $G(F(x)) = x$ in einer x-Menge $\mathfrak{M}$ mit der in II., 2., g) angegebenen Eigenschaft. Wie man sieht, kommt dies darauf heraus, daß das Φ-meßbare $F(x)$ eine Φ-meßbare Inverse $G(x)$ haben soll — mit Ausnahme einer x-Menge, für die die Urbilder aller Abbildungen $\varrho(\psi_f(y)) = x$ Nullmengen sind (vgl. II., 2., g)). (Die Beschränktheit von $G(x)$ ist unwesentlich: es kommt ja nur auf die x des Spektrums von A an, also in einem endlichen Intervalle, d. h. soweit der Wertverlauf von $G(u)$ überhaupt vorgeschrieben ist, liegt er zwischen endlichen Schranken.) Es wäre naheliegend, sich von der Annahme frei zu machen zu suchen, also etwa mit solchen $F(x)$ zu operieren, die jeden (reellen) Wert einmal und nur einmal annehmen. Indessen müßte dann auch noch die weitere Frage klargestellt werden, wann die Inverse einer Φ-meßbaren Funktion (die eine Inverse besitzt) auch Φ-meßbar ist? Diese Frage scheint aber ziemlich kompliziert zu sein[40]).

IV. Mehrvariablen-Funktionen und normale Operatoren.

1. Zunächst führen wir den Begriff der Φ-meßbarkeit für derartige (komplexwertige) Funktionen $F(x_1, x_2, \cdots)$ ein, die von einer endlichen oder abzählbar unendlichen Zahl komplexer Variablen $x_1, x_2, \cdots$ abhängen (auch bei einer endlichen Zahl von Variablen, wo es $x_1, \cdots, x_n$

heißen sollte, schreiben wir der Einheitlichkeit halber $x_1, x_2, \cdots$). Ein solches $F(x_1, x_2, \cdots)$ nennen wir Φ-meßbar, wenn es die folgende Eigenschaft hat: Wenn $f_1(u), f_2(u), \cdots$ ein beliebiges System für alle reellen u definierter und nach rechts halbstetiger Funktionen ist[41]), so ist $F(f_1(u), f_2(u), \cdots)$ als komplexwertige Funktion des reellen u aufgefaßt!) meßbar. Offenbar sind alle Funktionen, die nur von endlich vielen Variablen abhängen, und in diesen stetig sind, Φ-meßbar; und wenn $F_1, F_2, \cdots$ eine Folge Φ-meßbarer Funktionen gilt, die überall gegen ein F konvergieren, so ist auch F Φ-meßbar. Statt dieses letzteren beweisen wir gleich mehr: Seien $F_1, F_2 \cdots$ Φ-meßbar, und F so definiert:

$$F(x_1, x_2, \cdots) = \begin{cases} \lim\limits_{n\to\infty} F_n(x_1, x_2, \cdots), & \text{wenn dieser Limes existiert} \\ 0 & \text{sonst.} \end{cases}$$

Da nämlich Φ-Meßbarkeit von F_n, F die gewöhnliche Meßbarkeit von $F_n(f_1(u), f_2(u), \cdots)$, $F(f_1(u), f_2(u), \cdots)$ (bei jeder Wahl der $f_1(u), f_2(u), \cdots$) bedeutet, genügt es, den obigen Satz für Funktionen einer reellen Variablen u und gewöhnliche Meßbarkeit zu beweisen — dort ist er aber bekanntlich richtig. Somit sind alle Funktionen aller Baireschen Klassen wiederum Φ-meßbar[42]). Aus den analogen Eigenschaften der Meßbarkeit für Funktionen einer (reellen) Variablen folgt wieder unmittelbar, daß mit F, G auch $F \pm G$, aF, FG Φ-meßbar sind. Etwas tiefer liegt der folgende Satz:

SATZ 7. *Wenn $F(x_1, x_2, \cdots)$ Φ-meßbar ist und $G_1(y_1, y_2, \cdots)$, $G_2(y_1, y_2, \cdots), \cdots$ meßbar bzw. Φ-meßbar (alle Funktionen komplexwertig und mit komplexen Argumenten), so ist auch $F(G_1(y_1, y_2, \cdots), G_2(y_1, y_2, \cdots), \cdots)$ meßbar bzw. Φ-meßbar. (Man beachte, daß die erste, die gewöhnliche Meßbarkeit betreffende, Behauptung nur bei endlicher Anzahl der Variablen einen Sinn hat — da die gewöhnliche Meßbarkeit, im Gegensatze zur Φ-Meßbarkeit, nur dann definiert ist.)*

Beweis: Die zweite Behauptung (Φ-Meßbarkeit) läuft darauf heraus, daß für alle nach rechts halbstetigen $f_1(u), f_2(u), \cdots$ die Funktion

$$F\big(G_1(f_1(u), f_2(u), \cdots), G_2(f_1(u), f_2(u), \cdots), \cdots\big)$$

meßbar ist — da aber die $G_1(f_1(u), f_2(u), \cdots)$, $G_2(f_1(u), f_2(u), \cdots), \cdots$ meßbar sind, ist das die erste Behauptung, für eine reelle Variable. Die erste Behauptung für endlich viele komplexe Variable $y_1, \cdots, y_n$ (dies ist ihre allgemeine Form) kommt auf dasselbe heraus, wie für doppelt so viele reelle ($\Re y_1, \Im y_1, \cdots, \Re y_n, \Im y_n$) — somit ist lediglich die erste Behauptung für endlich viele reelle Variable zu beweisen. Diese reellen Variablen nennen wir wieder $y_1, y_2, \cdots$. Indem wir ferner die Variablen $x_1, x_2, \cdots$ von F durch $\Re x_1, \Im x_1, \Re x_2, \Im x_2, \cdots$ ersetzen, und entsprechend die

$G_1, G_2, \cdots$ durch $\mathfrak{R}(G_1)$, $\mathfrak{J}(G_1)$, $\mathfrak{R}(G_2)$, $\mathfrak{J}(G_2)$, $\cdots$, erreichen wir, daß auch F reelle Argumente hat — auch diese nennen wir wieder $x_1, x_2, \cdots$, und die einzusetzenden Funktionen wieder $G_1, G_2, \cdots$. Schließlich können wir, durch denselben Kunstgriff, wie bei Beginn des Beweises von Satz 2, alle Variablen $x_1, x_2, \cdots$ und $y_1, \cdots, y_n$ auf endliche Intervalle beschränken. Anstatt des dortigen Intervalles $0 < \xi < 2$ wählen wir jetzt $0 < \xi < 1$, daß natürlich zu $0 \leqq \xi < 1$ erweitert werden kann. Also: $F(x_1, x_2, \cdots)$ ist in $0 \leqq x_1 < 1, 0 \leqq x_2 < 1, \cdots$ definiert, die $G_1(y_1, \cdots, y_n)$, $G_2(y_1, \cdots, y_n), \cdots$ in $0 \leqq y_1 < 1, \cdots, 0 \leqq y_n < 1$, und die Werte der $G_1, G_2, \cdots$ sind (da sie in F substituiert werden), stets $\geqq 0, < 1$.

Nehmen wir nun an, der Beweis wäre für $n = 1$ geführt. Der allgemeine Fall mit $y_1, \cdots, y_n$ wird dann folgendermaßen auf denjenigen mit nur einem y zurückgeführt: Sei $y = \alpha(y_1, \cdots, y_n)$, $y_1 = \beta_1(y), \cdots, y_n = \beta(y_n)$ eine ein-eindeutige Abbildung des Intervalles $0 \leqq y < 1$ auf den Würfel $0 \leqq y_1 < 1, \cdots, 0 \leqq y_n < 1$ (bei beiden darf je eine, lineare bzw. n-dimensionale Lebesguesche Nullmenge ausgelassen werden), die maßtreu ist, d. h.: y-Urbild und $\{y_1, \cdots, y_n\}$-Bild sind beide meßbar oder beide unmeßbar, und haben im ersteren Falle dasselbe Maß (im linearen bzw. im n-dimensionalen Sinne)[43]). Dann ist mit $G_1(y_1, \cdots, y_n)$, $G_2(y_1, \cdots, y_n), \cdots$ auch $G_1(\alpha_1(y), \cdots, \alpha_n(y))$, $G_2(\alpha_1(y), \cdots, \alpha_n(y)), \cdots$ meßbar, also, wenn der Satz für eine Variable feststeht, auch $F\big(G_1(\alpha_1(y)), \cdots \cdots, \alpha_n(y)), G_2(\alpha_1(y), \cdots, \alpha_n(y)), \cdots\big)$, und dies bleibt meßbar, wenn darin y durch $\beta(y_1, \cdots, y_n)$ ersetzt wird — d. h. $F(G_1(y_1, \cdots, y_n)$, $G_2(y_1, \cdots, y_n), \cdots)$ ist meßbar. Wir gehen daher zur Erledigung des Falles mit einer einzigen reellen Variablen y über.

Seien $f_1(u), f_2(u), \cdots$ rechts halbstetig, $\varphi(y)$ monoton nichtfallend und halbstetig. Dann sind alle $f_1(\varphi(y)), f_2(\varphi(y)), \cdots$ rechts halbstetig, also ist $F\big(f_1(\varphi(y)), f_2(\varphi(y)), \cdots\big)$ meßbar. Da dies für alle genannten $\varphi(y)$ gilt, ist $F(f_1(u), f_2(u), \cdots)$ Φ-meßbar, also nach Satz 2 für jedes meßbare $G(y)$ $F\big(f_1(G(y)), f_2(G(y)), \cdots\big)$ meßbar. Wenn nun zu jedem System $G_1(y), G_2(y), \cdots$ meßbarer Funktionen eine meßbare Funktion $G(y)$ und rechts halbstetige $f_1(u), f_2(u), \cdots$ gefunden werden können, so daß $G_1(y) = f_1(G(y))$, $G_2(y) = f_2(G(y)), \cdots$ ist, so sind wir demnach am Ziele. Diese Konstruktion führen wir nun so durch: Sei $x = \alpha(x_1, x_2, \cdots)$, $x_1 = \beta_1(x)$, $x_2 = \beta_2(x), \cdots$ eine ein-eindeutige Abbildung einer Teilmenge des Intervalles $0 \leqq x < 1$ auf den (ganzen) Würfel $0 \leqq x_1 < 1, 0 \leqq x_2 < 1, \cdots$. Dabei seien alle $\beta_1(x), \beta_2(x), \cdots$ nach rechts halbstetig, und sie mögen die folgende Eigenschaft besitzen: wenn x den in einem Intervalle $\frac{p}{2^q} \leqq x < \frac{p+1}{2^q}$ $(q = 0, 1, \cdots, p = 0, 1, \cdots, 2^q - 1)$ gelegenen Teil der Bildmenge durchläuft, so durchläuft der Punkt $x_1 = \beta_1(x)$, $x_2 = \beta_2(x), \cdots$

einen Würfel (d.i. eine Menge $a_1 \leqq x_1 < b_1$, $a_2 \leqq x_2 < b_2, \cdots$)[44]). Betrachten wir nun $G(y) = \alpha(G_1(y), G_2(y), \cdots)$. $\frac{p}{2^q} \leqq G(y) < \frac{p+1}{2^q}$ kommt $a_1 \leqq G_1(y) < b_1$, $a_2 \leqq G_2(y) < b_2, \cdots$ gleich, findet also in einer y-Menge statt, die Durchschnitt abzählbar vieler meßbarer Mengen ist (die $G_1(y)$, $G_2(y), \cdots$ sind ja meßbar), folglich selbst meßbar ist. Nun ist die y-Menge von $G(y) < a$ (a beliebig) als Summe abzählbar vieler solcher Mengen herstellbar, also auch meßbar — somit ist $G(y)$ meßbar. Wenn wir also $f_1 \equiv \beta_1$, $f_2 \equiv \beta_2, \cdots$ setzen, so sind wir am Ziele.

Aus Satz 7 folgt sofort: ist die Zahl der $x_1, x_2, \cdots$ endlich, so ist ein Φ-meßbares $F(x_1, x_2, \cdots)$ meßbar — es genügt, die $y_1, y_2, \cdots$ mit den $x_1, x_2, \cdots$ zusammenfallen zu lassen, und $G_1(y_1, y_2, \cdots) = y_1$, $G_2(y_1, y_2, \cdots) = y_2, \cdots$ zu setzen. Der Vergleich des neuen Φ-Meßbarkeits-Begriffes mit dem alten (I., 3.) ist auch bei einer einzigen Variablen x unmöglich, da diese jetzt komplex ist, und früher reell war. Hängt aber $F(x)$ etwa nur von $\Re(x)$ ab, so sind beide gleichwertig: nach Definition beider folgt die alte Φ-Meßbarkeit aus der neuen, nach Satz 2 und der Definition der neuen Φ-Meßbarkeit gilt aber auch die Umkehrung.

Wenn $F(x_1, x_2, \cdots)$ im neuen Sinne Φ-meßbar ist, und $G_1(x)$, $G_2(x), \cdots$ im alten (x reell!), so sind es, nach dem soeben Gesagten, $G_1(\Re x)$, $G_2(\Re x), \cdots$ (x komplex!) auch im neuen Sinne. Nach Satz 7 ist es daher $F(G_1(\Re x), G_2(\Re x), \cdots)$ auch, und für reelles x ist es $F(G_1(x), G_2(x), \cdots)$ wieder im alten Sinne.

Auf Grund des bisher Gesagten können wir nunmehr die Unterscheidung zwischen beiden Definitionen der Φ-Meßbarkeit ohne Verwechslungsgefahr fallen lassen.

2. Sei nun $A_1, A_2, \cdots$ eine endliche oder abzählbar unendliche Abelsche Menge linear-beschränkter Operatoren (vgl. A, Seite 389), d. h. es soll jedes A_m mit jedem A_n und A_n^* vertauschbar sein — folglich sind insbesondere alle A_n normal. $F(x_1, x_2, \cdots)$ sei eine Φ-meßbare Funktion mit so vielen Variablen, als die Zahl der $A_1, A_2, \cdots$ beträgt.

Nach einer der Bemerkungen in III., 1. existieren solche (beschränkte) Hermitesche Operatoren, von denen alle $A_1, A_2, \cdots$ Funktionen sind. Sei etwa A ein solcher, also $A_1 = G_1(A)$, $A_2 = G_2(A), \cdots$, $G_1(x)$, $G_2(x), \cdots$ alle Φ-meßbar (x reell). Die Funktion $H(x) = F(G_1(x), G_2(x), \cdots)$ ist somit auch Φ-meßbar, und wenn sie in einer Menge $\mathfrak{M}$ mit den in II., 2., g) angegebenen Eigenschaften (mit dem Operator A) beschränkt ist, so können wir $H(A)$ bilden — dies ist ein beschränkt-linearer, normaler Operator. Ihn wollen wir als $F(A_1, A_2, \cdots)$ definieren, müssen aber zu diesem Zwecke zunächst zeigen, daß diese Definition vom hilfsweise eingeführten A nur scheinbar abhängt: d. h. daß sie für alle genannten A ausführbar ist, oder

für keines, und daß sie im ersteren Falle für alle dasselbe Resultat ergibt.

Der Ring $\boldsymbol{R}(A_1, A_2, \cdots)$ ist Abelsch, also nach A, Seite 401, Satz 10, ein $\boldsymbol{R}(B)$ mit Hermiteschem B; nach Satz 6 sind somit alle $A_1, A_2, \cdots$ Funktionen von B — somit ist B als ein A im Sinne von vorhin verwendbar. Wir wollen nun zeigen, daß sich jedes der genannten A für die Zwecke unserer Definition genau so benimmt, wie dieses B, und zum selben Resultate führt — damit ist dann alles Erforderliche geschehen.

Es ist $G_1(A) = A_1$, $G_2(A) = A_2, \cdots$ und $G'_1(B) = A_1$, $G'_2(B) = A_2, \cdots$. $\boldsymbol{R}(A)$ enthält alle Funktionen von A, also auch $A_1, A_2, \cdots$, also umfaßt es $\boldsymbol{R}(A_1, A_2, \cdots) = \boldsymbol{R}(B)$, also enthält es B. Daher ist $B = K(A)$. Hieraus folgt $G_1(A) = G'_1(K(A))$, $G_2 = G'_2(K(A)), \cdots$, und nach II., 2., f) sowie der Schlußbemerkung von II., 3. $G_1(x) = G'_1(K(x))$ in einer Menge $\mathfrak{M}$ mit den Eigenschaften II., 2., g) (für A), $G_2(x) = G'_2(K(x))$ in einer ebensolchen Menge, $\cdots$. Alle diese Gleichungen gelten gleichzeitig im Durchschnitt dieser Mengenfolge, d. h. wieder in einer Menge $\mathfrak{M}$ nach II., 2., g). Wir haben nun im Sinne unserer Definition bei A $H(x) = F(G_1(x), G_2(x), \cdots)$ zu bilden, bei B $H'(x) = F(G'_1(x), G'_2(x), \cdots)$. Nach obigem ist $H(x) = H'(K(x))$ in einer Menge $\mathfrak{M}$ nach II., 2., g). Aus II., 2., f) folgt sofort, daß $H'(B) = H'(K(A)) = H(A)$ ist, d. h. daß beide Definitionswege dasselbe ergeben, falls $H(x)$ für A und $H'(x)$ für B die gewünschten Beschränktheitseigenschaften haben. Es bleibt übrig, die Gleichwertigkeit dieser zu beweisen; wir beweisen aber lieber mehr: die Gültigkeit von $|H(x)| \leqq C$ in einem $\mathfrak{M}$ nach II., 2., g) für A ist derjenigen von $|H'(x)| \leqq C$ in einem $\mathfrak{M}$ nach II., 2., g) für B gleichwertig.

Ersetzen wir $F(x_1, x_2, \cdots)$ durch

$$F_D(x_1, x_2, \cdots) = \begin{cases} F(x_1, x_2, \cdots) & \text{für } |F(x_1, x_2, \cdots)| \leqq D, \\ D & \text{sonst,} \end{cases}$$

dann gehen $H(x)$, $H'(x)$ in

$$H_D(x) = \begin{cases} H(x) & \text{für } |H(x)| \leqq D, \\ D & \text{sonst,} \end{cases}$$

$$H'_D(x) = \begin{cases} H'(x) & \text{für } |H(x)| \leqq D, \\ D & \text{sonst,} \end{cases}$$

über. Da H_D, H'_D stets beschränkt sind, können wir $H_D(A)$, $H'_D(B)$ jedenfalls bilden, und sie sind einander gewiß gleich. Die genannte Beschränktheitseigenschaft von $H(x)$ bzw. $H'(x)$ besagt nun, daß $H_C(x) = H_{C+1}(x)$ in einem $\mathfrak{M}$ nach II., 2., g) für A gilt, bzw. $H'_C(x) = H'_{C+1}(x)$

in einem $\mathfrak{M}$ nach II., 2., g) für B, d. h. (nach der Schlußbemerkung von II., 3.) $H_C(A) = H_{C+1}(A)$ bzw. $H'_C(B) = H'_{C+1}(B)$. Und da die linken Seiten einander gleich sind, und die rechten auch, ist beides gleichwertig, wie behauptet wurde.

Damit ist unsere Definition von $F(A_1, A_2, \cdots)$ als einwandfrei erwiesen. Es ist noch zu erwähnen, daß sie der alten, soweit diese anwendbar ist, gleichwertig ist: wenn $F(x)$ eine Variable $x = x_1$ hat, und für diese ein Hermitesches A_1 einzusetzen ist, so können wir $A = A_1$ setzen, $G_1(x) = x$ ($G(A_1) = A_1 = A$), und das $F(A_1)$ im neuen Sinne ist $= F(A) = F(A_1)$ im alten Sinne, und auch die Beschränktheitsforderungen (d. h. die Bedingungen der Sinnvollheit) sind gleichlautend.

3. Wir verifizieren die Grundeigenschaften des neuen Funktionsbegriffes.

a) Für $F(x_1, x_2, \cdots) = \overline{G(x_1, x_2, \cdots)}$ ist $F(A_1, A_2, \cdots) = G(A_1^*, A_2^*, \cdots)^*$.

b) Für $F(x_1, x_2, \cdots) = aG(x_1, x_2, \cdots)$ ist $F(A_1, A_2, \cdots) = aG(A_1, A_2, \cdots)$.

c) Für $F(x_1, x_2, \cdots) = G(x_1, x_2, \cdots) + H(x_1, x_2, \cdots)$ ist $F(A_1, A_2, \cdots) = G(A_1, A_2, \cdots) + H(A_1, A_2, \cdots)$.

d) Für $F(x_1, x_2, \cdots) = G(x_1, x_2, \cdots) \cdot H(x_1, x_2, \cdots)$ ist $F(A_1, A_2, \cdots) = G(A_1, A_2, \cdots) \cdot H(A_1, A_2, \cdots)$.

(In a)—d) steht es mit der Sinnvollheit so: immer wenn die rechte Seite Sinn hat, hat auch die linke Sinn.)

e) Für $F(x_1, x_2, \cdots) = G(H_1(x_1, x_2, \cdots), H_2(x_1, x_2, \cdots), \cdots)$ ist $F(A_1, A_2, \cdots) = G(H_1(A_1, A_2, \cdots), H_2(A_1, A_2, \cdots), \cdots)$.

(Nach a), d) sind nämlich alle $H_1(A_1, A_2, \cdots)$, $H_2(A_1, A_2, \cdots)$, $\cdots$, wenn sie überhaupt Sinn haben, normal — in diesem Falle sind dann beide Seiten sinnvoll, oder keine.) Man erkennt, daß alle diese Eigenschaften mühelos durch Anwendung der Definition von $F(x_1, x_2, \cdots)$ und Vergegenwärtigung der entsprechenden Eigenschaften des alten Funktionsbegriffes (vgl. II., 2., b)—f)) beweisbar sind.

Genaue Kriterien für die Gleichheit und Konvergenz beim neuen Funktionsbegriff anzugeben, so wie wir sie beim alten hatten (II., 2., g) und h), oder ausführlicher in der Schlußbemerkung von II., 3. und Satz 4, 5) ist nicht nötig: es genügt jeweils die definitorische Zurückführung des neuen Funktionsbegriffes auf den alten. Immerhin sei die folgende (roheste) Form des Konvergenzsatzes erwähnt:

f) Es gelte überall $F_n(x_1, x_2, \cdots) \to F(x_1, x_2, \cdots)$, die $F_n(x_1, x_2, \cdots)$ seien gleichmäßig beschränkt. (Dann ist auch $F(x_1, x_2, \cdots)$ beschränkt, also alle $F_n(A_1, A_2, \cdots)$, $F(A_1, A_2 \cdots)$ sinnvoll.) Dann ist, im Sinne der starken Doppelkonvergenz in $\mathfrak{B}$, $F_n(A_1, A_2, \cdots) \to F(A_1, A_2, \cdots)$.

Dies ist evident, da der alte Funktionsbegriff diese Eigenschaft auch hatte.

4. Wir beweisen nun die folgende Verschärfung der vorletzten Behauptung von III, 1:

Satz 8. *$\mathbf{M}$ sei eine Abelsche Menge von Operatoren aus $\mathbf{B}$ (d. h. ihre Elemente seien linear-beschränkt, und für irgend zwei von ihnen, A, B, sei A mit B und mit B^* vertauschbar — also sind insbesondere alle normal). Dann gibt es einen Operator R derart, daß einerseits jedes A von $\mathbf{M}$ Funktion von R ist: $A = F_A(R)$, und andererseits eine endliche oder abzählbar unendliche Teilmenge $A_1, A_2, \cdots$ von $\mathbf{M}$ existiert, so daß R Funktion von $A_1, A_2, \cdots$ ist: $R = G(A_1, A_2, \cdots)$.*

Dabei kann R Hermitesch gewählt werden, und alle Funktionen an der Stelle 0 verschwindend: $F_A(0) = 0$ (für alle A von $\mathbf{M}$), $G(0, 0, \cdots) = 0$.

Beweis: $\mathbf{R}(\mathbf{M})$ ist, wie $\mathbf{M}$ selbst, Abelsch, kann also (wie wir mehrfach erwähnten) als $\mathbf{R}(R)$, mit Hermiteschem R, geschrieben werden. Da also alle A von $\mathbf{M}$ zu $\mathbf{R}(\mathbf{M}) = \mathbf{R}(R)$ gehören, sind sie nach Satz 6 gleich $F_A(R)$, mit $F_A(0) = 0$. Umgekehrt gehört R zu $\mathbf{R}(R) = \mathbf{R}(\mathbf{M})$, ist also nach A, Seite 398, Satz 9, Limes einer stark doppelkonvergenten Folge solcher Ausdrücke, deren jeder durch endlich viele Operationen A^*, aA, $A+B$, AB aus Elementen von $\mathbf{M}$ entsteht. Da in einen jeden dieser Ausdrücke nur endlich viele Elemente von $\mathbf{M}$ eingehen können, kommen für die gesamte Folge endlich oder abzählbar unendlich viele in Frage, wir nennen sie $A_1, A_2, \cdots$. Jedes Element der Folge ist daher ein Polynom endlich vieler $A_1, A_2, \cdots$ und $A_1^*, A_2^*, \cdots$, ohne Absolutglied — wenn wir $A_1, A_2, \cdots$ durch Variable $x_1, x_2, \cdots$ und die $A_1^*, A_2^*, \cdots$ durch deren Konjugierte $\bar{x}_1, \bar{x}_2, \cdots$ ersetzen, so heiße das n-te $p_n(x_1, x_2, \cdots)$. Nach Anm. [42]) ist $p_n(x_1, x_2, \cdots)$ Φ-meßbar, ferner ist, da das Absolutglied fehlt, $p_n(0, 0, \cdots) = 0$. Nach II., 2., α) sowie IV., 2., a)—d) ist dann das n-te Glied unserer Operatorenfolge $p_n(A_1, A_2, \cdots)$, folglich haben wir $p_n(A_1, A_2, \cdots) \to R$. Hieraus folgt $p_n(F_{A_1}(R), F_{A_2}(R), \cdots) \to R$, aus Satz 5 folgt daher, daß es eine Teilfolge $n_1, n_2, \cdots$ $(n_1 < n_2 < \cdots)$ von $n = 1, 2, \cdots$ gibt, derart, daß $p_{n_\nu}(F_{A_1}(x), F_{A_2}(x), \cdots) \to x$ (für $\nu \to \infty$) in einer x-Menge $\mathfrak{M}$ nach II., 2., g) (für R) stattfindet.

Sei nun

$$G(x_1, x_2, \cdots) = \begin{cases} \lim\limits_{\nu \to \infty} p_{n_\nu}(x_1, x_2, \cdots), & \text{wenn dieser Limes existiert,} \\ 0 & \text{sonst.} \end{cases}$$

Nach Anm. [42]) ist dieses $G(x_1, x_2, \cdots)$ Φ-meßbar, und nach dem vorhin Bemerkten gilt $G(F_{A_1}(x), F_{A_2}(x), \cdots) = x$ in einer Menge $\mathfrak{M}$ nach II., 2., g) (für R), daher ist nach Definition $G(A_1, A_2, \cdots) = R$. Aus $p_{n_\nu}(0, 0, \cdots) = 0$ folgt $G(0, 0, \cdots) = 0$.

Ferner gilt:

Satz 9. *$\mathbf{M}$ sei wie in Satz 8. $\mathbf{R}(\mathbf{M})$ ist dann die Menge aller $F(A_1, A_2, \cdots)$, wobei $F(x_1, x_2, \cdots)$ alle Φ-meßbaren Funktionen mit $F(0, 0, \cdots) = 0$ durchläuft, $A_1, A_2, \cdots$ aber jede endliche oder abzählbar unendliche Teil-*

menge von $\mathbf{M}$ *sein darf. Das Resultat bleibt aber auch bestehen, wenn für* $A_1, A_2, \cdots$ *nur eine einzige derartige Menge zugelassen wird, falls diese geeignet gewählt wird (was immer möglich ist).*

Beweis: Seien R, $F_A(x)$, $A_1, A_2, \cdots, G(x_1, x_2, \cdots)$ wie beim Beweise von Satz 8. Sind $B_1, B_2, \cdots$ irgendwelche (endlich oder abzählbar unendlich viele) Elemente von $\mathbf{M}$, so ist jedes $F(B_1, B_2, \cdots)$ wegen $B_1 = F_{B_1}(R)$, $B_2 = F_{B_2}(R), \cdots$ gleich $F(F_{B_1}(R), F_{B_2}(R), \cdots) = H(R)$ mit $H(x) = F(F_{B_1}(x), F_{B_2}(x), \cdots)$, $H(0) = F(0, 0, \cdots) = 0$, — also gehört es zu $\mathbf{R}(R) = \mathbf{R}(\mathbf{M})$. Umgekehrt ist jedes Element von $\mathbf{R}(\mathbf{M})$ eines von $\mathbf{R}(R)$, also gleich $J(R)$ mit $J(0) = 0$. Somit ist es gleich $J(G(A_1, A_2, \cdots)) = K(A_1, A_2, \cdots)$ mit $K(x_1, x_2, \cdots) = J(G(x_1, x_2, \cdots))$, $K(0) = J(0) = 0$. Somit hat es die gewünschte Form, sogar bei dieser einen speziellen Wahl der $B_1, B_2, \cdots$ (nämlich gleich $A_1, A_2, \cdots$).

Die in Satz 8, 9 vorkommenden Funktionen $G(x_1, x_2, \cdots)$ bzw. $F(x_1, x_2, \cdots)$ können, wie aus dem Beweise von Satz 8 hervorgeht, auf die Klasse derer eingeschränkt werden, die durch einen einzigen Grenzübergang im Sinne von Anm. [42]) aus Polynomen endlich vieler Variablen und ihrer konjugierten, $p(x_1, x_2, \cdots)$, entstehen. Im Sinne von Anm. [42]) sind sie also von der ersten Baireschen Klasse, auch dann, wenn man die nullte auf die genannten $p(x_1, x_2, \cdots)$ beschränkt. Definiert man dagegen die Baireschen Klassen durch überall konvergente Funktionenfolgen (vgl. IV., 2., f)), so braucht man, um diese Funktionen zu erreichen, drei sukzessive Grenzprozesse, wie man leicht erkennt. In diesem (allgemein üblichen) Sinne braucht man also alle Funktionen bis zur dritten Baireschen Klasse. Da also jede Funktion durch eine solche ersetzt werden kann, gilt dies auch für die $F_A(x)$ in Satz 8.

Schließlich machen wir noch eine, der Schlußbemerkung von III., 1. entsprechende, Feststellung. Wenn wir in der Aussage von Satz 9 die Bedingung $F(0, 0, \cdots) = 0$ fallen lassen, so erhalten wir im Allgemeinen nicht $\mathbf{R}(\mathbf{M})$, sondern die Gesamtheit aller $a \cdot 1 + B$ (a eine komplexe Zahl, B ein Element von $\mathbf{R}(\mathbf{M})$), und dies ist nach A, Seite 397, Satz 7, die Menge $\mathbf{M}''$, das ist die Menge aller A, die mit allen C vertauschbar sind, welche mit B und B^* für alle B von $\mathbf{M}$ vertauschbar sind. Diese A sind also mit sämtlichen $F(A_1, A_2, \cdots)$ (die $A_1, A_2, \cdots$ nach Satz 9) identisch.

Anmerkungen.

[1]) Im Laufe dieser Abhandlung werden zwei frühere Arbeiten des Verfassers mehrfach zitiert werden, und zwar: „Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren", Math. Ann., Bd. 102/1, Seite 49–131 (1929), und „Zur Algebra der Funktionaloperatoren und der Theorie der normalen Operatoren", Math. Ann., Bd. 102/3, Seite 370—427 (1929). Wir werden sie mit E bzw. A bezeichnen. — Hier handelt es sich um A, insbesondere um den Schluß von Kap. III, § 2.

Es sei noch darauf hingewiesen, daß Funktionen von Operatoren bereits von F. Rieß betrachtet wurden, und u. a. bei seinem Beweise des Hilbertschen Satzes über die Spektralform beschränkter Hermitescher Operatoren eine wichtige Rolle spielten. Neuerdings kündigte M. Stone eine allgemeine Definition von Funktionen von Operatoren an (Proc. Nat. Ac., Bd. 16/2, Seite 172—175, 1930), ferner spielen besondere Operatorenfunktionen-Konstruktionen in zwei Arbeiten von A. Haar eine sehr bemerkenswerte Rolle (Math. Ztschr., Bd. 31/5, Seite 769—798, 1930, bzw. ebenda 1931 erscheinend).

[2]) Die Hermiteschen (beschränkten, vgl. hierzu A, Einleitung, 1.) Operatoren A, B heißen vertauschbar, wenn $AB = BA$ gilt.

[3]) Vgl. hierzu die Beschreibung des Hilbertschen Raumes in E, Kap. I (ferner Einleitung I—IV sowie Anhang I). Über Operatoren vgl. E, Kap. II, sowie Anhang II, § 1. Alles Notwendige ist übrigens in A, Einleitung, 1., nochmals zusammengestellt. Der gleich zu erwähnende Ring aller beschränkten Operatoren wird in A, Einleitung 2, eingeführt. — Punkte des Hilbertschen Raumes nennen wir $f, g, \cdots, \varphi, \psi, \cdots$, beschränkte Operatoren $A, B, \cdots$, komplexe Zahlen $a, \cdots$ (wie in E und A).

[4]) A^* ist in der Matrizen-Terminologie die transponiert-konjugierte zu A, vgl. E, Anhang II, § 1 oder A, Ende von Einleitung, 1.

[5]) Die Definition steht in A, Kap. II, § 1, Definition 1. Vgl. auch A, Kap. I, § 3, Zusatz zu Satz 5.

[6]) Vgl. A, Kap. I, § 1, Definition 2.

[7]) Vgl. A, Ende von Kap. III, § 2.

[8]) D. h. wenn $A = A^*$ ist, bzw. A, A^* vertauschbar sind. Letzteres ist das Kriterium dafür, daß $\mathfrak{R}(A)$ Abelsch ist, d. h. aus lauter miteinander vertauschbaren Operatoren besteht. Vgl. A, Kap. III, Anfang von § 1 und von § 2.

[9]) Vgl. A, gegen Ende von Kap. II, § 1.

[10]) Kap. III, § 2, Satz 10.

[11]) Vgl. W. H. Young, Proc. Cambridge Phil. Soc., Bd. 14, Seite 520—529, 1908; H. Lebesgue, Thèse, Ann. di Math., Bd. 7 (3), 1902.

[12]) $g(u)$ braucht nicht mehr meßbar zu sein!

[13]) Da $\psi(b)$ nach rechts halbstetig ist, nimmt es am linken Ende des Intervalles den im Inneren angenommenen Wert gewiß an, am rechten dagegen nur, wenn es dort stetig ist. D. h. wenn der Wert $\psi(b) = a$ nicht der Anfang eines Konstanzintervalles von $\varphi(a)$ ist. Das genannte Intervall ist also allenfalls inkl. linkes Ende zu verstehen, inkl. rechtes Ende aber nur im soeben genannten Falle.

[14]) Dabei ist es wesentlich, daß $\mathfrak{P}_b$ alle I_n umfaßte, also $\mathfrak{P}_b'$ auch: wäre $\mathfrak{P}_b$ zu einigen I_n fremd gewesen, so brauchte $\mathfrak{P}_b'$ keins von beiden zu tun.

[15]) Da $\Phi(a) = a$ gesetzt werden kann, folgt aus dieser Φ-Meßbarkeit die gewöhnliche Meßbarkeit.

[16]) Für ein einziges $f_1(x)$, wobei dieses $= x$ ist (aber mit der zweiten Baireschen Klasse an Stelle der dritten), ist dies ein bekannter Vitalischer Satz. Das Wesentliche ist eben die Generalisation auf beliebige meßbare $f_1(x), f_2(x), \cdots$. Zum Vitalischen Satz vgl. Rend. Lomb., Bd. 38, Seite 599 (1905).

[17]) Sei etwa $H(x)$ von zweiter Klasse, also $H_1(x), H_2(x), \cdots$ von erster, und $H_n(x) \to H(x)$. Wenn nun $H(x)$ in den Intervallen $I_1, I_2, \cdots$ konstant gemacht werden soll, so erreichen wir dies, indem wir $H_1(x)$ nur in I_1, $H_2(x)$ nur in I_1, I_2, usw. konstant machen. — In unserem Falle ($H(x) = \overline{G}(x)$ und nach der Änderung $= \overline{\overline{G}}(x)$) enthalten die I_n allenfalls ihre linken Enden, ob sie die rechten auch enthalten, ist ungewiß (vgl. Anm.[13]).

[18]) Durch Zerlegung der Zahlengeraden in endlich viele Teile, deren jeder genau eine solche Stelle enthält, erkennt man, daß es genügt, sich mit einer einzigen solchen Stelle auseinanderzusetzen. Eine triviale Variablentransformation erlaubt noch, das Definitionsintervall zu $-1 < x < 1$, und die fragliche Stelle zu $x = 0$ zu normieren. Sei nunmehr die genannte Funktionenfolge $f_1(x), f_2(x), \cdots$. Sei dann $\bar{f}_n(x) = f_n(x)$ für $|x| \geqq \frac{1}{n}$ und für $x = 0$, in $-\frac{1}{n} \leqq x \leqq 0$ und in $0 \leqq x \leqq \frac{1}{n}$ aber linear. Dann sind die $\bar{f}_1(x), \bar{f}_2(x), \cdots$ stetig, und haben dieselben Limites wie die $f_1(x), f_2(x), \cdots$.

[19]) $f_{m,n}(x) \to g_m(x)$ für $n \to \infty$, $g_m(x) \to \overline{\overline{G}}(x)$ für $m \to \infty$.

[20]) Sei $f(x)$ nach rechts halbstetig, dann sind alle $\bar{f}_n(y) = n \int\limits_y^{y+\frac{1}{n}} f(x)\,dx$ stetig, und $\bar{f}_n(x) \to f(x)$.

[21]) Wenn $\mathfrak{M}$ alle im Definitionsintervalle von $\varphi(x)$ gelegenen Zahlen, ausschließlich seiner Konstanzintervalle, umfaßt, so ist dies erfüllt: denn es enthält dann alle Werte des $\psi(y)$. Übrigens erkennt man leicht, daß die y, für die $\psi(y)$ nicht in $\mathfrak{M}$ liegt, die folgenden sind: man nehme für jedes außerhalb von $\mathfrak{M}$ (aber im Definitionsintervalle von $\varphi(x)$) gelegene x die Zahl $y = \varphi(x)$, wenn x eine Stetigkeitsstelle von $\varphi(x)$ ist, und das ganze „Sprungintervall" $\lim\limits_{\substack{x' \to x \\ x' < x}} \varphi(x') \leqq y \leqq \varphi(x) = \lim\limits_{\substack{x' \to x \\ x' > x}} \varphi(x')$, wenn x eine Unstetigkeitsstelle ist. Daher ist von $\mathfrak{M}$-s Komplementärmenge im Definitionsintervalle zu fordern: sie darf keine Unstetigkeitsstelle enthalten, und ihr durch $y = \varphi(x)$ vermitteltes Bild muß eine Lebesguesche Nullmenge sein.

[22]) $\overline{\mathfrak{M}}$ soll Φ-meßbar sein, d. h. die Funktion, die in $\overline{\mathfrak{M}} = 1$ und in der Komplementärmenge $= 0$ ist, soll es sein. Oder: für jedes monoton nichtfallende und nach rechts halbstetige $\varphi(x)$ soll die Menge der x mit $\varphi(x)$ aus $\overline{\mathfrak{M}}$ im Lebesgueschen Sinne meßbar sein.

[23]) Man kann bekanntlich die obere und die untere Grenze des Realteils sowie des Imaginärteils aller $\varphi(x_1') - \varphi(x_1'') + \cdots + \varphi(x_n') - \varphi(x_n'')$ mit $x_1' < x_1'' < \cdots < x_n' < x_n'' < x$ (x fest, alles andere variabel) bilden, und diese bzw. gleich $\psi_1(x) - \Re\varphi(-\infty)$, $-\psi_2(x)$, $\psi_3(x) - \Im\varphi(-\infty)$, $-\psi_4(x)$ setzen. Das sind dann die gewünschten Funktionen.

[24]) Hilbert, Gött. Nachr., 1906, insbesondere Seite 189—190, Satz II. Unsere Bezeichnungen weichen etwas von den Hilbertschen ab, sie sind die in E und A verwendeten. Vgl. insbesondere E, Seite 91—92, Definition 17 und Satz 36, ferner Seite 114, Anhang II, Satz 9*.

[25]) Der Integrand, λ, ist ja stetig, und $(E(\lambda) f, g)$ von beschränkter Schwankung. Das letztere folgt daraus, daß

$$\Re(E(\lambda) f, g) = \left(E(\lambda)\frac{f+g}{2}, \frac{f+g}{2}\right) - \left(E(\lambda)\frac{f-g}{2}, \frac{f-g}{2}\right)$$

und $\Im(E(\lambda) f, g)$ entsprechend mit ig statt g ist, also $(E(\lambda) f, g)$ ein Aggregat von vier $(E(\lambda)h, h)$ mit den Koeffizienten ± 1, $\pm i$; und $(E(\lambda)h, h) = |E(\lambda)h|^2$ ist in λ monoton nichtfallend und beschränkt (vgl. E, Seite 114).

[26]) Vgl. A, Seite 418, Anhang I.

[27]) Vgl. E, Seite 91, Definition 17. $E \leqq F$ bedeutet $|Ef| \leqq |Ff|$ für alle f, $E_n \to F$ bedeutet $E_n f \to Ef$ für alle f (im Sinne der „starken" Konvergenz).

[28]) Da $\mathfrak{S}$ immer beschränkt ist, ist dies sicher der Fall, wenn $F(x)$ in jedem endlichen Intervalle beschränkt ist. $F(x) = x$ tut dies z. B.

[29]) Für $E \leqq F$ gilt allgemein $|Ef|^2 - |Ff|^2 = |(E-F)f|^2$, vgl. E, Seite 78.

[30]) $E(\lambda_n) - E(\lambda_{n-1}) = E$ ist ein Projektionsoperator, und für diese gilt

$$|(Ef, g)| \leqq |Ef| \cdot |Eg|$$

allgemein (vgl. E, Seite 93).

[31]) Vgl. das Zitat und den Beweis in E, Seite 94, Anm. [52]).

[32]) Für ein reelles $F(x)$ folgt also hieraus, daß $F(A)$ Hermitesch ist. Auf Grund des in 1. Gesagten genügt es sogar, wenn $F(x)$ im Spektrum von A, in $\mathfrak{S}$, reell ist.

[33]) Da $B = H(A)$ in $G(B)$ Hermitesch sein muß, und $y = H(x)$ in $G(y)$ reell, muß $H(x)$ stets reell sein (vgl. Anm. [32])). Nach Satz 2 wird $F(x)$ Φ-meßbar sein, wenn $G(x)$, $H(x)$ es sind.

[34]) Genau wie in Anm. [21]) erkennt man, daß dies mit folgendem gleichbedeutend ist: die Komplementärmenge von $\mathfrak{M}$ (ihr allgemeines Element heiße ξ) darf, nachdem sie der Abbildung $x = \varrho^{-1}(\xi)$ unterworfen wurde, keine Unstetigkeitsstelle von $\Phi_f(x)$ enthalten, also sie selbst keine von $|E(\xi)f|^2$ (was für alle f der Fall ist, wenn sie zum Punktspektrum $\mathfrak{P}$ von A fremd ist) — ferner muß ihr durch $x = \varrho^{-1}(\xi)$ und $y = \Phi_f(x)$, d. h. ihr durch $y = |E(\xi)f|^2$ vermitteltes Bild muß eine Lebesguesche Nullmenge sein. Da diese Funktion in den Komplementärintervallen des Spektrums stets konstant ist, brauchen wir nur die im Spektrum $\mathfrak{S}$ von A liegenden Teile der Komplementärmenge von $\mathfrak{M}$ zu berücksichtigen (vgl. auch 1.). Wenn also $\mathfrak{M}$ ganz $\mathfrak{S}$ umfaßt, so ist alles erfüllt.

[35]) Vgl. A, Seite 383.

[36]) Vgl. E, Seite 65, Bedingung C.

[37]) Sei $F_n(A)f = f_n$, $F(A)f = \bar{f}$, dann ist zu zeigen: wenn „schwach" $f_n \to \bar{f}$ ist, und $|f_n| \to |\bar{f}|$, so gilt $f_n \to \bar{f}$ „stark" (zur Terminologie vgl. z. B. A, Seite 378—379). Es ist nämlich:

$$|f_n - \bar{f}|^2 = (f_n - \bar{f}, f_n - \bar{f}) = |f_n|^2 + |\bar{f}|^2 - 2\Re(f_n, \bar{f}) \to |\bar{f}|^2 + |\bar{f}|^2 - 2\Re(\bar{f}, \bar{f}) = 0,$$

also wirklich „stark" $f_n \to \bar{f}$.

[38]) Die Prämisse von 2., h) wird durch die Prämisse von γ) geliefert. Zwar erfolgt in 2., h) (bei gegebenem A und $F_n(x)$, $F(x)$) der Schluß von allen f auf alle f, aber man erkennt mühelos, daß ebensogut von einem gegebenen f (ohne Rücksicht auf die übrigen) auf dieses f geschlossen werden kann.

[39]) Er lautet so: Sei $\eta > 0$, $\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{N}$ nach Lebesgue meßbare Mengen, die Masse der $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ alle $\geqq \eta$, das Maß von $\mathfrak{N}$ endlich, $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ alle Teilmengen von $\mathfrak{N}$. Dann hat die Menge aller Punkte, die unendlich vielen unter den $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ angehören, ein Maß $\geqq \eta$. Vgl. z. B. de la Vallée Poussin, Cours d'Analyse infinitésimale, Ausg. 1/2 (Louvain-Paris, 1909), Seite 68—69.

[40]) Unter Voraussetzung der Richtigkeit der sog. Kontinuumhypothese (wonach jede unabzählbare Zahlenmenge die Mächtigkeit des Kontinuums hat) läßt sich ein Beispiel

einer Φ-meßbaren Funktion mit einer Inversen, die nicht Φ-meßbar ist, angeben. Wir gehen hier nicht näher darauf ein. (Wird die Φ-Meßbarkeit durch gewöhnliche Meßbarkeit ersetzt, so existiert ein Gegenbeispiel gewiß.)

[41]) Es ist leicht beweisbar, daß man sich ebensogut auf überall stetige $f_1(u), f_2(u), \cdots$ beschränken könnte.

[42]) Als nullte Klasse definieren wir jetzt die Gesamtheit aller Funktionen, die nur von endlich vielen Variablen tatsächlich abhängen, und von diesen stetig. Den Grenzprozeß interpretieren wir im soeben angegebenen weiteren Sinne. So wird z. B. das durch

$$F(x_1, x_2, \cdots) = \begin{cases} x_1 + x_2 + \cdots, & \text{wenn diese Reihe konvergiert} \\ 0 & \text{sonst} \end{cases}$$

definierte $F(x_1, x_2, \cdots)$ zur ersten Klasse gehören.

[43]) Dies ist eine Art der Peanoschen Kurve, die von Steinhaus (Studia Mathematica, Bd. II, Seite 24—27, 1930) für solche Zwecke verwendet wurde. Sie beruht auf der dyadischen Entwicklung $z = 0, \zeta_1 \zeta_2 \cdots = \sum_1^\infty{}_\nu \frac{\zeta_\nu}{2^\nu}$, $\zeta_\nu = 0, 1$, aber unendlich oft 0, die für alle Zahlen $0 \leqq z < 1$ eindeutig möglich ist, und wird durch

$$y_1 = 0, \eta_1^{(1)} \eta_2^{(1)} \cdots, \quad \cdots, \quad y_n = 0, \eta_1^{(n)} \eta_2^{(n)} \cdots, \quad y = 0, \eta_1^{(1)} \cdots \eta_1^{(n)} \eta_2^{(1)} \cdots \eta_2^{(n)} \cdots$$

definiert. Sie bildet den ganzen Würfel $0 \leqq y_1 < 1, \cdots, 0 \leqq y_n < 1$ ein-eindeutig auf alle Punkte $y = 0, \xi_1 \xi_2 \cdots$ des Intervalles $0 \leqq y < 1$ ab, für die jede der Folgen $\xi_k, \xi_{k+n}, \xi_{k+2n}, \cdots$ $(k = 1, \cdots, n)$ unendlich viele 0-en enthält — d. h. auf alle Punkte mit Ausnahme einer Lebesgueschen Nullmenge. Daß nun diese Abbildung die erwähnte Maß-erhaltende Eigenschaft hat, beweist man unschwer analog zu den Überlegungen, die Steinhaus a. a. O. anstellt.

[44]) Auch diese Abbildung läßt sich am einfachsten mit Hilfe der dyadischen Entwicklung angeben. Wenn die Zahl der $x_1, x_2, \cdots$ endlich ist, etwa m, so sei

$$x_1 = 0, \xi_1^{(1)} \xi_2^{(1)} \cdots, \quad \cdots, \quad x_m = 0, \xi_1^{(m)} \xi_2^{(m)} \cdots, \quad x = 0, \xi_1^{(1)} \cdots \xi_1^{(m)} \xi_2^{(1)} \cdots \xi_2^{(m)} \cdots,$$

ist sie aber unendlich, so sei

$$x_1 = 0, \xi_1^{(1)} \xi_2^{(1)} \cdots, \quad x_2 = 0, \xi_1^{(2)} \xi_2^{(2)} \cdots, \quad \cdots, \quad x = 0, \xi_1^{(1)} \xi_1^{(2)} \xi_2^{(1)} \xi_1^{(3)} \xi_2^{(2)} \xi_3^{(1)} \cdots.$$

Man verifiziert mühelos alle im Text formulierten Eigenschaften.

THE FOURIER TRANSFORM IDENTITY THEOREM.[1]

By Andrew C. Berry.[2]

It is known that a function $f(x)$ of integrable square possesses a Fourier transform $F(x)$ also of integrable square and that the Fourier transform of $F(x)$ is precisely $f(-x)$. The present paper establishes this identity under more general assumptions and makes use of the Titchmarsh theory only for the derivation of a corollary to the main result. The one-dimensional case will be discussed in detail, and the method of extending the results to n dimensions indicated.

1. Throughout this paper "region" shall mean "point-set, one-dimensional or n-dimensional, measurable in the sense of Lebesgue". A function shall be said to be integrable if it is integrable in the sense of Lebesgue over all finite regions. To simplify the presentation we introduce the following notations.

Definition 1. *An integrable function $f(x)$ will be said to be in L_ϱ for some ϱ such that $1 \leqq \varrho < \infty$ if there exists, as a finite number,*

$$l_\varrho(f) = \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^\varrho \, dx \right\}^{\frac{1}{\varrho}}.$$

Definition 2. *An integrable function $f(x)$ will be said to be in L_∞ if there exists, as a finite number,*

$$l_\infty(f) = \textit{upper measurable bound of } |f(x)|,$$

i. e. a number such that the sets of points x for which

$$|f(x)| > l_\infty(f) + \varepsilon, \qquad |f(x)| > l_\infty(f) - \varepsilon$$

are respectively of zero measure and of measure greater than zero for all $\varepsilon > 0$.

Definition 3. *The notation*

$$f(x) = \underset{\nu \to \infty}{\mathrm{l.m.}}\,(\varrho)\; f_\nu(x)$$

shall mean that $f(x)$ and all the functions $f_\nu(x)$ are in the same L_ϱ for some ϱ such that $1 \leqq \varrho \leqq \infty$, and that

[1] Received September 5, 1930.

[2] National Research Fellow.

$$\lim_{\nu\to\infty} l_\varrho(f-f_\nu) = 0.$$

DEFINITION 4. *For any integrable* $f(x)$ *we shall set*

$$f^{(\nu)}(x) = \begin{cases} f(x), & -\nu < x < \nu, \\ 0, & \textit{otherwise}. \end{cases}$$

THEOREM 1a. *If* $f(x)$ *is in some* L_ϱ *where* $1 \leqq \varrho < \infty$, *then*

$$f(x) = \underset{\nu\to\infty}{\mathrm{l.m.}}\,(\varrho)\; f^{(\nu)}(x).$$

Proof. This theorem is an immediate consequence of the additivity of the Lebesgue integral as a function of the region over which it is extended.

THEOREM 1b. *If* $f(x)$ *is in* L_∞ *and if* $\lim_{|x|\to\infty} f(x) = 0$, *then*

$$f(x) = \underset{\nu\to\infty}{\mathrm{l.m.}}\,(\infty)\; f^{(\nu)}(x).$$

DEFINITION 5. *We set*

$$\varrho' = \begin{cases} \dfrac{\varrho}{\varrho-1}, & 1 < \varrho < \infty, \\ \infty, & \varrho = 1, \\ 1, & \varrho = \infty. \end{cases}$$

DEFINITION 6. *We shall set*

$$Q(f, g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

when this expression exists.

We recall the following theorem from the theory of convergence in mean.

THEOREM 2. *If, for some* ϱ *such that* $1 \leqq \varrho \leqq \infty$,

$$f(x) = \underset{\nu\to\infty}{\mathrm{l.m.}}\,(\varrho)\, f_\nu(x), \qquad g(x) = \underset{\nu\to\infty}{\mathrm{l.m.}}\,(\varrho')\, g_\nu(x),$$

then

$$Q(f, g) = \lim_{\nu\to\infty} Q(f_\nu, g_\nu).$$

2. We turn now to the study of a class Φ whose elements are the functions

$$g(x) = g_{a,x_0}(x) = \begin{cases} 2a - |x-x_0|, & |x-x_0| < 2a, \\ 0, & \text{otherwise}. \end{cases} \quad \left(\begin{matrix} a > 0, \\ -\infty < x_0 < \infty \end{matrix}\right).$$

It is readily seen that

$$g_{a,x_0}(x) = \int_{-\infty}^{\infty} \varphi_a\left(t - \frac{x_0}{2}\right) \varphi_a\left(x - \frac{x_0}{2} - t\right) dt$$

where

$$\varphi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & \text{otherwise.} \end{cases}$$

Noting that the functions φ_a differ from zero only in finite regions we can justify the inversion of an order of integration and the replacement of an iterated integral by a double integral and so find that

$$Q(f, g_{a,x_0}) = \frac{1}{\sqrt{2\pi}} \int_{\frac{x_0}{2}-a,\, \frac{x_0}{2}-a}^{\frac{x_0}{2}+a,\, \frac{x_0}{2}+a} \int f(x+y)\, dx\, dy$$

for any integrable $f(x)$.

THEOREM 3. *If $f(x)$ is integrable and if $Q(f, g) = 0$ for all functions g in Φ, then $f(x) = 0$ almost everywhere.*

Proof. If $f(x_0) \neq 0$, then $f(x+y)$ as a function of the two variables x, y differs from zero along the line $x+y = x_0$. Hence if $f(x) \neq 0$ on a set of one-dimensional measure greater than zero, $f(x+y) \neq 0$ on a set of two-dimensional measure greater than zero. The latter situation is impossible since, under the hypotheses of the theorem the double integral of $f(x+y)$ must vanish over all squares.

THEOREM 4. *If we set*

$$G(u) = G_{a,x_0}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_{a,x_0}(x)\, e^{-ixu}\, dx,$$

then

$$g(-x) = g_{a,x_0}(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_{a,x_0}(u)\, e^{-ixu}\, du,$$

and

$$G(u) = \underset{\nu\to\infty}{\text{l. m.}}\, (\varrho)\, G^{(\nu)}(u)$$

for all ϱ, $1 \leqq \varrho \leqq \infty$.

Proof. We readily make the calculation

$$G(u) = \frac{1}{\sqrt{2\pi}} \left(\frac{2 \sin a u}{u}\right)^2 e^{-ix_0 u}.$$

From the trigonometric identity

$$2 \sin^2 a \cos 2b = \sin^2(a+b) + \sin^2(a-b) - 2\sin^2 b$$

and the fact that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a u}{u}\right)^2 du = |a|$$

we deduce the result

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(u)\, e^{-ixu}\, du$$

$$= \left|a+\frac{x+x_0}{2}\right| + \left|a-\frac{x+x_0}{2}\right| - |x+x_0|$$

$$= \begin{cases} 2a-|x+x_0|, & |x+x_0|<2a, \\ 0 & , \text{ otherwise,} \end{cases}$$

$$= g_{a,x_0}(-x) = g(-x).$$

Finally we note that the hypotheses of theorems 1 a and 1 b are satisfied for G. This completes the proof of the present theorem.

3. To establish the identity theorem it is essential to use some such precise definition as the following.

DEFINITION 7. *An integrable function $f(x)$ shall be said to have a Fourier transform $F(u)$ in L_ϱ for some ϱ such that $1 \leqq \varrho \leqq \infty$, if*

$$F(u) = \underset{\nu\to\infty}{\text{l. m.}}\,(\varrho)\, F_\nu(u)$$

where

$$F_\nu(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{(\nu)}(x)\, e^{-ixu}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\nu}^{\nu} f(x)\, e^{-ixu}\, dx.$$

THEOREM A. *If f has a Fourier transform F in some L_ϱ, $1 \leqq \varrho \leqq \infty$, if g is an arbitrary function of Φ and if $g_-(x)$ denotes $g(-x)$, then*

$$Q(g_-, f) = Q(G, F).$$

Proof.

$$Q(g_-^{(\nu)}, f) = \frac{1}{\sqrt{2\pi}}\int_{-\nu}^{\nu} f(x)\, g(-x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\nu}^{\nu} f(x)\, dx \int_{-\infty}^{\infty} G(u)\, e^{-ixu}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u)\, du \int_{-\nu}^{\nu} f(x)\, e^{-ixu}\, dx$$

$$= Q(G, F_\nu).$$

The inversion of the order of integration is justified by the absolute integrability of $G(u)$ over $(-\infty, \infty)$. On the one hand, the fact that g

differs from zero only on a finite interval and, on the other hand, theorems 1a, 1b and 4 enable us to proceed to the limit as $\nu \to \infty$ and thus establish the present theorem.

THEOREM B. *If F is in some L_ϱ, $1 \leqq \varrho \leqq \infty$, and has a Fourier transform $\mathfrak{F}$ in some L_μ, $1 \leqq \mu \leqq \infty$, and if g is an arbitrary function in Φ, then*

$$Q(G, F) = Q(g, \mathfrak{F}).$$

Proof.

$$\begin{aligned} Q(G^{(\nu)}, F) &= \frac{1}{\sqrt{2\pi}} \int_{-\nu}^{\nu} F(u)\, G(u)\, du \\ &= \frac{1}{2\pi} \int_{-\nu}^{\nu} F(u)\, du \int_{-\infty}^{\infty} g(x)\, e^{-ixu}\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)\, dx \int_{-\nu}^{\nu} F(u)\, e^{-ixu}\, du \\ &= Q(g, \mathfrak{F}_\nu). \end{aligned}$$

Here the change in order of integration is justified by the fact that all the integrals involved are finite integrals. Again we may proceed to the limit and establish our theorem.

THEOREM C. *If f has a Fourier transform F in some L_ϱ, $1 \leqq \varrho \leqq \infty$, and if F has a Fourier transform $\mathfrak{F}$ in some L_μ, $1 \leqq \mu \leqq \infty$, then $\mathfrak{F}(x) = f(-x)$ almost everywhere.*

Proof. By Theorems A and B,

$$Q(g_-, f) = Q(g, \mathfrak{F}),$$

i. e.

$$Q(g, \mathfrak{F} - f_-) = 0$$

for all g in Φ. By Theorem 3, then, $\mathfrak{F}(x) = f_-(x) = f(-x)$ almost everywhere.

This theorem and the known fundamental result of the Titchmarsh Fourier transform theory,[3] namely that if f is in some L_ϱ, $1 \leqq \varrho \leqq 2$, it automatically possesses a Fourier transform in the corresponding $L_{\varrho'}$, establish the following

COROLLARY. *If F possesses a Fourier transform f in some L_ϱ, $1 \leqq \varrho \leqq 2$, then F is in $L_{\varrho'}$, and F_- is the Fourier transform in $L_{\varrho'}$ of f.*

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[3] E. C. Titchmarsh, A contribution to the theory of Fourier transforms, Proc. London Math. Soc. (2), 23 (1924), 279–289. For the case $\varrho = 1$ which is not discussed here see A. C. Berry, The Fourier transform theorem in n dimensions, which is to appear in the London Math. Soc. Proc.

4. All results stated extend immediately to n dimensions, the class Φ now having as its elements the functions

$$
\begin{aligned}
g(x_1, \cdots, x_n) &= g_{a, x_{10}, \cdots, x_{n0}}(x_1, \cdots, x_n) \\
&= \prod_{j=1}^{n} \begin{Bmatrix} 2a - |x_j - x_{j0}|, & |x_j - x_{j0}| < 2a, \\ 0 & , \text{ otherwise}, \end{Bmatrix} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_a\left(t_1 - \frac{x_{10}}{2}, \cdots, t_n - \frac{x_{n0}}{2}\right) \\
&\qquad \times \varphi_a\left(x_1 - \frac{x_{10}}{2} - t_1, \cdots, x_n - \frac{x_{n0}}{2} - t_n\right) dt_1 \cdots dt_n
\end{aligned}
$$

where

$$
\varphi_a(x_1, \cdots, x_n) = \begin{cases} 1, & \text{all } |x_j| < a, \\ 0, & \text{otherwise.} \end{cases}
$$

After the analogues of Theorems 3 and 4 are established all reference to any Cartesian coördinates may be dropped and vectorial methods employed.

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A NOTE ON THE THEORY OF INFINITE SERIES.[1]

By M. H. Stone.

Among the theorems which deal with the behavior of general trigonometric series, some of the most interesting are the elementary ones connecting convergence properties of the series with asymptotic properties of the coefficients. The four theorems which suggested the results embodied in this note read as follows:

Theorem A. (Cantor).[2] *If the series* $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *converges everywhere on the interval* $0 \leqq x \leqq 2\pi$, *then* a_n *and* b_n *tend to zero with* $1/n$.

Theorem B. (Lebesgue).[3] *If* a_n *and* b_n *do not both tend to zero with* $1/n$, *then the series* $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *diverges almost everywhere on the interval* $0 \leqq x \leqq 2\pi$.

Theorem C. (Fatou).[4] *If the series* $\sum_{n=1}^{\infty}|a_n \cos nx + b_n \sin nx|$ *converges on any closed interval, then the series* $\sum_{n=1}^{\infty}(|a_n| + |b_n|)$ *converges.*

Theorem D. (Denjoy-Lusin).[5] *If the series* $\sum_{n=1}^{\infty}(|a_n| + |b_n|)$ *diverges, then the series* $\sum_{n=1}^{\infty}|a_n \cos nx + b_n \sin nx|$ *diverges almost everywhere on the interval* $0 \leqq x \leqq 2\pi$.

It is evident that Theorem A follows from Theorem B, Theorem C from Theorem D. In those discussions of Theorems B and D which I have consulted, the method of proof consists in the application of a suitable elementary property of the trigonometric functions selected *ad hoc*. Theorem B is ordinarily established on the basis of a property summarized in its sharpest form by the equation $\limsup_{n\to\infty}|a_n \cos nx + b_n \sin nx| = \limsup_{n\to\infty}\sqrt{a_n^2 + b_n^2}$, which holds almost everywhere on the interval $0 \leqq x \leqq 2\pi$.[6] The demonstration of Theorem D is made to depend upon

[1] Received October 30, 1930.

[2] Georg Cantor, Journal für Mathematik, 57 (1870), pp. 130–138.

[3] Lebesgue, Leçons sur les séries trigonométriques, p. 110.

[4] Fatou, Acta Mathematica, 30 (1906), pp. 335–400, especially pp. 399–400.

[5] Denjoy, Comptes rendus, 155 (1912), pp. 135–136; Lusin, 155 (1912), pp. 580–582. Lusin had published his results in an earlier Russian paper antedating Denjoy's.

[6] Steinhaus, Wiadomości Matematyczne, 24 (1920), pp. 197–201.

an appraisal of the integral $\int_e |\cos nx|\,dx$ taken over a measurable set e. As a matter of fact, this integral behaves in such a manner that it can be used in constructing a proof of Theorem B as well. In view of the fact that both Theorems B and D thus have the same source, it seemed desirable to examine more carefully the logical relations connecting all four theorems. The four theorems in the theory of infinite series which resulted from this examination are the subject-matter of the present note.

Let E be a bounded Lebesgue-measurable set of real numbers and let $\{u_n(x)\}$ be a uniformly bounded sequence of Lebesgue-measurable functions defined on the set E. We shall denote by M the least upper bound of the set of numbers $|u_n(x)|$; by e an arbitrary Lebesgue-measurable subset of E; by $\mu(e)$ the measure of the set e; and by $\lambda(e)$ the number $\liminf_{n\to\infty} \int_e |u_n(x)|\,dx \geqq 0$. In order to state our theorems in simple form it will be convenient for us to introduce certain preliminary definitions. With a view to their immediate usefulness we express the first and third definitions in terms of an arbitrary set e, the second and fourth in terms of the given set E. This differentiation in form is superficial rather than substantial.

Definition 1. *The sequence* $\{u_n\}$ *is said to have the Cantor property on a set* e *if, whenever the series* $\sum_{n=1}^{\infty} a_n u_n$ *converges almost everywhere on* e, *the coefficient* a_n *tends to zero with* $1/n$.

Definition 2. *The sequence* $\{u_n\}$ *is said to have the Lebesgue property on the set* E *if, whenever* $\{a_n\}$ *is a sequence for which* $\limsup_{n\to\infty} |a_n|$ *differs from zero, the series* $\sum_{n=1}^{\infty} a_n u_n$ *diverges almost everywhere on* E.

Definition 3. *The sequence* $\{u_n\}$ *is said to have the Fatou property on a set* e, *if whenever the series* $\sum_{n=1}^{\infty} |a_n u_n|$ *converges almost everywhere on* e, *the series* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

Definition 4. *The sequence* $\{u_n\}$ *is said to have the Denjoy-Lusin property on the set* E, *if, whenever the series* $\sum_{n=1}^{\infty} |a_n|$ *diverges, the series* $\sum_{n=1}^{\infty} |a_n u_n|$ *diverges almost everywhere on the set* E.

We shall now discuss necessary and sufficient conditions for each of these four properties in as many corresponding theorems.

Theorem 1. *A necessary and sufficient condition that the sequence* $\{u_n\}$ *have the Cantor property on a set* e *of positive measure is that* $\lambda(e)$ *be positive.*

The condition is sufficient. In order that the series $\sum_{n=1}^{\infty} a_n u_n$ converge almost everywhere on the set e it is necessary that the sequence $\{|a_n|\,|u_n|\}$ and hence also the sequence $\left\{\frac{|a_n|}{1+|a_n|}|u_n|\right\}$ should converge almost everywhere on the set e to the limit zero. By virtue of the inequality $0 \leqq \frac{|a_n|}{1+|a_n|}|u_n| \leqq |u_n| \leqq M$ the second sequence can be integrated term-by-term with the result that $\lim_{n\to\infty} \frac{|a_n|}{1+|a_n|}\int_e |u_n|\,dx = 0$. In view of our assumption that $\lambda(e)$ is positive, we must have $\limsup_{n\to\infty}\frac{|a_n|}{1+|a_n|} = 0$. This relation implies that a_n tends to zero with $1/n$.

The condition is necessary. If $\lambda(e)$ vanishes, there exists a sequence of integers $\{n(r)\}$ such that

$$n(r+1) > n(r), \qquad \int_e |u_{n(r)}|\,dx < 1/4^r, \qquad r = 1, 2, 3, \cdots.$$

The set of points e_r contained in e such that $|u_{n(r)}| \geqq 1/2^r$ is a measurable set; by a familiar argument its measure cannot exceed $1/2^r$. We define the set $e_\nu^* = \sum_{\nu+1}^{\infty} e_r$; it is measurable and its measure satisfies the inequality $\mu(e_\nu^*) \leqq \sum_{\nu+1}^{\infty} \mu(e_r) \leqq 1/2^\nu$. Finally we denote by e^* the limiting set of the sequence $\{e - e_\nu^*\}$; since $e_{\nu+1}^*$ is a subset of e_ν^*, the set e^* exists and is a measurable set whose measure is determined from the relations $\mu(e^*) = \lim_{\nu\to\infty} \mu(e - e_\nu^*) = \mu(e) - \lim_{\nu\to\infty}\mu(e_\nu^*) = \mu(e)$. If x is an arbitrary point in e^*, then there exists an integer $\nu = \nu(x)$ such that x belongs to $e - e_\nu^*$ and is therefore in none of the sets $e_{\nu+1}, e_{\nu+2}, e_{\nu+3}, \cdots$. Consequently, for this value of x and for $r = \nu+1, \nu+2, \nu+3, \cdots$, the function $|u_{n(r)}(x)|$ is less than $1/2^r$. It is now clear that the series $\sum_{r=1}^{\infty} u_{n(r)}(x)$ converges at every point x of the set e^*, in spite of the fact that its coefficients do not tend to zero with $1/n$. Since $\mu(e^*)$ is equal to $\mu(e)$, the sequence $\{u_n\}$ does not have the Cantor property on the set e.

THEOREM 2. *A necessary and sufficient condition that the sequence $\{u_n\}$ have the Lebesgue property on the set E is that $\lambda(e)$ be positive whenever $\mu(e)$ is positive.*

This theorem can be restated in the form: a necessary and sufficient condition that the sequence $\{u_n\}$ have the Lebesgue property on the set E is that it have the Cantor property on every set e of positive measure. When the theorem is viewed in this light, its truth is evident.

THEOREM 3. *A necessary and sufficient condition that the sequence $\{u_n\}$ have the Fatou property on a set e of positive measure is that $\lambda(e)$ be positive.*

The condition is sufficient. Let the series $\sum_{n=1}^{\infty} |a_n u_n|$ converge almost everywhere on e and suppose that the series $\sum_{n=1}^{\infty} |a_n|$ diverges. Then there exists an integer l such that $|a_l| > 0$, $\int_e |u_n| \, dx > \lambda(e)/2$ for $n \geqq l$. We form the sequence of functions $\{v_m\}$ where $v_m(x) = \sum_{n=l}^{n=m} |a_n| |u_n| / \sum_{n=l}^{n=m} |a_n|$. Since v_m is a weighted mean of the functions $|u_l|, \cdots, |u_m|$, none of which exceeds M in value, it must satisfy the inequality $0 \leqq v_m \leqq M$. Our hypotheses concerning the two sums which appear in the definition of v_m require that v_m should tend to zero with $1/m$ almost everywhere on e. In consequence the integral $\int_e v_m \, dx$ tends to zero with $1/m$. On the other hand, this integral is evidently a weighted average of the integrals $\int_e |u_n| \, dx$, $n = l, \cdots, m$, each of which exceeds $\lambda(e)/2$, and must itself have a value greater than $\lambda(e)/2$. If $\lambda(e)$ is positive we have a contradiction and are compelled to discard our assumption that the series $\sum_{n=1}^{\infty} |a_n|$ is divergent.

The condition is necessary. In the proof of Theorem 1 we showed that whenever $\lambda(e)$ vanishes it is possible to construct a series of the form $\sum_{r=1}^{\infty} u_{n(r)}$ which converges absolutely at almost every point of e. This series obviously does not have the behavior required by the Fatou property.

THEOREM 4. *A necessary and sufficient condition that the sequence* $\{u_n\}$ *have the Denjoy-Lusin property on the set* E *is that* $\lambda(e)$ *be positive whenever* $\mu(e)$ *is positive.*

This theorem can be restated in a form in which it is evident: a necessary and sufficient condition that the sequence $\{u_n\}$ have the Denjoy-Lusin property on the set E is that it have the Fatou property on every set e of positive measure.

By observing the manner in which the number $\lambda(e)$ enters into each of these four theorems we can ascertain at once the logical relations between them. If we use the notation $\alpha \to \beta$, where α and β are any two distinct integers chosen from the set 1, 2, 3, 4, to signify that when the sequence $\{u_n\}$ has the property of Definition α on the set E it necessarily has the property of Definition β on the set E, we can represent these relations by the scheme

$$\begin{array}{ccc} 1 & \rightleftarrows & 3 \\ \uparrow & & \uparrow \\ 2 & \rightleftarrows & 4. \end{array}$$

The further relations between 1 and 2 and those between 3 and 4 are stated explicitly above; they do not lend themselves to such a diagrammatic representation.

Among various modifications and generalisations of these four theorems which can be formulated, there is one which is of sufficient interest to require some comment. It is evident that the rôle played by the number $\lambda(e)$ can be assigned quite as well to any other number $\lambda^*(e)$ which has the property that the equations $\lambda(e) = 0$ and $\lambda^*(e) = 0$ are equivalent. For example, if $\Theta(z)$ is a continuous convex function defined on the interval $0 \leqq z \leqq M$ which vanishes for $z = 0$ and assumes positive values elsewhere, then the number $\lambda^*(e) = \liminf_{n \to \infty} \int_e \Theta(|u_n|)\,dx$ has the property in question: for the well-known inequalities[7]

$$0 \leqq \Theta(z) \leqq \frac{\Theta(M)}{M} z, \qquad \Theta\left(\frac{\int_e |u_n|\,dx}{\mu(e)}\right) \leqq \frac{\int_e \Theta(|u_n|)\,dx}{\mu(e)}$$

enable us to write

$$\Theta\left(\frac{\lambda(e)}{\mu(e)}\right) \leqq \frac{\lambda^*(e)}{\mu(e)} \leqq \frac{\Theta(M)\,\lambda(e)}{M\,\mu(e)}.$$

In practice the special case $\Theta(z) = z^2$ is particularly important.

The application of our four theorems to the theory of trigonometric series is now obvious. By writing $a_n \cos nx + b_n \sin nx = A_n \cos n(x - \alpha_n)$, we see that every trigonometric series can be treated as a series of functions of the type $\cos n(x - \alpha_n)$ and conversely. Thus the study of trigonometric series reduces to the study of all sequences $\{\cos n(x - \alpha_n)\}$ and leads therefore to special instances of the general results given in Theorems 1, 2, 3, 4: these results are stated in Theorems A, B, C, D respectively; their logical relations must conform to the remarks made above. The actual proof of Theorems A, B, C, D must rest upon an appraisal of the number $\lambda(e)$ associated with the sequence $\{\cos n(x - \alpha_n)\}$ and an arbitrary measurable subset of the interval $0 \leqq x \leqq 2\pi$. It is not difficult to establish the following exact result: *if* $\{\alpha_n\}$ *is an arbitrary sequence of real numbers,* $\{\lambda_n\}$ *a sequence of positive real numbers such that* $1/\lambda_n$ *tends to zero with* $1/n$, *and e an arbitrary set of finite measure, then*

$$\lim_{n \to \infty} \int_e |\cos \lambda_n (x - \alpha_n)|\,dx = 2\mu(e)/\pi.$$

It is still easier, however, to apply the remarks of the preceding paragraph, for we can show at once that

[7] Jensen, Acta Mathematica, 30 (1906), pp. 175-193.

$$\lim_{n\to\infty} \int_e \cos^2 \lambda_n (x - \alpha_n)\, dx$$

$$= \lim_{n\to\infty} \int_e (1 - \cos 2\lambda_n (x - \alpha_n))\, dx/2 = \mu(e)/2$$

by means of the theorem of Riemann-Lebesgue.[8]

Another specialization of our general theorems which is of some interest is the following result: *if the sequence* $\{u_n\}$ *is a uniformly bounded normal orthogonal set of functions on the set* E *then it has the Cantor property and the Fatou property on the set* E. This result is an immediate consequence of the equation $\int_E u_n^2\, dx = 1$. It is clear from very simple examples that such a sequence does not need to possess the Lebesgue and Denjoy-Lusin properties.

[8] Lebesgue, Leçons sur les séries trigonométriques, pp. 60–61.

THE UNIFORM APPROXIMATION OF A SUMMABLE FUNCTION BY STEP FUNCTIONS.[1]

By R. L. Jeffery.

1. **Introduction.** Let R be a closed and bounded domain in n dimensions, and $L_i = L_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ the linear section of R for each fixed $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. Let the function f be summable on R, and a summable function for each $L_i (i = 1, 2, \cdots, n)$. It is of some interest to know when there exists a step function K such that for a given $\epsilon > 0$ we have

$$\text{(A)} \qquad \int_{L_i} |f - K|\, dx < \epsilon \qquad (i = 1, 2, \cdots, n)$$

uniformly in L_i.[2] If for every ϵ there exists K satisfying (A), then for a given ϵ there exists K such that (A) holds, and at the same time

$$\int_R |f - K|\, dx_1 \cdots dx_n < \epsilon.$$

This follows from the fact that the multiple integral can be computed by repeated integration. In this paper some sufficient conditions for the existence of K are determined, and for a certain class of functions conditions which are both necessary and sufficient. Incidental to these results the necessary and sufficient conditions for the uniform approximation of a summable function by means of a bounded function are obtained.

2. **Approximation of a bounded function.** Let f be bounded and measurable on R, and a measurable function of x_i on each L_i. We prove

THEOREM I. *Let η be an arbitrary positive number and E_η the part of R at which the saltus of f is greater than or equal to η. It is then sufficient for* (A) *that on each L_i $m(L_i E_\eta) = 0$ $(i = 1, 2, \cdots, n)$.*

We note that since E_η is closed the part of this set on each L_i can be put in a finite number of cells[3] c_i so that $m(L_i c_i)$ is arbitrarily small. It is not immediately evident, however, that there is a finite set of cells c containing E_η for which $m(L_i c)$ is arbitrarily small uniformly in L_i. In this connection we prove the following lemma concerning closed sets in general.

[1] Received August 2, 1930. — Presented to the American Mathematical Society, April 18, 1930 (Abstract No. 36-5-226).

[2] This question came up in a paper on Integral Equations by Hille and Tamarkin, read before the society Dec. 27, 1929. See also these Annals, (2) 31, pp. 479–528.

[3] Here, and elsewhere, we shall understand by cell a rectangular domain with sides parallel to the coördinate axes.

LEMMA I. *Let E be any closed set on an n-dimensional domain R. If the measure of the part of E on each L_i is zero, then for a given ϵ the set E can be put in a finite set of open cells c such that $m(L_i c) < \epsilon$ uniformly in L_i $(i = 1, 2, \cdots, n)$.*

Suppose the lemma is false. Then there exists a number $\lambda > 0$ such that for every possible enclosure c there is some L_i for which $m(L_i c) > \lambda$. Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers tending to zero. Put E in c_j where d_j, the greatest diameter of any cell of c_j is less than δ_j. For each j there is some line L_i^j for which $m(L_i^j c_j) > \lambda$. From the infinite sequence of lines L_i^j thus defined there is at least one sub-sequence for which i does not change. This sub-sequence has a limit line L_i, and from this sequence with a fixed i it is possible to select a sub-sequence L_i^k with a single limit L_i. Let $\bar{\beta}_i{}^k$ be the intervals defined on L_i^k by the corresponding sub-sequence c_k of c_j, and β_i^k the projection of $\bar{\beta}_i{}^k$ on L_i. Then $m\beta_i^k > \lambda$ for each k, and there exists on L_i a set of points β with $m\beta > \lambda$ each of which belongs to an infinite number of the sets of intervals β_i^k.[4] The line L_i^k approaches L_i, the diameter of c_k approaches zero, and each c_k contains at least one point of E. From these it follows that each point of β is a limit point of E and consequently a point of E. Thus the part of E on L_i has measure not less than λ. But this is a contradiction, and the lemma follows.

We can now employ Lemma 1 to enclose the set E_η in a finite number of open cells c where $m(L_i c) < \eta$ $(i = 1, 2, \cdots, n)$ uniformly in L_i. Let $\Sigma = R - c$. Then Σ is closed, and at each point of Σ the saltus of f is less than η. Consequently about each point p of Σ there exists a cell σ with p as midpoint for which

$$|f(p) - f(p')| < \eta$$

provided p' is a point of Σ on this cell. From the infinite set of cells thus defined it is possible to choose, by means of the Heine-Borel theorem, a finite set $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_m$ containing the points of Σ. Let p_j be the mid-point of σ_j. On σ_1 let $K(p) = f(p_1)$, and on the part of σ_j not on $\sigma_1, \sigma_2, \cdots, \sigma_{j-1}$ let $K(p) = f(p_j)$ $(j = 2, 3, \cdots, m)$. On $R - \sigma$ let $K = 0$. If then M is the least upper bound of $|f - K|$ on R, it easily follows that

$$\int_{L_i} |f - K|\, dx_i < \eta\, m L_i + \eta M \qquad (i = 1, 2, \cdots, n)$$

uniformly in L_i.

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[4] W. H. Young, Proceedings of the London Mathematical Society, (2), vol. 2, p. 26. Also Borel, Comptes Rendus, December, 1903.

A simple example shows that the conditions of Theorem I are not necessary. Let $f(x, y) = 0$ on the unit square except on the line $y = \frac{1}{2}$. On this line let $f(x, y) = 1$ for x rational, and $f(x, y) = 0$ for x irrational. For $\eta < 1$ every point on the line $y = \frac{1}{2}$ is a point of E_η. Nevertheless the step function $K = 0$ satisfies (A).

As a consequence of a known theorem[5] there exists a perfect sub-set P of R with measure arbitrarily near to that of R relatively to which f is continuous. Since R is closed and P is perfect it follows that $L_i P$ is closed, and consequently measurable. Making use of these facts we prove

THEOREM II. *If for an arbitrary* $\eta > 0$ *the perfect set* P *can be so chosen that* $m L_i - m(L_i P) < \eta$ *uniformly in* L_i *then there exists a step function* K *satisfying* (A).

Since f is continuous at each point of P relatively to this set, there exists for each point p of P a cell with p as mid-point such that

$$(1) \qquad |f(p) - f(p')| < \eta$$

for p' any other point of P on this cell. Since P is perfect we can, by virtue of the Heine-Borel theorem, select from this infinite set of cells containing P a finite set $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_m$ containing all the points of P. On σ_1 let K have the value of f at the mid-point of σ_1, and on the part of σ_j not on $\sigma_1, \cdots, \sigma_{j-1}$ let K have the value of f at the mid-point of σ_j $(j = 2, 3, \cdots, m)$. On $R - \sigma$ let $K = 0$. It then follows from (1) that at every point of P we have

$$|f(p) - K(p)| < \eta.$$

This, and the fact that $m(L_i CP) < \eta$ gives

$$\int_{L_i} |f - K|\, dx_i = \int_{L_i P} |f - K|\, dx_i + \int_{L_i CP} |f - K|\, dx_i < \eta\, m L_i + \eta M,$$

where M is the least upper bound of $|f - K|$. Since η is arbitrary and $m L_i$ is uniformly bounded, the theorem is established.

We now state

LEMMA II. *Let* f *be bounded and measurable on the bounded domain* R. *Let* p *be a point of* R, *and let* ω_k^p *be a sequence of cells with equal sides each of which contains* p, *and such that* $m\,\omega_k^p$ *tends to zero. Let* E *be the part of* R *for which the ratio*

$$\int_{\omega_k^p} f\, dx / m\, \omega_p^k$$

converges to $f(p)$. *Then* $m E = m R$.

[5] See e. g. Hobson, Real Variable, second ed., vol. II, § 179.

This lemma follows immediately from a known theorem.[6] The cell ω_k^p containing p and the smallest cell with p as center containing ω_k^p can be taken as Caratheodory's set $\sigma(p; a_k)$ and cell $q(p; a_k)$ respectively. Then for every k we have $m\sigma(p; a_k)/mq(p; a_k) \geqq 1/2^{n-1}$, and our lemma and the theorem referred to are then identical.

With the existence of the set E mentioned in Lemma II established, it is possible to prove

THEOREM III. *Let f be bounded and measurable on the closed domain R, and such that at each point of R the saltus of f relatively to E is zero. It is then necessary and sufficient for the existence of K satisfying (A) that for each L_i $m(L_iCE) = 0$.*

Since the saltus of f relatively to the set E is zero at each point of R, there exists about each point of R a cell such that for p and p' any two points of E on this cell we have

$$|f(p)-f(p')| < \eta. \tag{1}$$

Since R is closed there exists a finite number of these cells containing all the points of R, and consequently all the points of E. Since $mE = mR$ a cell of this finite set that contains no points of E contains at most a part of R the measure of which is zero. We can, therefore, select from this finite set of cells a set $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_m$ such that σ contains all of E, and all of R except a set of zero measure, and such that each cell σ_j contains at least one point p_j of E. On σ_1 let $K = f(p_1)$, and on the part of σ_j not on $\sigma_1, \sigma_2, \cdots, \sigma_{j-1}$ let $K = f(p_j)$. On $R-\sigma$ let $K = 0$. Then at each point p of E we have from (1)

$$|f(p)-K(p)| < \eta,$$

and since $m(L_iCE) = 0$, this gives

$$\int_{L_i} |f-K|\, dx_i = \int_{L_iE} |f-K|\, dx_i < \eta\, mL_i.$$

Since mL_i is uniformly bounded and η is arbitrary, it follows that the conditions of the theorem are sufficient.

As a first step in proving that the conditions are necessary we show that a point of CE is a point of discontinuity of f relatively to E. Let ω_k^p be any sequence of cells with equal sides containing a point p of CE for which $m\omega_k^p$ converges to zero. Since $mE = mR$ we can find on ω_k^p two points p_k' and p_k'' of E for which

$$m\omega_k^p f(p_k') \leqq \int_{\omega_k^p} f\,dx \leqq m\omega_k^p f(p_k'').$$

[6] See e.g. Carathéodory, Vorlesungen über Reelle Funktionen, Berlin. 1927, § 446, Theorem III.

Divide the members of this inequality by $m\omega_k^p$. If p is a point of continuity of f relatively to E it follows that the first and third members of the inequality thus obtained converge to $f(p)$. Consequently the second member converges to $f(p)$. But this makes p a point of E, which is a contradiction.

If then for some $L_i = L_i'$ $m(L_i' CE) > 0$, it easily follows that there exists a number $d > 0$, a number $\lambda > 0$, and on L_i' a part C_i' of CE such that $m(L_i' C_i') > \lambda$, and for p' a point of C_i' we have

$$(1) \qquad |f(p') - f(p)| > d$$

for all points p of E sufficiently near to p'. Let K be any step function on R. For every point p' of C_i' there is a cell c on which K is constant and for points p of which we have

$$(2) \qquad K(p) = K(p').$$

Since the cells on which K is defined have their sides parallel to the coördinate axes, it is not difficult to show that we can choose from these cells c a set c' containing a part $\bar{C}_i$ of C_i' with $m(L_i' \bar{C}_i) > \lambda/2^{n-1}$, and a sequence of lines L_i^j approaching L_i' such that every line L_i^j is on each cell of c'. For a given η the set of lines L_i for which $m(L_i CE) > \eta$ has zero measure on the space of $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$. If this were not the case it would then be true that $mCE > 0$ on R. It follows from this observation that we can choose the sequence L_i^j so that $m(L_i^j CE) < \eta$. Let $\bar{e}_i^j$ be the projection of $\bar{C}_i$ on E_i^j, the part of E on L_i^j. Then $m\bar{e}_i^j > \lambda/2^{n-1} - \eta$. Let e_i^j be the part of $\bar{e}_i^j$ for which

$$(3) \qquad |f(p_i^j) - f(\bar{p}_i)| > d - \eta'$$

where η' is arbitrary. Since e_i^j belongs to E, and $\bar{p}_i$ to C_i', it follows from (1) and the fact that at each $\bar{p}_i$ the saltus of f relatively to E is zero, that for j sufficiently large and η'' arbitrary we have

$$(4) \qquad m e_i^j > m\bar{e}_i^j - \eta'' > \frac{\lambda}{2^{n-1}} - \eta - \eta''.$$

It is also evident that we have

$$(5) \qquad \int_{L_i^j} |f - K| \, dx_i \geqq \int_{e_i^j} |f - K| \, dx_i,$$

and

$$(6) \qquad \int_{L_i} |f - K| \, dx_i \geqq \int_{\bar{C}_i} |f - K| \, dx_i.$$

But from (2) $K(p_i^j) = K(\bar{p}_i)$. It then follows from (3) and (4) that for ε sufficiently small the right hand side of (5) and (6) cannot both be less than ε for all j sufficiently large.

It is desirable to obtain results which do not require that the saltus of f relatively to E be zero at each point of R. However, the following example seems to indicate that nothing very general in the shape of necessary conditions can be obtained without this or a similar requirement.

On $0 \leqq x \leqq 1$, $0 \leqq y < 1/2$ let $f = 0$. On $0 \leqq x \leqq 1$, $1/2 \leqq y \leqq 1$ let $f = 1$. In this case f is itself a step function. But every point on the line $y = 1/2$ is a point of CE.

3. Approximation of unbounded functions. Let f be summable on R, and summable in x_i for each $L_i \, (i = 1, 2, \cdots, n)$. Then for every f there is a bounded function φ such that for a given $\varepsilon > 0$ we have

$$\int_R |f - \varphi| \, dx < \varepsilon.$$

But unless φ can be determined in such a way that we have

$$\text{(B)} \qquad \int_{L_i} |f - \varphi| \, dx_i < \varepsilon \qquad (i = 1, 2, \cdots, n)$$

uniformly in L_i there will exist for this function f no step function K satisfying (A) of § 1. We are thus led to seek the necessary and sufficient conditions for the existence of a bounded function φ satisfying (B).

Let a_i be the smallest value of x_i on R for a fixed L_i. Let

$$F(x_i) = \int_{a_i}^{x_i} f \, dx.$$

It is well known that $F(x_i)$ is an absolutely continuous function of x_i. We now prove

THEOREM IV. *It is necessary and sufficient for the existence of a bounded function φ satisfying* (B) *that $F(x_i)$ be absolutely continuous uniformly in L_i* $(i = 1, 2, \cdots, n)$.

Suppose the condition does not hold. Then there exists a number $\lambda > 0$ such that for every $\delta > 0$ it is possible to find l_i, a measurable part of some L_i, with $m l_i < \delta$, for which we have

$$\text{(1)} \qquad \left| \int_{l_i} f \, dx_i \right| > \lambda.$$

Let φ be any bounded function on R. Then there exists $\delta' > 0$ for which

$$\text{(2)} \qquad \left| \int_{l_i} \varphi \, dx_i \right| < \frac{\lambda}{2} \qquad (i = 1, 2, \cdots, n)$$

when $m l_i < \delta'$. But

$$\int_{l_i} |f - \varphi| \, dx_i \geqq \left| \int_{l_i} f \, dx_i - \int_{l_i} \varphi \, dx_i \right| \geqq \left| \int_{l_i} f \, dx_i \right| - \left| \int_{l_i} \varphi \, dx_i \right|.$$

From (1) and (2) the last member, and consequently the first member, of this inequality is greater than or equal to $\lambda/2$. We conclude, therefore, that the condition is necessary.

For N a positive integer let E_N denote the part of R for which $-N \leqq f \leqq N$. Suppose there exists a number $\lambda > 0$ and some $L_i = L_i'$ for which

$$m(L_i' C E_N) > \lambda \tag{1}$$

for all N. We show that this cannot be true if f satisfies the conditions of the theorem. Let η be arbitrarily fixed, and choose a positive number $\delta < \lambda/2$ and such that when $ml_i < \delta$ we have

$$\left| \int_{l_i} f\,dx_i \right| < \eta \qquad (i = 1, 2, \cdots, n) \tag{2}$$

uniformly in L_i. Let $\overline{C}_i'(N)$ be the part of $(L_i' C E_N)$ at which $f \geqq 0$, and $\underline{C}_i'(N)$ the part at which $f < 0$. Then from (1) either $m\overline{C}_i'(N) > \lambda/2$ or $m\underline{C}_i'(N) > \lambda/2$. Suppose the first is the case. If η' is arbitrary it is then possible to choose for each N a part $C_i'(N)$ of $\overline{C}_i'(N)$ for which we have

$$\delta - \eta' < mC_i'(N) < \delta. \tag{3}$$

But on CE_N $|f| > N$. Hence

$$\int_{C_i'(N)} f\,dx_i > NmC_i'(N),$$

and for N sufficiently large this, with (3), contradicts (2). We conclude, therefore, that if f satisfies the conditions of the theorem it is possible for a given δ to find N such that

$$m(L_i C E_N) < \delta \qquad (i = 1, 2, \cdots, n)$$

independent of L_i. Let $\varphi = f$ on E_N and $\varphi = 0$ elsewhere on R. Then

$$\int_{L_i} |f - \varphi|\,dx_i = \int_{L_i C E_N} |f - \varphi|\,dx_i + \int_{L_i C E_N} |f - \varphi|\,dx_i = \int_{L_i C E_N} f\,dx_i.$$

We thus see that the conditions of the theorem are sufficient.

If f is summable in x_i for each L_i then for L_i fixed

$$\int_{x_i}^{x_i+h} f\,dx_i$$

converges to zero with h independent of x_i. If the conditions of Theorem IV are satisfied this convergence is uniform in L_i. We give an example to show that the converse of this is not true, and thus establish the fact that the uniformity of this convergence is not sufficient to insure the possibility of the uniform approximation of f by a bounded function φ in the sense of (B).

Let R be the unit square. Let $y_1, y_2, \cdots$ be a sequence of values of y tending monotonically to $y=1$. Let each y_n be the mid-point of an interval δ_n on the interval $0 \leqq y \leqq 1$ where the intervals δ_n have no points in common. Let $f(x, y)=0$ on R except when y is on an interval of the set δ_n. For y on an interval of this set let $f(x, y)=n$ when $k/n-1/n^2<x<k/n$ $(k=1, 2, \cdots, n)$, and $f(x, y)=0$ for the remaining values of x on $0 \leqq x \leqq 1$. For y on δ_n let e_y be the set of x-points for which $f(x, y)=n$. Then $me_y=1/n$, and

$$\int_{e_y} f(x, y)\, dx = 1.$$

But for such values of y

$$\int_x^{x+h} f(x, y)\, dx < h + 2/n,$$

independent of x.

Let f be summable on R. Let α be a point in the space of the coördinates $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$. Let

$$\Delta_f(\alpha, h) = \int_{L_i} |f(\alpha, x_i+h) - f(\alpha, x_i)|\, dx_i.$$

Lebesgue shows[7] that for a fixed α $\Delta f(\alpha, h)$ converges to zero with h. Hille and Tamarkin have shown[8] that a necessary condition for the uniform approximation of f in the sense of (B) by a continuous function is that $\Delta_f(\alpha, h)$ converge to zero with h uniformly in α. An analogous method can be applied to the problem of approximation by step functions. We state the result without proof:

THEOREM V. *Let f be summable on R. It is then necessary for the uniform approximation of f in the sense of (A) by a step function K that $\Delta_f(\alpha, h)$ converge to zero with h uniformly in α.*

A simple example shows that this condition is not sufficient. Let $f(x, y)=0$ on the unit square except on the line $y=\frac{1}{2}$. On this line let $f(x, y)=0$ for x rational, and $f(x, y)=1$ for x irrational. Exterior to the unit square let $f(x, y)=0$.

It is easily verified that the condition of Theorem V is satisfied. But it is evident that no step function exists which approximates $f(x, y)$ in the sense of (A).

An example has been exhibited by Hille and Tamarkin[9] which establishes the existence of bounded functions for which $\Delta_f(\alpha, h)$ does not converge to zero with h uniformly in α.

[7] Leçons sur les séries trigonométriques, p. 15.

[8] loc. cit., pp. 511–512.

[9] loc. cit., p. 512.

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ON THE INVERSE FUNCTION OF AN ANALYTIC ALMOST PERIODIC FUNCTION.[1]

By Harald Bohr.

In the present paper I shall prove a general theorem stating *that under certain general conditions the inverse function of an analytic almost periodic function will again be almost periodic.*

I have divided the paper into five sections. In § 1 I give a short resumé, omitting proofs, of that part of the theory of almost periodic functions which underlies the present investigation. In § 2 I give the exact formulation of the inversion theorem, and in § 3 I give another version of this theorem which is better adapted to the proof. § 4 deals with a lemma which is closely connected with our inversion theorem and which is of special interest in itself. Finally in § 5 I give the proof of the inversion theorem; after establishing the lemma in § 4 this proof presents no further difficulty.

1. I shall begin the resumé of the theory of almost periodic functions by considering functions of a real variable t. Here the definition of almost periodicity runs as follows:

A function $f(t) = u(t) + iv(t)$, continuous for $-\infty < t < \infty$, is called an almost periodic function if to any given $\varepsilon > 0$ there exists a length $l = l(\varepsilon)$ such that every interval $t_1 < t < t_1 + l$ of this length contains at least one translation number $\tau = \tau(\varepsilon)$, i. e. a number τ satisfying for $-\infty < t < +\infty$ the inequality

$$|f(t+\tau) - f(t)| \leqq \varepsilon.$$

A main theorem in the theory of almost periodic functions states that the class of these functions, characterised in the above definition by "translation properties", can also be characterised in a quite other way, namely by "oscillation properties". Infact, denoting as an "exponential polynomial" every finite sum of pure oscillations, i. e. every sum of the form

$$P(t) = \sum_{n=1}^{N} a_n e^{i\lambda_n t}$$

with real exponents λ_n and complex coefficients a_n, we have the following proposition, the so called approximation theorem:

[1] Received December 1, 1930.

The necessary and sufficient condition that a function $f(t)$, defined for all values of t, may be approached uniformly for $-\infty < t < \infty$ by exponential polynomials is that $f(t)$ is almost periodic.

That the condition of almost periodicity is a necessary condition for the approximation in question is fairly easy to prove; the real difficulty involved in the approximation theorem is to prove, that the condition is also a sufficient one, i. e. that every almost periodic function can be uniformly approached by exponential polynomials. The key to the proof of this latter fact is the theory of Fourier series of almost periodic functions:

With each almost periodic function $f(t)$ is associated a certain infinite series of the form $\sum A_n e^{i\Lambda_n t}$ called its Fourier series; we may write

$$f(t) \sim \sum A_n e^{i\Lambda_n t}.$$

Here the numbers Λ_n occurring as exponents are real numbers (which may for instance lie everywhere dense on the real axis). These numbers Λ_n are called the Fourier exponents of the function $f(t)$. In several investigations it is not just the set of Fourier exponents Λ_n but the larger set consisting of all numbers of the form

$$g_1\Lambda_1 + g_2\Lambda_2 + \cdots + g_n\Lambda_n,$$

where the coefficients $g_1, g_2, \cdots$ are integers, which is of importance. This latter set we call the module of the Fourier exponents of the function $f(t)$, or more shortly the module M_f of the function $f(t)$ itself.

There exist interesting relations between the translation numbers $\tau(\varepsilon)$ of an almost periodic function and its Fourier exponents Λ_n. We shall here only mention a single consequence of these relations which we have to use later on: The necessary and sufficient condition, that the module M_g of a given almost periodic function $g(t)$ is contained in the module M_f of another given almost periodic function $f(t)$, is that the almost periodicity of $g(t)$ is "majorised" by the almost periodicity of $f(t)$ in the following sense: to each $\varepsilon > 0$ there exists a $\delta > 0$ such that every translation number τ of $f(t)$ belonging to δ is at the same time a translation number of $g(t)$ belonging to ε.

Many simple operations with almost periodic functions lead again to almost periodic functions and run parallel to the corresponding formal operations with the Fourier series of the functions; for instance the sum and the product of two or more almost periodic functions are again almost periodic functions and their Fourier series are obtained from the Fourier series of the given functions by formal addition and multiplication. Further

the limit function $f(t)$ of a sequence of almost periodic functions $f_n(t)$, uniformly convergent in $-\infty < t < \infty$, is again an almost periodic function and its Fourier series can be deduced from the Fourier series of $f_n(t)$ by a formal limiting process; in particular we observe that $f(t)$ cannot have any number as its Fourier exponent which is not also a Fourier exponent of one of the functions $f_n(t)$ (in fact of all the functions $f_n(t)$ from a certain stage onward).

We now consider analytic almost periodic functions of a complex variable $z = r \cdot e^{i\theta}$. We denote by S_z the Riemann surface of the function $\log z$, i. e. the surface with infinitely many sheets characterised by $0 < r < \infty$, $-\infty < \theta < +\infty$, and (when $0 \leqq r_1 < r_2 \leqq \infty$) we shall denote by (r_1, r_2) the part of the Riemann surface S_z determined by $r_1 < r < r_2$, $-\infty < \theta < +\infty$.

A function $f(z)$ analytic in (r_1, r_2) is called almost periodic in (r_1, r_2) if to any $\varepsilon > 0$ there exists a length $l = l(\varepsilon)$ such that every interval $\theta_1 < \theta < \theta_1 + l$ contains at least one translation number $\tau = \tau(\varepsilon)$, i. e. a number τ satisfying for all z in (r_1, r_2) the inequality

$$|f(z \cdot e^{i\tau}) - f(z)| \leqq \varepsilon.$$

In other words: for every fixed r in the interval $r_1 < r < r_2$ the function $F_r(\theta) = f(r e^{i\theta})$ shall be an almost periodic function of the real variable θ, and the almost periodicity shall hold "uniformly" in $r_1 < r < r_2$.

For the sake of abbreviation we shall say about a function $f(z)$ that it has a certain property (for instance almost periodicity) "in $[r_1, r_2]$" if for every choice of r', r'' satisfying the inequalities $r_1 < r' < r'' < r_2$ it has the mentioned property in (r', r''). Similarly we shall say that $f(z)$ has a certain property in $[r_1, r_2)$ or in $(r_1, r_2]$ if for every choice of r' in the interval $r_1 < r' < r_2$ it has the property in question in (r', r_2) or in (r_1, r') respectively.

In the case of a complex variable the approximation theorem runs as follows:

The necessary and sufficient condition for an analytic function $f(z)$ to be almost periodic in $[r_1, r_2]$ is that $f(z)$ can be approached uniformly in $[r_1, r_2]$ by means of "irregular polynomials", i. e. finite sums of the form

$$\sum_{n=1}^{N} a_n z^{\lambda_n}$$

with real exponents λ_n.

In the proof of this approximation theorem the notion of the "irregular Laurent series" — or, as it is usually called by the author, the Dirichlet series — of an almost periodic function $f(z)$, plays a fundamental role. With each analytic function $f(z)$, almost periodic in $[r_1, r_2]$, is associated a certain irregular Laurent series

$$\sum A_n z^{\Lambda_n}$$

with real exponents Λ_n which series (just as in the special case of an ordinary Laurent series $\sum_{-\infty}^{\infty} A_n z^n$) for any fixed r in $r_1 < r < r_2$ gives us the Fourier series of the function $f(re^{i\theta})$ regarded as an almost periodic function of the real variable θ:

$$f(re^{i\theta}) \sim \sum A_n r^{\Lambda_n} \cdot e^{i\Lambda_n \theta}.$$

Thus the exponents Λ_n of the function $f(z)$ are completely known if we only know the Fourier exponents of the function $f(re^{i\theta})$ for one single value of r in $r_1 < r < r_2$. In the case of an almost periodic function $f(z)$ of a complex variable we shall still denote the module of the exponents Λ_n by M_f.

For our present purpose we are only interested in the case $r_1 = 0$ (or in the analogous case $r_2 = \infty$). Here we have, as proved by the author, the following theorem completely analogous to a fundamental theorem of Weierstrass on ordinary Laurent series.

For analytic functions $f(z)$ almost periodic in $[0, r_2]$ there are just three possibilities:

1°. As $r \to 0$ the function $f(z)$ will tend to a finite limit c, uniformly in $-\infty < \theta < \infty$. This case will occur if and only if the exponents Λ_n of the function $f(z)$ are all $\geqq 0$, the limit c will then, as might be expected, be the constant term in the Dirichlet series of $f(z)$ (this means of course the number 0 if there is no constant term, i. e. if all the exponents Λ_n are > 0). Furthermore $f(z)$ will then be almost periodic not only in $[0, r_2]$ but even in $(0, r_2]$.

2°. The function $|f(z)|$ will, as $r \to 0$, tend to infinity uniformly in $-\infty < \theta < \infty$. This case will occur if and only if there are negative numbers among the exponents Λ_n and among these a numerically greatest one.

3°. For every arbitrary small $r^* > 0$ the values of the function $f(z)$ in $(0, r^*)$ lie everywhere dense in the whole complex plane. This case will occur if there exist negative exponents Λ_n but among these no numerically greatest (i. e. if either the lower bound of the exponents is $-\infty$, or this lower bound is a finite negative number which however does not itself belong to the exponents). In this case we have the analogue of Picard's theorem; in every domain $(0, r^*)$ the function $f(z)$ takes every complex value with at most one exception.

2. It is with case 1° of the theorem mentioned above that our inversion theorem deals. In order to state the content of our theorem as clearly as possible we shall introduce the following notation:

If $f(z)$ is an analytic almost periodic function in $(0, r_2]$ *whose exponents Λ_n are all positive* (i. e. >0 and not merely $\geqq 0$) *and which has a smallest exponent $\Lambda > 0$*, we shall say that it has *a "normal" almost periodic singularity in the point $z = 0$.* In this case $f(z)$ can be written in the form $f(z) = z^{\Lambda} \cdot f_1(z)$ $(\Lambda > 0)$, where the function $f_1(z)$ is almost periodic in $(0, r_2]$, has all its exponents $\geqq 0$ and possesses a constant term $\neq 0$.

Furthermore — in contrast to what is generally done in the theory of analytic functions — we shall attach the branch point $z = 0$ itself to the Riemann surface S_z where (as in the simple case of an algebraic singularity) it shall count as just o n e point; and, in case of a function $f(z)$ which has a normal almost periodic singularity at $z = 0$, we shall say that $f(z)$ in this point $z = 0$ has the value 0 (i. e. the limit value of $f(z)$ for $z \to 0$).

Finally, for any $r^* > 0$, we shall mean by *the neighbourhood E_{r^*} of the point $z = 0$* the domain obtained from $(0, r^*)$ by adding to it the point $z = 0$. Similarly, when in our theorem below we denote another complex variable by $\zeta = \varrho e^{i\mu}$ and its logaritmic Riemann surface by S_ζ, we shall mean by the neighbourhood E_{ϱ^*} $(\varrho^* > 0)$ of the point $\zeta = 0$ the set of points on S_ζ consisting of all points $0 < \varrho < \varrho^*$, $-\infty < \mu < \infty$ together with the branch point $\zeta = 0$ itself.

With the help of these notations we can express the proposition, whose proof is the purpose of the present paper, in the following way:

Inversion theorem. *Let $\zeta = f(z)$ be an analytic almost periodic function in a domain $(0, r_2]$ which has a normal almost periodic singularity at the point $z = 0$. Then for sufficiently small $r^* > 0$ the function $\zeta = f(z)$ will represent the neighbourhood E_{r^*} of $z = 0$ on a simply covered ("schlicht") domain D of the Riemann surface S_ζ, which domain D will contain a complete neighbourhood E_{ϱ^*} of the point $\zeta = 0$. Hence inside this latter neighbourhood E_{ϱ^*}, we can speak of the function $z = g(\zeta)$ inverse to the given function $\zeta = f(z)$.*

This inverse function $z = g(\zeta)$ will (at any rate for sufficiently small ϱ^*) be analytic and almost periodic in $(0, \varrho^*]$ and have a normal almost periodic singularity at $\zeta = 0$.

Furthermore, if $\Lambda > 0$ denotes the smallest exponent of $f(z)$ and $M > 0$ the smallest exponent of $g(\zeta)$, then

$$M = \frac{1}{\Lambda},$$

and if we write $f(z)$ in the form $f(z) = z^{\Lambda} \cdot f_1(z)$ and $g(\zeta)$ in the form $g(\zeta) = \zeta^{M} \cdot g_1(\zeta)$, the modules of $f_1(z)$ and of $g_1(\zeta)$ will be connected by the equation

$$M_{g_1} = \frac{1}{\Lambda} M_{f_1}.$$

3. In investigations on almost periodic functions of a complex variable — and in particular in the proof which follows of the inversion theorem just stated — it is convenient, in order to get the independent variable to vary in a "schlicht" plane instead of on a Riemann surface, to apply the transformation $z = e^s$ (or $\log z = s = \sigma + it$) by which the Riemann surface S_z will be represented on the simply covered s-plane (the two points $z = 0$ and $z = \infty$ corresponding to the "points" $\sigma = -\infty$ and $\sigma = +\infty$ respectively). If we regard our functions in this way as functions of s instead of as functions of z the notion of almost periodicity will of course be slightly changed. The definition in the new form (which is, by the way, the definition usually adopted by the author) runs as follows:

Let $-\infty \leqq \sigma_1 < \sigma_2 \leqq +\infty$ and let $f(s) = f(\sigma + it)$ be an analytical function of s in (σ_1, σ_2) i.e. in the strip $\sigma_1 < \sigma < \sigma_2$. Then $f(s)$ is said to be almost periodic in (σ_1, σ_2) if to any $\varepsilon > 0$ corresponds a length $l = l(\varepsilon)$ such that each interval $t_1 < t < t_1 + l$ on the t-axis contains at least one translation number $\tau = \tau(\varepsilon)$, i.e. a number τ which for all s in (σ_1, σ_2) satisfies the inequality

$$|f(s + i\tau) - f(s)| \leqq \varepsilon.$$

Furthermore, if for every $(\sigma_1 <)\, \sigma'' < \sigma_2$ the function $f(s)$ is almost periodic in (σ_1, σ'') we shall say that $f(s)$ is almost periodic in $(\sigma_1, \sigma_2]$. To every function $f(s)$, almost periodic in a certain strip, belongs a Dirichlet series

$$f(s) \sim \sum A_n e^{\Lambda_n s}.$$

In complete analogy with the notation introduced in § 2 we shall say that a function $f(s)$, analytical and almost periodic in a certain halfplane $(-\infty, \sigma_2]$, has a "normal almost periodic singularity at $\sigma = -\infty$" if all the exponents Λ_n are positive and there exists a smallest exponent $\Lambda > 0$, in other words, if $f(s)$ can be written in the form $e^{\Lambda s} \cdot f_1(s)$ $(\Lambda > 0)$ where $f_1(s)$ is again almost periodic in $(-\infty, \sigma_2]$, has all its exponents $\geqq 0$ and one exponent just equal to 0. Then we have the following alternative version of the theorem stated in § 2, which is in fact the same theorem expressed in other terms, as the reader will immediately see by replacing s by $\log z$ and w by $\log \zeta$.

Alternative form of the inversion theorem. *Let $\zeta = f(s)$ be almost periodic in $(-\infty, \sigma_2]$ and have a normal almost periodic singularity at $\sigma = -\infty$. Then for a sufficiently large negative σ^* the function $f(s)$ will be $\neq 0$ in the halfplane $(-\infty, \sigma^*)$, and the function $w = \log f(s)$ (i.e. an arbitrarily chosen*

regular branch) will represent this halfplane $\sigma < \sigma^*$ on a non overlapping domain in the plane of the variable $w = u + iv$ which will contain a certain halfplane $u < u^*$, such that for $u < u^*$ we can speak of the inverse function $s = h(w)$ of the function $w = \log f(s)$ and hence also of the function $z = e^s = e^{h(w)} = g(w)$.

This latter function $z = g(w)$ will (at any rate for sufficiently large negative u^*) be an analytic almost periodic function of w in $(-\infty, u^*]$ with a normal almost periodic singularity at $u = -\infty$.

Furthermore, if $\Lambda > 0$ and $M > 0$ denote the smallest exponents of $f(s)$ and $g(w)$ respectively, and $f(s) = e^{\Lambda s} \cdot f_1(s)$, $g(w) = e^{Mw} \cdot g_1(w)$, we shall have

$$M = \frac{1}{\Lambda} \text{ and } M_{g_1} = \frac{1}{\Lambda} M_{f_1}.$$

We conclude this section by mentioning some wellknown facts concerning analytic almost periodic functions $f(s)$ which we shall have to use in the following proofs. Some of these facts are in themselves of minor importance and some are implicitly contained in the short general resumé given in § 1. It may however be convenient for the reader to have these facts stated in the form in which we have to apply them later on.

If $f(s)$ is almost periodic in $(-\infty, \sigma_2]$ and its exponents are all $\geqq 0$, then $f(s)$ is uniformly continuous and bounded in $(-\infty, \sigma_2]$ and its derivative $f'(s)$ will tend to 0, uniformly in t, as $\sigma \to -\infty$. On the other hand, if a function $f(s)$, almost periodic in $(-\infty, \sigma_2]$, is known to be bounded in $(-\infty, \sigma_2]$, all its exponents must be $\geqq 0$. Moreover, if $f(s)$ is analytic and bounded in $(-\infty, \sigma_2]$ and for some value $\sigma_2' < \sigma_2$ is almost periodic in $(-\infty, \sigma_2']$, it must necessarily be almost periodic in the whole half plane $(-\infty, \sigma_2]$.

If $f(s)$ is almost periodic in $(-\infty, \sigma_2]$ and all its exponents are $\geqq 0$, then the function $g(s) = e^{f(s)}$ will also be almost periodic in $(-\infty, \sigma_2]$, have all its exponents $\geqq 0$ and have one exponent equal to 0; furthermore, the module M_g will coincide with the module M_f. On the other hand, if $f(s)$ is almost periodic in $(-\infty, \sigma_2]$, has all exponents $\geqq 0$ and one exponent $= 0$, then for sufficiently large negative σ^* the function $f(s)$ will be $\neq 0$ in the halfplane $(-\infty, \sigma^*)$ and the function $h(s) = \log f(s)$ (i. e. an arbitrary regular branch) will be almost periodic in $(-\infty, \sigma^*]$, have all exponents $\geqq 0$ and $M_h = M_f$.

Let $f(s)$ be an almost periodic function in the halfplane $(-\infty, \alpha + \eta)$ and let $g(s)$ be an analytic function given in some other halfplane $(-\infty, \beta)$. Then $g(s)$ will certainly be almost periodic in $(-\infty, \beta)$ and its module M_g contained in the module M_f, if $g(s)$ is "majorised" by $f(s)$ in the sense that to any $\varepsilon > 0$ corresponds a $\delta > 0$ such that every translation number

of $f(s)$ $(\sigma < \alpha)$ belonging to δ is at the same time a translation number of $g(s)$ $(\sigma < \beta)$ belonging to ε.

Finally, if a sequence of analytic functions $f_n(s)$ $(n = 1, 2, \cdots)$, each of which is almost periodic in $(-\infty, \sigma_2)$, tends to a limit-function $f(s)$, uniformly in $(-\infty, \sigma_2)$, then this latter function $f(s)$ will again be almost periodic in $(-\infty, \sigma_2)$, and every module M_g which contains all the modules M_{f_n} will also contain the module M_f.

4. In this section we shall prove a lemma from which our main theorem easily follows.

LEMMA. *Let a be a finite real number, and $\varphi(s)$ an analytic function in the half plane $\sigma < a$ which is almost periodic in $(-\infty, a]$ and whose exponents are all $\geqq 0$. For every $\alpha < a$, let $k(\alpha)$ denote the upper bound of $|\varphi(s)|$ in $\sigma \leqq \alpha$. We consider the function*

$$z = f(s) = s + \varphi(s) \qquad (\sigma < a).$$

Then for each $\alpha < a$ the following assertions will be true:

1° *The values $z = x + iy$ which the function $z = f(s)$ takes in the half plane $\sigma < \alpha$ all lie in the half plane $x < \alpha + k(\alpha)$, and every value z in the smaller half plane $x < \alpha - k(\alpha)$ is taken by the function $z = f(s)$ in one and only one point s of the halfplane $\sigma < \alpha$. Hence for z varying in the half plane $x < \alpha - k(\alpha)$ we can speak of the inverse function $s = g(z)$ to $z = f(s)$, which we write in the form*

$$s = g(z) = z - \psi(z) \qquad (x < \alpha - k(\alpha)).$$

2° *This latter function $\psi(z)$, which eo ipso is analytic in $x < \alpha - k(\alpha)$, will be almost periodic in $(-\infty, \alpha - k(\alpha)]$ and have all its exponents $\geqq 0$.*

3° *The module M_ψ of the function $\psi(z)$ will be identical with the module M_φ of the given function $\varphi(s)$.*

Proof. The statements in 1. are very easily proved. In the first place, since $|\varphi(s)| \leqq k(\alpha)$ in $\sigma < \alpha$, it is clear that the value of the function $z = f(s) = s + \varphi(s)$ at an arbitrary point s in $\sigma < \alpha$ lies in $x < \alpha + k(\alpha)$. Secondly it follows at once from a well known theorem of Rouché that the equation

$$s + \varphi(s) = z_0, \tag{1}$$

for an arbitrary given point $z_0 = x_0 + iy_0$ in $x < \alpha - k(\alpha)$, has just one root s in $\sigma < \alpha$. In fact, since $|\varphi(s)| \leqq k(\alpha)$ in $\sigma < \alpha$, it is clear that any point $s = s_0$ in $\sigma < \alpha$ which satisfies (1) must lie inside the circle $|s - z_0| = \alpha - x_0$ $(> k(\alpha))$ which touches the line $\sigma = \alpha$. But at each point s on this circle $|s - z_0| = \alpha - x_0$ we have $|\varphi(s)| \leqq k(\alpha) < |s - z_0|$.

Hence by Rouché's theorem the function $(s-z_0)+\varphi(s)$ has just as many roots inside $|s-z_0| = \alpha - x_0$ as has the function $s-z_0$, namely one.

Before proving the remainder of the lemma, I shall make some preliminary remarks.

1) When 1° is once proved—and thus the existence of the inverse function $s = g(z)$ $(x < \alpha - k(\alpha))$ established—we may in our proof of 2° confine ourselves without loss of generality to considering arbitrary large negative values of α instead of considering all values $\alpha < a$. In fact, when we have proved that $\psi(z)$ is almost periodic in the half plane $(-\infty, d)$ for one (arbitrary large) negative d we can conclude that $\psi(z)$, for every $\alpha < a$, must be almost periodic in the whole half plane $(-\infty, \alpha - k(\alpha)]$ because we know that $\psi(z)$ $(= \varphi(s))$ is bounded in this latter half plane, namely absolutely $\leqq k(\alpha)$. In our proof it will be convenient to take $\alpha < \alpha_0$ $(< a)$ where α_0 has been chosen so near to $-\infty$ that

$$|\varphi'(s)| < \frac{1}{2} \quad \text{for } \sigma < \alpha_0;$$

this is possible since $\varphi'(s)$ tends uniformly to zero as $\sigma \to -\infty$.

2) Further, we need not bother about the part of the assertion in 2° stating that the function $\psi(z)$ has all its exponents $\geqq 0$. For when we have proved that $\psi(z)$ is almost periodic in $(-\infty, \alpha - k(\alpha)]$ it is clear that its exponents must be $\geqq 0$ since $\psi(z)$ is bounded in $(-\infty, \alpha - k(\alpha)]$.

3) Finally, as regards 3°, we need only to prove that M_ψ is contained in M_φ (instead of proving that M_ψ is identical with M_φ); in fact if we can show that M_φ contains M_ψ it follows by symmetry that M_ψ must also contain M_φ, and hence that $M_\psi = M_\varphi$.

And now back to the proof! After the foregoing remarks all that we have to prove is that, for every fixed $\alpha < \alpha_0$, $\psi(z)$ is almost periodic in $(-\infty, \alpha - k(\alpha)]$, and M_ψ is contained in M_φ.

To establish these properties of $\psi(z)$ we shall first give an explicit determination (by a limiting process) of the function $s = g(z) = z - \psi(z)$ inverse to the given function $z = f(s) = s + \varphi(s)$. In fact we shall solve the equation

$$z = s + \varphi(s) \qquad (x < \alpha - k(\alpha);\ \alpha < \alpha_0)$$

for s by means of the method of successive approximation. To this purpose we write our equation $z = s + \varphi(s)$ in the form

$$s = z - \varphi(s)$$

and, starting from the function

$$s = g_0(z) = z = z - \psi_0(z)$$

where $\psi_0(z)$ is identically zero, we construct the sequence of functions

$$\begin{aligned}
g_1(z) &= z - \varphi(g_0(z)) = z - \varphi\{z - \psi_0(z)\} = z - \psi_1(z),\\
g_2(z) &= z - \varphi(g_1(z)) = z - \varphi\{z - \psi_1(z)\} = z - \psi_2(z),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
g_{n+1}(z) &= z - \varphi(g_n(z)) = z - \varphi\{z - \psi_n(z)\} = z - \psi_{n+1}(z),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

It is clear that, for an arbitrary fixed point z in the half plane $x < \alpha - k(\alpha)$, the successive determination of the numbers $g_0(z) = z$, $g_1(z)$, $g_2(z)$, $\cdots$ can be carried out. In fact, the function $\varphi(s)$ is defined in the whole half plane $\sigma < \alpha$, and if for some n the number $g_n(z)$ lies in this half plane $\sigma < \alpha$ so that we can build the next number $g_{n+1}(z) = z - \varphi(g_n(z))$, this latter number $g_{n+1}(z)$ will again lie in $\sigma < \alpha$ because $|\varphi(g_n(z))| \leqq k(\alpha)$.

Furthermore it is easy to see that the sequence of analytic functions $\psi_n(z)$ (and hence also the sequence $g_n(z)$) converges uniformly in the whole half plane $x < \alpha - k(\alpha)$. In fact for an arbitrary point z in $x < \alpha - k(\alpha)$ we find

$$\begin{aligned}
|\psi_{n+1}(z) - \psi_n(z)| &= |\varphi\{z - \psi_n(z)\} - \varphi\{z - \psi_{n-1}(z)\}|\\
&\leqq |\psi_n(z) - \psi_{n-1}(z)| \cdot \underset{\sigma<\alpha}{\text{Upper bound}}\, |\varphi'(s)|\\
&\leqq |\psi_n(z) - \psi_{n-1}(z)| \cdot \frac{1}{2}
\end{aligned}$$

and hence

$$\begin{aligned}
|\psi_{n+1}(z) - \psi_n(z)| &\leqq \frac{1}{2^2}|\psi_{n-1}(z) - \psi_{n-2}(z)| \leqq \frac{1}{2^3}|\psi_{n-2}(z) - \psi_{n-3}(z)|\\
&\leqq \cdots \leqq \frac{1}{2^n}|\psi_1(z) - \psi_0(z)| = \frac{1}{2^n}|\psi_1(z)| = \frac{1}{2^n}|\varphi(z)|
\end{aligned}$$

such that for every $n \geqq 0$

$$|\psi_{n+1}(z) - \psi_n(z)| \leqq \frac{1}{2^n} \cdot k$$

where $k = k(\alpha)$ denotes the upper bound of $|\varphi(s)|$ in $\sigma < \alpha$.

Since the value $g_n(z)$, for every n and every z in $x < \alpha - k(\alpha)$, lies in the half plane $\sigma < \alpha$, it is clear that the limit $g(z) = \lim g_n(z)$ must certainly lie in the closed half plane $\sigma \leqq \alpha$.

The limit function $s = g(z) = \lim g_n(z)$ thus obtained satisfies our equation

$$z = s + \varphi(s)$$

for every z in $x < \alpha - k(\alpha)$; to see this we have only to let $n \to \infty$ in the identity

$$z = g_{n+1}(z) + \varphi(g_n(z)).$$

Further, for any point z in $x < \alpha - k(\alpha)$, the above solution $s = g(z) = \lim g_n(z)$ of our equation $z = s + \varphi(s)$ must be the "right" solution, i. e. the unique one which lies in the halfplane $\sigma < \alpha$. For we know already that $g(z)$ lies in the closed halfplane $\sigma \leqq \alpha$, and it certainly cannot lie on the line $\sigma = \alpha$, because no point s on this line can satisfy the equation $z = s + \varphi(s)$ since $|\varphi(\alpha + it)| \leqq k(\alpha)$.

It remains to show that $\psi(z)$ really fulfills our assertions, i. e. that $\psi(z)$ is almost periodic in $(-\infty, \alpha - k(\alpha)]$ and that M_ψ is contained in M_φ.

We first observe that it suffices to prove that, for each fixed n, the function $\psi_n(z)$ has these properties—i. e. that $\psi_n(z)$ is almost periodic in $(-\infty, \alpha - k(\alpha)]$ and that M_{ψ_n} is contained in M_φ—because, from the theorems stated at the end of § 3, it follows that $\psi(z)$, being the uniform-limit function of $\psi_n(z)$ in the whole half plane $x < \alpha - k(\alpha)$, will then also have the properties in question.

Again, to prove that $\psi_n(z)$, for a fixed n, has the mentioned properties, it suffices—also by the theorems stated in § 3—to prove that $\psi_n(z)$ is almost periodic in $(-\infty, \alpha - k(\alpha))$ in such a way that its almost periodicity is "majorised" by the almost periodicity of the given function $\varphi(s)$ in $(-\infty, \alpha)$.

And that this is the case we prove by induction. It is certainly true for $n = 0$, since the function $\psi_0(z)$ is identically zero. Assume our assertion to be true for $\psi_n(z)$; we have to show that it will then also be true for $\psi_{n+1}(z)$, i. e. that to any given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ (which may of course depend on n also) such that every real number τ satisfying the inequality

$$|\varphi(s + i\tau) - \varphi(s)| \leqq \delta \qquad (\text{for } \sigma < \alpha) \tag{2}$$

will also satisfy the inequality

$$|\psi_{n+1}(z + i\tau) - \psi_{n+1}(z)| \leqq \varepsilon \qquad (\text{for } x < \alpha - k(\alpha)). \tag{3}$$

To this end we first determine (as we can by the uniform continuity of $\varphi(s)$ in $\sigma < \alpha$) a number $\varepsilon_1 > 0$ so small that the inequality

$$|\varphi(s') - \varphi(s'')| \leqq \varepsilon$$

holds for any pair of points s' and s'' in $\sigma < \alpha$ whose difference $s' - s''$ can be written in the form

$$s' - s'' = i\tau + \eta$$

where τ is a translation number of $\varphi(s)$ $(\sigma < \alpha)$ belonging to $\frac{\varepsilon}{2}$, and $|\eta| < \varepsilon_1$.

Next, from the assumption made on $\psi_n(z)$ (namely that it is almost periodic and that its almost periodicity is "majorised" by the almost periodicity of $\varphi(s)$), we determine to the above number ε_1 a number $\delta_1 > 0$ such that any τ satisfying the inequality

$$(4) \qquad |\varphi(s+i\tau) - \varphi(s)| \leqq \delta_1 \qquad (\text{for } \sigma < \alpha)$$

will also satisfy the inequality

$$(5) \qquad |\psi_n(z+i\tau) - \psi_n(z)| \leqq \varepsilon_1 \qquad (\text{for } x < \alpha - k(\alpha)).$$

Then the number

$$\delta = \operatorname{Min}\left\{\delta_1, \frac{\varepsilon}{2}\right\}$$

will have the desired property; if τ is an arbitrary number satisfying (2) it will also satisfy (3). In fact let τ be an arbitrary number satisfying (2) and let z be an arbitrary chosen point in the half plane $x < \alpha - k(\alpha)$; we shall show that

$$|\psi_{n+1}(z+i\tau) - \psi_{n+1}(z)| \leqq \varepsilon.$$

We have

$$\psi_{n+1}(z+i\tau) - \psi_{n+1}(z) = \varphi\{z + i\tau - \psi_n(z+i\tau)\} - \varphi\{z - \psi_n(z)\}$$

where the two arguments on the right hand side $s' = z + i\tau - \psi_n(z+i\tau)$ and $s'' = z - \psi_n(z)$ both lie in the halfplane $\sigma < \alpha$ and where their difference $s' - s''$ is equal to

$$s' - s'' = i\tau - \{\psi_n(z+i\tau) - \psi_n(z)\} = i\tau + \eta.$$

Here τ is, by assumption, a translation number of $\varphi(s)$ belonging to δ and hence a fortiori to $\frac{\varepsilon}{2}$, and η satisfies the inequality

$$|\eta| = |\psi_n(z+i\tau) - \psi_n(z)| \leqq \varepsilon_1$$

because τ, being a translation number of $\varphi(s)$ belonging to δ, is a fortiori a translation number of $\varphi(s)$ belonging to δ_1, and we know that every number τ satisfying (4) will also satisfy (5). Hence we conclude that

$$|\varphi(s') - \varphi(s'')| \leqq \varepsilon,$$

i. e.,

$$|\psi_{n+1}(z+i\tau) - \psi_{n+1}(z)| \leqq \varepsilon$$

q. e. d.

5. By means of the lemma in § 4 we can now easily prove our inversion theorem in the form stated in § 3.

Let $\zeta = f(s)$ be the given function of the theorem in § 3, almost periodic in $(-\infty, \sigma_2]$ and with a normal almost periodic singularity at $\sigma = -\infty$. As in the enunciation we denote by $\Lambda > 0$ the smallest exponent of $f(s)$ and write $f(s)$ in the form

$$\zeta = f(s) = e^{\Lambda} \cdot f_1(s),$$

where $f_1(s)$ is again almost periodic in $(-\infty, \sigma_2]$, has all its exponents $\geqq 0$ and has one exponent equal to 0.

By theorems stated in the end of § 3 we know that $f_1(s)$ and $f(s)$, for a sufficiently large negative a, will be $\neq 0$ in the halfplane $\sigma < a$, such that for $\sigma < a$ we can speak of the function

$$w = \log \zeta = \log f(s) = \Lambda s + \log f_1(s) = \Lambda \cdot (s + \varphi(s))$$

where $\log f_1(s)$ denotes an arbitrarily chosen regular branch of the logarithm. Here, again by the theorems in § 3, the function $\varphi(s) = \frac{1}{\Lambda} \log f_1(s)$ will be almost periodic in $(-\infty, a]$, its exponents will all be $\geqq 0$ and its module M_φ will be equal to the module M_{f_1}. Let us for a moment denote $\frac{w}{\Lambda}$ by z, so that we have an equation of the type considered in our lemma in § 4

$$z = s + \varphi(s).$$

Then the lemma tells us that, for a sufficiently large negative σ^*, the function $z = s + \varphi(s)$ will represent the half plane $\sigma < \sigma^*$ on a simply covered domain in the plane of the variable $z = x + iy$ containing a certain half plane $x < x^*$, inside of which the (analytic) inverse function will have the form

$$s = z - \psi(z),$$

where $\psi(z)$ is almost periodic in $(-\infty, x^*]$, has all its exponents $\geqq 0$ and has its module $M_\psi = M_\varphi$. Hence, writing again $\frac{w}{\Lambda}$ for z and denoting Λx^* by u^*, we have

$$s = \frac{w}{\Lambda} - \psi\left(\frac{w}{\Lambda}\right) = \frac{w}{\Lambda} - \chi(w),$$

where $\chi(w)$ is almost periodic in $(-\infty, u^*]$, has all its exponents $\geqq 0$ and has its module given by

$$M_\chi = \frac{1}{\Lambda} M_\psi = \frac{1}{\Lambda} M_\varphi = \frac{1}{\Lambda} M_{f_1}.$$

The inversion theorem deals with the function

$$g(w) = e^{s} = e^{\frac{w}{\Lambda}} \cdot e^{-\chi(w)} = e^{\frac{w}{\Lambda}} \cdot g_1(w).$$

From the expression on the right hand side we can now immediately deduce the properties of this function $g(w)$ as asserted in the theorem. In fact, by the theorems of § 3, the function $g_1(w) = e^{-\chi(w)}$ is almost periodic in $(-\infty, u^*]$, has all its exponents $\geqq 0$ and one exponent $= 0$ and has its module $M_{g_1} = M_{\chi}$. Hence $g(w) = e^{\frac{w}{\Lambda}} \cdot g_1(w)$ is itself almost periodic in $(-\infty, u^*]$, has a normal almost periodic singularity at $u = -\infty$, has its smallest exponent $M = \frac{1}{\Lambda}$, and the module M_{g_1} is given by

$$M_{g_1} = M_{\chi} = \frac{1}{\Lambda} M_{f_1}$$

which completes the proof.

Stanford University, California.
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ON A PROBLEM IN THE ADDITIVE THEORY OF NUMBERS.

(FOURTH PAPER.)

By C. J. A. Evelyn and E. H. Linfoot.

1. **Introductory.** In three recent papers[1] we considered the representation of a large number as the sum of the M-numbers defined as follows: The integer

$$n = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_\varrho^{\lambda_\varrho}$$

is said to be a number of class N or an "M-number" when $\lambda_1, \lambda_2, \cdots, \lambda_\varrho \leq N-1$. In (I) we stated the following theorem:

Theorem 1.1. *The number of M-numbers which do not exceed x is asymptotic to $\dfrac{x}{\zeta(N)}$ as $x \to \infty$; more precisely, if*

$$\sum_{M \leq x} 1 = \frac{x}{\zeta(N)} + R(x)$$

then[2]

$$R(x) = O\left(x^{1/N} e^{-\alpha N^{-3/2}\sqrt{\log x \log\log x}}\right),$$

where $\alpha > 0$ is an absolute constant.

Proof. Following Landau[3], let $x \geqq 1$ and

$$\eta(x) = \max_{t \geqq x^{1/2N}} \left(\frac{M(t)}{t}\right)^{1/N}$$

where $M(t) = \sum_{m=1}^{t} \mu(m)$ and $\mu(m)$ is Möbius's function. Since[4]

$$M(t) = O\left(t e^{-2\alpha\sqrt{\log t \log\log t}}\right)$$

as $t \to \infty$, it is clear that the decreasing function

$$\eta(x) = O\left(e^{-\alpha N^{-3/2}\sqrt{\log x \log\log x}}\right).$$

Let

$$\delta(x) = \max\left(\frac{1}{x^{1/2N}}, \eta(x)\right)$$

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* Received September 20, 1930.

[1] Math. Zeitschrift 30 (1929) 433–448; Journal für Math. 164 (1931), Math. Zeitschrift (1931), referred to throughout as (I), (II) and (III).

[2] In (I) the factor $\alpha N^{-3/2}$ was wrongly given as $\alpha/2N^2$.

[3] Handbuch II, p. 306, § 162.

[4] "Vorlesungen über Zahlentheorie" II, p. 157.

so that $\delta(x) > 0$ and $\delta(x) = O(e^{-\alpha N^{-3/2}\sqrt{\log x \log\log x}})$. Let x be so big that $\delta < 1$. We have

$$\begin{aligned}
\sum_{M \leqq x} 1 &= \sum_{n^N m \leqq x}^{m,n} \mu(n) \\
&= \sum_{n=1}^{\delta x^{1/N}} \mu(n) \sum_{m=1}^{x/n^N} 1 + \sum_{m=1}^{1/\delta^N} \sum_{n=1}^{(x/m)^{1/N}} \mu(n) - \sum_{n=1}^{\delta x^{1/N}} \mu(n) \sum_{m=1}^{1/\delta^N} 1 \\
&= \sum_{n=1}^{\delta x^{1/N}} \mu(n) \left[\frac{x}{n^N}\right] + \sum_{m=1}^{1/\delta^N} M\left(\left(\frac{x}{m}\right)^{1/N}\right) - M(\delta x^{1/N}) \left[\frac{1}{\delta^N}\right].
\end{aligned}$$

In this sum

$$\begin{aligned}
\sum_{n=1}^{\delta x^{1/N}} \mu(n) \left[\frac{x}{n^N}\right] &= \sum_{n=1}^{\delta x^{1/N}} \mu(n) \frac{x}{n^N} + O(\delta x^{1/N}) \\
&= x \sum_{n=1}^{\delta x^{1/N}} \frac{\mu(n)}{n^N} + O(\delta x^{1/N}) \\
&= \frac{x}{\zeta(N)} + O(\delta x^{1/N}) - x \sum_{n > \delta x^{1/N}} \frac{M(n) - M(n-1)}{n^N} \\
&= \frac{x}{\zeta(N)} + O(\delta x^{1/N}) - x \sum_{n > \delta x^{1/N}} M(n) \left(\frac{1}{n^N} - \frac{1}{(n+1)^N}\right) \\
&\quad + x \frac{M(\delta x^{1/N})}{([\delta x^{1/N}] + 1)^N}.
\end{aligned}$$

Since in the sum on the right

$$n > \delta x^{1/N} \geqq x^{1/2N},$$

we have there

$$|M(n)| = \left|\frac{M(n)}{n}\right| n \leqq (\eta(x))^N n \leqq \delta^N n;$$

in the last term

$$|M(\delta x^{1/N})| \leqq \delta^N \cdot \delta x^{1/N} = \delta^{N+1} x^{1/N}.$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\delta x^{1/N}} \mu(n) \left[\frac{x}{n^N}\right] &= \frac{x}{\zeta(N)} + O(\delta x^{1/N}) + O\left(x\, \delta^N \sum_{n > \delta x^{1/N}} n \cdot \frac{1}{n^{N+1}}\right) \\
&\quad + O\left(x \frac{\delta^{N+1} x^{1/N}}{\delta^N x}\right) \\
&= \frac{x}{\zeta(N)} + O(\delta x^{1/N}) + O\left(x \delta^N \frac{1}{\delta^{N-1} x^{\frac{N-1}{N}}}\right) + O(\delta x^{1/N}) \qquad (1.11) \\
&= \frac{x}{\zeta(N)} + O(\delta x^{1/N}).
\end{aligned}$$

Secondly, in $\sum_{m=1}^{1/\delta^N} M\left(\left(\frac{x}{m}\right)^{1/N}\right)$ each argument

$$\left(\frac{x}{m}\right)^{1/N} \geqq \delta x^{1/N} \geqq x^{1/2N}$$

and thus each term

$$\left|M\left(\left(\frac{x}{m}\right)^{1/N}\right)\right| \leqq \delta^N \left(\frac{x}{m}\right)^{1/N}.$$

Whence

$$\begin{aligned}\sum_{m=1}^{1/\delta^N} M\left(\left(\frac{x}{m}\right)^{1/N}\right) &= O\left(\delta^N x^{1/N} \sum_{m=1}^{1/\delta^N} \frac{1}{m^{1/N}}\right)\\ &= O\left(\delta^N x^{1/N} \cdot \frac{1}{\delta^{N-1}}\right) \qquad (1.12)\\ &= O(\delta x^{1/N}).\end{aligned}$$

Finally

$$\begin{aligned}M(\delta x^{1/N})\left[\frac{1}{\delta^N}\right] &= O\left(\delta^{N+1} x^{1/N} \cdot \frac{1}{\delta^N}\right) \qquad (1.13)\\ &= O(\delta x^{1/N}).\end{aligned}$$

From (1.11), (1.12), (1.13) we have

$$\sum_{M \leqq x} 1 = \frac{x}{\zeta(N)} + O(\delta x^{1/N})$$

and since $\delta = O(e^{-\alpha N^{-3/2}\sqrt{\log x \log\log x}})$ the theorem follows.

The determination of a lower bound for $R(x)$ is of interest. We first prove

THEOREM 1.2. *As* $x \to \infty$,

$$R(x) \neq O\left(x^{\frac{1}{2N}-\delta}\right)$$

for any $\delta > 0$.

Proof. We may suppose $0 < \delta < 1$. Let

$$\begin{aligned} e(n) &= 1 \quad n \text{ an } M\text{-number},\\ &= 0 \quad \text{otherwise}\end{aligned}$$

and suppose

$$E(x) = \sum_{n=1}^{x} e(n) = \frac{x}{\zeta(N)} + O\left(x^{\frac{1}{2N}-\delta}\right).$$

Then will

$$\sum_{n=1}^{m} \frac{e(n)}{n^s} = \sum_{n=1}^{m} \frac{E(n) - E(n-1)}{n^s}$$

$$= \sum_{n=1}^{m} E(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + \frac{1}{(m+1)^s} E(m)$$

$$= \sum_{n=1}^{m} \frac{n}{\zeta(N)} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + \frac{1}{(m+1)^s} \frac{m}{\zeta(N)}$$
$$+ \sum_{n=1}^{m} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) O\left(n^{\frac{1}{2N}-\delta}\right) + \frac{1}{(m+1)^s} O\left(m^{\frac{1}{2N}-\delta}\right)$$
$$= \frac{1}{\zeta(N)} \sum_{n=1}^{m} \frac{1}{n^s} + s \sum_{n=1}^{m} O\left(n^{\frac{1}{2N}-\delta}\right) \int_n^{n+1} \frac{dt}{t^{s+1}} + \frac{1}{(m+1)^s} O\left(m^{\frac{1}{2N}-\delta}\right);$$

$$\sum_{n=1}^{m} \frac{e(n)}{n^s} - \frac{1}{\zeta(N)} \sum_{n=1}^{m} \frac{1}{n^s} = s \sum_{n=1}^{m} O\left(\frac{1}{n^{1+\sigma-\frac{1}{2N}+\delta}}\right) + O\left(m^{\frac{1}{2N}-\sigma-\delta}\right),$$

where the terms on the right are all analytic functions of s. If $\sigma \geqq \frac{1}{2N} - \frac{\delta}{2}$ the right side converges uniformly near every s as $m \to \infty$ und so represents an analytic function. The left hand side has for $\sigma > 1$ the limit $\frac{\zeta(s)}{\zeta(Ns)} - \frac{\zeta(s)}{\zeta(N)}$; this function must therefore be regular in $\sigma \geqq \frac{1}{2N} - \frac{\delta}{2}$. It follows that $\frac{\zeta(s)}{\zeta(Ns)}$ is likewise regular in this half-plane except for a pole at $s = 1$. This is only possible if all the zeros of $\zeta(Ns)$ in the half-plane are zeros of $\zeta(s)$. But in any strip $\left(\frac{1-\varepsilon}{2N}, \frac{1+\varepsilon}{2N}\right)$ with $0 < \varepsilon < \frac{\delta}{2}$, $\zeta(Ns)$ has more zeros than $\zeta(s)$.[5] Thus we have a contradiction and the theorem is proved.

Using a slightly deeper result about the distribution of the zeros of $\zeta(s)$ we can prove the sharper

THEOREM 1.3.

$$R(x) \neq o\left(x^{\frac{1}{2N}}\right).$$

Proof. 1. There is a zero of $\zeta(Ns)$ on $\sigma = \frac{1}{2N}$ which is not a zero of $\zeta(s)$. For[6] the number of zeros of $\zeta(Ns)$ on $\sigma = \frac{1}{2N}$ with ordinates lying in $(0, T)$ is greater than $c_1 T$. While[7] the number of zeros of $\zeta(s)$ in $0 < \sigma < \frac{1}{2} - \varepsilon$ with ordinates lying in $(0, T)$ is $o(T)$ as $T \to \infty$. *A fortiori* this is true on $\sigma = \frac{1}{2N}$.

[5] H. Bohr and E. Landau, C. R. Paris 158 (1914) 106–110.
[6] Hardy and Littlewood, Math. Zeitschrift 10 (1921), 283–317.
[7] Bohr and Landau, l. c. [5].

2. Now suppose $R(x) = o\left(x^{\frac{1}{2N}}\right)$. Then[8] for $\sigma > 1$

$$\frac{\zeta(s)}{\zeta(Ns)} - \frac{\zeta(s)}{\zeta(N)} = \sum_{n=1}^{\infty} \frac{e(n) - \frac{1}{\zeta(N)}}{n^s} = \sum_{n=1}^{\infty} \frac{R(n) - R(n-1)}{n^s}$$
$$= \sum_{n=1}^{\infty} R(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = \sum_{n=1}^{\infty} o\left(n^{-1-\sigma+\frac{1}{2N}}\right).$$

This series converges uniformly for $\sigma \geqq \sigma_0 > \frac{1}{2N}$. Hence $\frac{\zeta(s)}{\zeta(Ns)} - \frac{\zeta(s)}{\zeta(N)}$ is regular in $\sigma > \frac{1}{2N}$ and in this region

$$\frac{\zeta(s)}{\zeta(Ns)} - \frac{\zeta(s)}{\zeta(N)} = \sum_{n=1}^{\infty} o\left(n^{-1-\sigma+\frac{1}{2N}}\right)$$
$$= o\left(\frac{1}{\sigma - \frac{1}{2N}}\right)$$

as $\sigma \to \frac{1}{2N}$, by an easy argument. This holds for every t; taking t to be the ordinate of a zero of $\zeta(Ns)$ on $\sigma = \frac{1}{2N}$ which is not a zero of $\zeta(s)$, we have a contradiction.

2. In our first three papers we attacked the problem of the representation of a large number as the sum of s M-numbers and proved in (I) that for $s \geqq 3$ the number of representations

$$\nu_s(n) = \frac{n^{s-1}}{(s-1)!} \frac{1}{\zeta^s(N)} S_s(n) + O\left(n^{s-\frac{3}{2}+\frac{1}{2N}+\varepsilon}\right) \tag{2.11}$$

as $n \to \infty$, for every $\varepsilon > 0$, where

$$S_s(n) = \prod_{p^N \nmid n} \left(1 + \frac{(-1)^{s+1}}{(p^N-1)^s}\right) \prod_{p^N \mid n} \left(1 + \frac{(-1)^s}{(p^N-1)^{s-1}}\right); \tag{2.12}$$

in (II) and (III) we extended this to the cases $s \geqq 2$ and sharpened the error term to

$$O\left(n^{s-2+\frac{2}{N+1}+\varepsilon}\right). \tag{2.13}$$

The later treatment of the problem was quite different from that of (I), being purely elementary, while (I) was an application of the analytic method of Hardy and Littlewood, which consisted in studying the function

$$f(x) = \sum_M x^M$$

[8] Compare E. C. Titchmarsh, "The Zeta Function of Riemann", Cambridge Tract No. 26 (1930), p. 80.

in the neighbourhood of its barrier of singularities $|x| = 1$. In this section we show how, by a refinement of the argument of (I), we can obtain the error term

$$O\left(n^{s-2+\frac{1}{N}+\frac{1}{s-1}\frac{N-1}{N}}\right), \tag{2.2}$$

which is a sharpening of (2.13) when $s \geqq N+3$. We assume that the reader is already familiar with the notation and argument of (I) and indicate briefly the modifications to be made.

Our new Farey dissection will ultimately be taken to be of order $[n^K]$, where

$$K = \frac{s-2}{s-1} + \frac{1}{N(s-1)}, \tag{2.31}$$

major arcs $\mathfrak{M}$ being those for which

$$k \leqq n^H,$$

where

$$H = 1 - K = \frac{1}{s-1} - \frac{1}{N(s-1)}; \tag{2.32}$$

minor arcs $\mathfrak{m}$ those for which

$$n^H < k \leqq n^K.$$

For the present, however, we consider quite general $H = H(s, N)$, $K = K(s, N)$ subject only to

$$0 \leqq H \leqq K = 1 - H \leqq 1.$$

Then § 3 of (I) holds with one modification; Lemma 3.3 must be replaced by

LEMMA A. *On a major arc* $\mathfrak{M}$

$$|f(x) - \psi_\varrho(x)| < A_{24}\, n^{H+\frac{1}{N}+\varepsilon}.$$

Proof. As before

$$\left| f(x) - (1-X)\frac{E_N(k)}{\zeta(N)} \prod_{p|k} \frac{-1}{p^N-1} \sum_{\nu=0}^{\infty} (\nu+1) X^\nu \right|$$

$$< A_{11}\, k^\varepsilon\, n^{H+\frac{1}{N}} + 8A_6\, k^{1+\varepsilon} \frac{n^H}{k}$$

$$< A_{11}\, n^{H+\frac{1}{N}+\varepsilon} + 8A_6\, n^{H+\varepsilon}$$

$$< (A_{11} + 8A_6)\, n^{H+\frac{1}{N}+\varepsilon}.$$

That is

$$\left| f(x) - \frac{1}{1-X} \frac{E_N(k)}{\zeta(N)} \prod_{p|k} \frac{-1}{p^N-1} \right| < A_{24}\, n^{H+\frac{1}{N}+\varepsilon}.$$

Proceeding to the main argument, we have

$$(2.41)\qquad \begin{aligned} \nu_s(n) &= \frac{1}{2\pi i}\int_C f^s(x)\frac{dx}{x^{n+1}} \\ &= \frac{1}{2\pi i}\sum_{\mathfrak{M}}\int_{\mathfrak{M}}\psi_\varrho^s(x)\frac{dx}{x^{n+1}}+\lambda_1(n), \end{aligned}$$

where

$$(2.42)\qquad \begin{aligned} |\lambda_1(n)| &\leqq \frac{1}{2\pi}\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|f^s(x)-\psi_\varrho^s(x)|\frac{|dx|}{|x|^{n+1}}+\frac{1}{2\pi}\sum_{\mathfrak{m}}\int_{\mathfrak{m}}|f(x)|^s\frac{|dx|}{|x|^{n+1}} \\ &= \frac{e}{2\pi}\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|f^s(x)-\psi_\varrho^s(x)|\,d\vartheta+\frac{e}{2\pi}\sum_{\mathfrak{m}}\int_{\mathfrak{m}}|f(x)|^s d\vartheta \\ &= \lambda_1^{(1)}(n)+\lambda_1^{(2)}(n). \end{aligned}$$

Since

$$|f^s-\psi^s|\leqq s\,|f-\psi|\,(|f|^{s-1}+|\psi|^{s-1}),$$

we have

$$(2.43)\qquad \begin{aligned} \lambda_1^{(1)}(n) &\leqq \frac{es}{2\pi}\max_{\mathfrak{M}}|f-\psi_\varrho|\left\{\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|f(x)|^{s-1}d\vartheta+\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|\psi_\varrho(x)|^{s-1}d\vartheta\right\} \\ &< A_{25}\,n^{s-2+H+\frac{1}{N}+\varepsilon}, \end{aligned}$$

by Lemma A and (3.22), (3.4) of (I). And by (3.7) of (I)

$$(2.44)\qquad \begin{aligned} \lambda_1^{(2)}(n) &\leqq \frac{e}{2\pi}\max_{\mathfrak{m}}|f(x)|^{s-2}\sum_{\mathfrak{m}}\int_{\mathfrak{m}}|f(x)|^2 d\vartheta \\ &< \frac{e}{2\pi}A_{13}^{s-2}\,n^{(s-2)(1-H+\varepsilon)}\int_C|f(x)|^2 d\vartheta \\ &< A_{26}\,n^{(s-2)(1-H)+1+\varepsilon} \\ &= A_{26}\,n^{(s-1)-(s-2)H+\varepsilon}. \end{aligned}$$

(2.43) and (2.44) together give us an upper bound for the error on replacing $\nu_s(n)$ by $\dfrac{1}{2\pi i}\sum_{\mathfrak{M}}\int_{\mathfrak{M}}\psi_\varrho^s(x)\dfrac{dx}{x^{n+1}}$. For the error

$$\lambda_2(n)=\frac{1}{2\pi i}\sum_{\mathfrak{M}}\int_{C-\mathfrak{M}}\psi_\varrho^s(x)\frac{dx}{x^{n+1}}$$

on replacing each $\int_{\mathfrak{M}}$ by the integral over the whole circle C we have as before the inequality

$$(2.45)\qquad |\lambda_2(n)|<c_{15}\,n^{(s-1)-(s-2)H}.$$

We have now replaced $\nu_s(n)$ by

$$\sum_{k\leqq n^H}\frac{1}{2\pi i}\int_C\psi_\varrho^s(x)\frac{dx}{x^{n+1}}.$$

The error on replacing this by the complete series

$$\sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_C \psi_\varrho^s(x) \frac{dx}{x^{n+1}}$$

is less than $A_{27}\, n^{s-1-(s-2)H+\varepsilon}$ by an easy calculation. Since for $H = \frac{1}{s-1} - \frac{1}{N(s-1)}$

$$s-2+H+\frac{1}{N} = s-2+\frac{1}{N}+\frac{1}{s-1}\,\frac{N-1}{N}$$

$$(s-2)(1-H)+1 = s-2+\frac{1}{N}+\frac{1}{s-1}\,\frac{N-1}{N}$$

we obtain on inserting our special values of H and K

$$(2.5) \qquad \nu_s(n) = \frac{n^{s-1}}{(s-1)!}\,\frac{1}{\zeta^s(N)}\,S_s(n) + O\left(n^{s-2+\frac{1}{N}+\frac{1}{s-1}\frac{N-1}{N}+\varepsilon}\right).$$

Since

$$\frac{2}{N+1} - \frac{1}{N} - \frac{1}{s-1}\,\frac{N-1}{N} = \frac{2N(s-1)-(N+1)(s-1)-(N-1)(N+1)}{N(N+1)(s-1)}$$

$$= \frac{(N-1)(s-N-2)}{N(N+1)(s-1)},$$

this error term is sharper than (2.13) when $s \geqq N+3$.

3. Finally we prove some identities which, though not directly concerned in the solution of the main problem, seem curious and interesting enough to deserve mention. The first is a recurrence formula for the function $E_N(n)$. We observe that the asymptotic equality

$$\nu_s(n) \sim \frac{n^{s-1}}{(s-1)!}\,\frac{1}{\zeta^s(N)}\,S_s(n),$$

which has been established for $s \geqq 2$, reduces to an identity when $s = 1$. For, setting $s = 1$ in § 5 of (I),

$$S_1(n) = \prod_{p^N \nmid n}\left(1+\frac{1}{p^N-1}\right)\prod_{p^N | n}(1-1)\begin{cases} = \zeta(N) & n \text{ an } M\text{-number}, \\ = 0 & \text{otherwise}. \end{cases}$$

This can be written

$$(3.1) \qquad E_{N-1}(n) = \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \varrho^{-n},$$

which is our recurrence formula.

Now suppose $|x| < 1$ and consider the expression

$$\frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \frac{1}{1-\frac{2}{\varrho}}$$

$$= \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\nu=0}^{\infty} x^\nu \sum_{\varrho(k)} \varrho^{-\nu};$$

it is dominated by

$$\frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{1}{p^N-1} \sum_{\nu=0}^{\infty} |x|^\nu \left| \sum_{\varrho(k)} \varrho^{-\nu} \right|$$

$$= \sum_{\nu=0}^{\infty} |x|^\nu \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{1}{p^N-1} \left| \sum_{d|(k,\nu)} d\mu\left(\frac{k}{d}\right) \right|$$

$$\leq \sum_{\nu=0}^{\infty} |x|^\nu \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{1}{p^N-1} \sum_{d|\nu} d$$

$$< \sum_{\nu=0}^{\infty} \nu^2 |x|^\nu \prod_{p} \left(1+\frac{N}{p^N-1}\right),$$

and so can be written as

$$\sum_{\nu=0}^{\infty} x^\nu \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \varrho^{-\nu}$$

$$= \sum_{\nu=0}^{\infty} E_{N-1}(\nu)\, x^\nu,$$

by (3.1),

$$= f(x).$$

Thus for $|x| < 1$ we have the identity

$$f(x) = \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \frac{1}{1-\frac{x}{\varrho}}$$

(3.2)

$$= \sum_{k=1}^{\infty} \sum_{\varrho(k)} \psi_\varrho(x),$$

which shows that $f(x)$ is, so to speak, merely the sum of its singularities. In the same way

$$\frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} f\left(\frac{x}{\varrho}\right)$$

$$= \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \sum_{M} \frac{x^M}{\varrho^M}$$

$$= \sum_{M} x^M \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \varrho^{-M}$$

$$= \sum_{M} x^M,$$

so that $f(x)$ satisfies the functional identity

$$(3.3) \qquad f(x) = \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} f\left(\frac{x}{\varrho}\right).$$

Next let

$$\tau_\varrho(x) = f(x) - \psi_\varrho(x).$$

Then will

$$(3.4) \qquad \tau_1(x) = \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} \tau_\varrho(x\varrho).$$

For the right hand side is

$$\frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} (f(x\varrho) - \psi_\varrho(x\varrho))$$

$$= \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} f\left(\frac{x}{\varrho}\right)$$

$$- \frac{1}{\zeta^2(N)} \frac{1}{1-x} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \left(\frac{-1}{p^N-1}\right)^2 \varphi(k)$$

$$= f(x) - \frac{1}{\zeta^2(N)} \frac{1}{1-x} \prod_p \left(1 + \frac{p-1+p^2-p+\cdots+p^N-p^{N-1}}{(p^N-1)^2}\right)$$

$$= f(x) - \frac{1}{\zeta^2(N)} \frac{1}{1-x} \prod_p \frac{p^N}{p^N-1}$$

$$= f(x) - \frac{1}{\zeta(N)} \frac{1}{1-x}$$

$$= \tau_1(x).$$

An exactly parallel argument shows that if

$$R_{\nu,\varrho} = \sum_{M \leq \nu} \varrho^M - \nu \frac{E_N(k)}{\zeta(N)} \prod_{p|k} \frac{-1}{p^N-1}$$

is the error term in lemma 2.8 (the "crucial lemma" of (I)), then R satisfies the identity

$$(3.5) \qquad R_{\nu,1} = \frac{1}{\zeta(N)} \sum_{k=1}^{\infty} E_N(k) \prod_{p|k} \frac{-1}{p^N-1} \sum_{\varrho(k)} R_{\nu,\varrho}.$$

THE MINIMA OF INDEFINITE QUATERNARY QUADRATIC FORMS.[1]

By Alexander Oppenheim.

1. **Introduction.** Let $f = \sum a_{ik} x_i x_k$ $(a_{ik} = a_{ki})$ be a real quadratic form with real coefficients in n variables $x_1, x_2, \cdots, x_n$. The variables assume integral values only, the set $0, 0, \cdots, 0$ excepted. One of the many problems associated with quadratic forms is that of finding the upper bound of $L(f)$, the absolute lower bound of f, when f runs over the set of forms which have a given hessian $a = \| a_{ik} \|$, a not zero.[2]

If f is a positive definite binary form, it is well known that $L^2 \leqq 4a/3$, and that, if equality holds, f is equivalent to the form $L(x^2 + xy + y^2)$. Further we can find a form f of hessian a with $4a/3 > L^2 > 4a/3 - e$, for any positive e, however small.

In sharp contradistinction to the last result is the behavior of indefinite forms. Thus, if f is an indefinite binary quadratic form, then $L^2 \leqq -4a/5$. Equality implies that $f \sim L(x^2 + xy - y^2)$. If equality does not hold, then necessarily $L^2 \leqq -\frac{1}{2}a$, a much stronger inequality. Equality here implies $f \sim L(x^2 + 2xy - y^2)$. And so on. The complete theorem is due to Markoff.[3] It is Theorem 1 in § 2.

Similar results hold for indefinite ternary and quaternary quadratic forms. The first three minima for the former were obtained by Markoff[4] for forms with commensurable coefficients, and extended to the fourth minimum by Dickson[5] for forms which attain their lower bound. The extension to forms which do not necessarily attain their lower bound was made by the writer.[6] The results are given in Theorem 2 in § 3.

In the case of quaternary forms, the problem splits up naturally into two parts, for indefinite quaternary forms can have signature zero or ± 2. One difference between quaternary forms on the one hand and ternary and binary forms on the other hand may be noted. In the latter case corresponding, say, to $L^2 = -\frac{1}{2}a$, we have just one class of indefinite binary forms with this value of L. For indefinite quaternary forms of

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[1] Received May 26, 1930.—Doctoral Dissertation, University of Chicago, 1930.

[2] If a is zero, it is easy to see that $L = 0$.

[3] A. Markoff, 1, 2. For an exposition see L. E. Dickson, 2.

[4] Markoff, 3; exposition in Dickson, 2.

[5] Dickson, 2.

[6] See Dickson, 2.

signature ± 2 with $L^4 = -\frac{4}{15}a$, on the contrary, we obtain *two non-equivalent* classes of forms (Theorem B below).

My results are contained in two theorems which follow.

THEOREM A. *Let f be an indefinite quaternary quadratic form of signature zero, so that $a > 0$. Let $L^4(f) > \frac{64}{875}a$. Then necessarily*[7]

(A 1) $$L^4 = \tfrac{4}{9}a \quad or \quad L^4 = \tfrac{4}{17}a,$$

and $\pm f/L$ is equivalent to

(A 2) $$\begin{gathered} x^2 + xt + t^2 - 2y^2 - 2yz - 2z^2 \quad or \\ x^2 - y^2 - z^2 + t^2 + xt + yt + 2zt + 3zx + 2xy + yz \end{gathered}$$

respectively.

THEOREM B. *Let f be an indefinite quaternary quadratic form of signature ± 2, so that $a < 0$. Let $L^4(f) \geqq -\frac{1}{5}a$. Then necessarily*

(B1) $$-L^4/a = \tfrac{4}{7} \quad or \quad \tfrac{4}{15} \quad or \quad \tfrac{2}{9}.$$

In the first case, $\pm f/L$ is equivalent to the form

(B2) $$t^2 - x^2 - y^2 - z^2 + xt + yt + zt, \qquad -L^4/a = \tfrac{4}{7}.$$

In the second case, $\pm f/L$ is equivalent to one of the forms

(B3) $$x^2 + xt - t^2 + 2(y^2 + yz + z^2), \quad 2(x^2 + xt - t^2) + y^2 + yz + z^2, \quad -L^4/a = \tfrac{4}{15}.$$

In the third case, $\pm f/L$ is equivalent to the form

(B4) $$x^2 + xy + y^2 - 6z^2 + t^2, \qquad -L^4/a = \tfrac{2}{9}.$$

We have immediately the

COROLLARY. *Let f be an indefinite quaternary quadratic form. If f does not represent both L and $-L$ (in particular, if L is not attained), or if the coefficients of f are incommensurable, then $L^4 < \frac{1}{5}|a|$.*

In an earlier (unpublished) paper I obtained the first minima and forms in Theorems A and B. An account of this paper appears in Dickson, 2, Ch. IX. To this treatise and to Dickson, 1, I refer for expositions of the results given in §§ 2 and 3 on indefinite binary and ternary quadratic forms.

In the case of indefinite forms in five or more variables, no such theorems hold, for it is known that all indefinite quadratic forms in five or more variables with *commensurable* coefficients are *null*, so that $L(f)$ is necessarily zero. Very likely $L(f)$ is zero also when the coefficients of f are incommensurable, but as yet this has not been proved.

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[7] Another deduction of (A 1.1) is given in Oppenheim, 1.

Method of proof. Since L is the absolute lower bound of f, we have $|f| \geqq L$ for all sets of values of $x_1, x_2, \cdots, x_n$ other than $0, 0, \cdots, 0$. By means of this condition, we can derive various inequalities between the coefficients of f and so determine a relation between L and a. Since a and L are homogeneous of degrees n and 1 respectively in the coefficients of f, such a relation, it is clear, will be of the form $L^n \leqq K(n, s) |a|$ where K is a number depending only on n and s, the signature of the set of forms under consideration.

In practice it is better to take linear combinations of the variables and so obtain binary or ternary forms, which are *sections* of f. Evidently any such section, g, will have a lower bound $\geqq L$, since every value of g is a value of f. To these sections g we can apply the known results concerning the lower bounds of binary and ternary quadratic forms. By proper choice of these sections, we derive information sufficient to prove Theorems A and B.

To avoid stretching this paper to inordinate length, I assume throughout that $L(f)$ is attained.[8] I have shown elsewhere[9] in the case of indefinite ternary forms how this restriction may be removed. Precisely the same type of argument will serve in the present case.

Sections 2 and 3 are devoted to the results and notions required from the theory of binary and ternary forms.

Section 4 deals with notation and the reduction of the quaternary form to a *standard* form, convenient for the analysis.

Section 5 is devoted to the proof of Theorem A. The proof of Theorem B occupies §§ 6 and 7.

2. **Properties of binary quadratic forms.**[10] Let $q(x, y) = ax^2 + bxy + cy^2$, which we also denote by $[a, b, c]$ or $(a, \frac{1}{2}b, c)$. The hessian or determinant of q we denote by $\Delta(q)$. The discriminant is defined by

$$d = b^2 - 4ac = -4\Delta(q). \tag{2.1}$$

2.1. *Definite forms.* $q = [a, b, c]$ is definite if $\Delta > 0$ being positive definite if $a > 0$, negative definite if $a < 0$.

LEMMA 1. *If q is definite, then q has a non-zero minimum which is properly represented.*

We say that the positive form q is *reduced* if

$$-a < b \leqq a \leqq c \quad \text{and} \quad b \geqq 0 \quad \text{if} \quad c = a. \tag{2.11}$$

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[8] If f is definite, the lower bound is always attained.

[9] Dickson, 2.

[10] For proofs of the results in § 2, see Dickson, 1 and 2.

We shall say that q is reduced in the wide sense if the last condition is omitted.

LEMMA 2. *Let $q = [a, b, c]$ be positive definite. Let a be the minimum of q. There is a unique integer k such that the parallel transformation $x = X + kY$, $y = Y$ carries q into $Q = [A, B, C]$, where Q is reduced in the wide sense.*

Plainly $A = a$, $B = 2ak + b$. There is a unique integer k such that $-a = -A < B \leqq a$. Since a is the minimum of q and so of Q, we have $C \geqq A = a$. The lemma follows.

LEMMA 3. *Every positive definite binary form is equivalent to a reduced form $[a, b, c]$.*

LEMMA 4. *The least value taken by a positive reduced form $[a, b, c]$ is a. Any other value, v, of q, with the possible exceptions of $4a$, $9a$, $\cdots$, is greater than or equal to c. In particular, if v/a is not the square of an integer, then $v \geqq c$. Further, in a reduced form,*

$$a^2 \leqq ac \leqq 4\Delta/3, \quad 3ac \leqq a(4c - a) \leqq 4\Delta. \tag{2.12}$$

If $y \neq 0$, we have

$$ax^2 + bxy + cy^2 - c \geqq ax^2 - a|xy| + a(y^2 - 1) = a(x^2 - |xy| + y^2 - 1) \geqq 0.$$

If $y = 0$, the values of q are a, $4a$, $9a$, $\cdots$. The first three statements of the lemma follow at once. The last follows from (2.01) and (2.111).

LEMMA 5. *Equivalent positive reduced forms are identical.*

Similar results hold for negative definite forms.

2.2. *Indefinite forms.* $q = [a, b, c]$ is indefinite if $d > 0$ (or $\Delta < 0$). Let $R^2 = d$, $R > 0$. We say that *q is reduced* if

$$0 < R - b < 2|a| < R + b. \tag{2.21}$$

Since $R^2 - b^2 = -4ac$, these inequalities are equivalent to

$$0 < R - b < 2|c| < R + b. \tag{2.22}$$

In a reduced form q, a and c have opposite signs.

Df. Right neighboring form. If the proper transformation $x = Y$, $y = -X + kY$ where k is an integer not zero, carries q into r, we say that r is a right hand neighbor of q.

q is called a left hand neighbor of r.

LEMMA 6. *Any indefinite binary form is equivalent to a reduced form.*

LEMMA 7. *Every reduced form has a unique reduced right (or left) neighbor.*

To any indefinite form $q = [a, b, c]$ of discriminant d corresponds a *chain* (φ_i), extending to infinity in both directions (if the numbers $(R+b)/2a$ are irrational), of equivalent reduced forms, each form being the right (left) hand neighbor of its predecessor (successor).

LEMMA 8. *Equivalent reduced forms belong to the same chain.*

Consequently the same chain is determined by any one of its links.

Notation for chain (φ_i). We write

$$(2.31) \quad \varphi_i = [(-)^i A_i,\ B_i,\ (-)^{i+1} A_{i+1}], \qquad A_i > 0,\ B_i > 0 \text{ (all } i).$$

$$(2.32) \quad B_{i-1} + B_i = 2 g_{i-1} A_i, \qquad g_i \text{ a positive integer,}$$

$$(2.33) \quad F_i = (g_i, g_{i+1}, \cdots), \qquad S_i = (0, g_{i-1}, g_{i-2}, \cdots),$$

$$(2.34) \quad K_i = F_i + S_i = R/A_{i+1}, \quad F_i - S_i = B_i/A_{i+1}, \quad F_i S_i = A_i/A_{i+1},$$

$$(2.35) \quad d = R^2 = B_i^2 + 4 A_i A_{i+1}, \qquad R > 0.$$

The roots of $\varphi_i(t, 1) = 0$ are $(-)^i/F_i$ and $(-)^{i+1}/S_i$. The transformation $x = Y$, $y = -X + k_i Y$, $k_i = (-)^i g_i$, carries φ_i into φ_{i+1}.

We suppose that the numbers $(R \pm B_i)/2A_i$ are irrational. Otherwise the chain would terminate. In other words we suppose that neither root of the quadratic in t, $q(t, 1) = 0$, is rational.

LEMMA 9. (Lagrange.) *The numbers $(-)^i A_i$ include all those numbers numerically $\leqq \frac{1}{2}R$ which are properly represented by q.*

Lemma 9 follows from Lemma 8 and

LEMMA 10. *Let $q = [a, b, c]$ be indefinite. Let $|a| \leqq \frac{1}{2}R$. There is a unique parallel transformation $x = X + kY$, $y = Y$, k integral, which makes q reduced.*

The transformation $x = X + kY$, $y = Y$ carries $[a, b, c]$ into $[A, B, C]$ where $A = a$, $B = 2ak + b$. There is a unique integer k such that

$$0 < R - B = R - 2ak - b < 2|A| = 2|a|.$$

Since $2|A| \leq R$, $R - B < R$, $B > 0$ and therefore $2|A| \leqq R < R + B$. The form $[A, B, C]$ is reduced.

Let $L(q)$ denote the lower bound of $|q|$ for all integral values of x, y, the pair $(0, 0)$ excepted. Let (φ_i) be the corresponding chain.

LEMMA 11. *$L(q)$ is the lower bound of the numbers A_i, and is therefore equal to R divided by the upper bound of the numbers K_i.*

2.5. *Markoff's Theorem on the minima of indefinite binary quadratic forms.* We consider all indefinite binary quadratic forms f of discriminant $d = R^2$, $R > 0$.

THEOREM 1. *We have the following possibilities*:

$$L(f) = \sqrt{\tfrac{d}{5}}, \quad f \sim f_0 = \sqrt{\tfrac{d}{5}}(x^2 + xy - y^2);$$
$$L(f) = \sqrt{\tfrac{d}{8}}, \quad f \sim f_1 = \sqrt{\tfrac{d}{8}}(x^2 + 2xy - y^2);$$
$$L(f) = \sqrt{\tfrac{25d}{221}}, \quad f \sim f_2 = \sqrt{\tfrac{d}{221}}(5x^2 + 11xy - 5y^2);$$

$\cdots$ *The set* $f_0, f_1, \cdots$ *continues indefinitely: each* f_i *has discriminant* d *and* $L(f_i) > \frac{1}{3}R$. *Each is proportional to an integral form. Further, if* $L(f) > \frac{1}{3}R$, *then* f *is equivalent properly or improperly to one of the* f_i. *Each* f_i *represents both* $L(f_i)$ *and* $-L(f_i)$.

LEMMA 12. *If the coefficients of* f *are incommensurable or if* f *does not represent both* $L(f)$ *and* $-L(f)$, *then* $L(f) \leqq \frac{1}{3}R$.

3. Ternary quadratic forms.

3.1. *Indefinite ternary quadratic forms.*

THEOREM 2. (Markoff)[11]. *Let* Φ *be an indefinite ternary quadratic form of non-zero hessian* Δ *and absolute lower bound* $L(\Phi)$. *We have the following possibilities:*

(3.11) $L^3 = \frac{2}{3}|\Delta|$ *and* $\Phi \sim \Phi_1 = (\frac{2}{3}\Delta)^{\frac{1}{3}}(x^2 - y^2 - z^2 + xy + xz)$;

(3.12) $L^3 = \frac{2}{5}|\Delta|$ *and* $\Phi \sim \Phi_2 = (\frac{2}{5}\Delta)^{\frac{1}{3}}(x^2 - y^2 - z^2 + yz + 2zx + xy)$;

(3.13) $L^3 = \frac{1}{3}|\Delta|$ *and* $\Phi \sim \Phi_3 = (\frac{1}{3}\Delta)^{\frac{1}{3}}(x^2 - y^2 - z^2 + 2xy + 2xz)$;

(3.14) $L^3 = \frac{8}{25}|\Delta|$ *and* $\Phi \sim \Phi_4 = (\frac{8}{25}\Delta)^{\frac{1}{3}}(x^2 - y^2 - \frac{3}{2}z^2 + yz + 2zx + xy)$;

or else

(3.15) $$L^3 < .30033\,|\Delta|.$$

LEMMA 13. *If the coefficients of* Φ *are incommensurable or if* Φ *does not represent both* L *and* $-L$, *then* $L^3 < .30033\,|\Delta|$.

For $\Phi_1, \cdots, \Phi_4$ have commensurable coefficients and represent both L and $-L$.

Suppose now that Φ attains its lower bound $L(\Phi)$. We may take

(3.21) $$\Phi = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \qquad a\Phi = (ax + hy + gz)^2 + \varphi(y, z),$$

(3.22) $$\varphi(y, z) = Cy^2 - 2Fyz + Bz^2, \qquad BC - F^2 = a\Delta,$$

where $|a| = L$. Let $L > 0$. The binary form φ is negative definite if $a\Delta > 0$ and indefinite if $a\Delta < 0$. We may suppose that C and $a\Delta$ have opposite signs and that φ is reduced. By a parallel transformation on x, we can reduce (Lemmas 2 and 10) both the binaries (a, h, b) and (a, g, c) without altering φ. Reduction here may be in the wide sense.

[11] Markoff, 2 for Φ_1, Φ_2 and Φ_3; Dickson, 2 for (3.14) and (3.15). Exposition of Theorem 2 in Dickson, 2.

Under these conditions we shall say that Φ is in *standard form.* The standard form is not necessarily unique.

LEMMA 14. *Let Φ be in standard form with $a>0$. Then, if $a^3=\frac{2}{3}\Delta$, Φ/a is equal to*

$$(3.31)\qquad (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2-\tfrac{5}{4}y^2-\tfrac{1}{2}yz-\tfrac{5}{4}z^2;$$

and, if $a^3=-\frac{2}{3}\Delta$, Φ/a is equal to

$$(3.32)\quad (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2+\tfrac{3}{4}y^2+\tfrac{3}{2}yz-\tfrac{5}{4}z^2 \quad \text{or} \quad (x+\tfrac{1}{2}z)^2+y^2+yz-\tfrac{5}{4}z^2.$$

We may take $a=2$ so that $\Delta=\pm 12$. Since Φ is equivalent to Φ_1 of Theorem 2, the coefficients of which are integral multiples of $\frac{1}{2}L=\frac{1}{2}a=1$, the coefficients $a,\cdots,h$ and therefore the cofactors B, C, F of Φ are integers.

If $\Delta=12$, φ is a reduced negative definite binary form so that

$$(3.33)\qquad |2F|\leqq|C|,\quad |2F|\leqq|B|,\quad BC-F^2=a\Delta=24.$$

The indefinite binary forms (a,h,b) and (a,g,c) have minimum a. Theorem 1 yields therefore

$$(3.34)\qquad -C=5 \text{ or } -C\geqq 8;\qquad -B=5 \text{ or } -B\geqq 8.$$

From (3.33) and (3.34) we find $B=C=-5$, $F=1$. Since (a,h,b) is reduced indefinite, of minimum a, we have

$$(3.35)\ 0<\sqrt{5}-h<a\leqq -b<\sqrt{5}+h,\qquad -5=ab-h^2=2b-h^2,$$

whence $h=1$, $b=-2$. Similarly $g=1$, $c=-2$. We obtain (3.31).

If $\Delta=-12$, φ is reduced indefinite with $C>0$ so that $B<0$ and

$$(3.36)\qquad 0<\sqrt{12}+F<C<\sqrt{12}-F,\qquad BC-F^2=-24.$$

The indefinite binary (a,g,c) gives (3.342). The definite binary (a,h,b) of minimum a gives $C\geqq 3$. Using (3.36), we obtain $C=4$, $F=-3$, $B=-5$ or $C=4$, $F=-2$, $B=-5$.

As before, $g=1$, $c=-2$. Since (a,h,b) is reduced positive definite and $2b-t^2=C=3$ or 4, we must have $b=2$, $t=1$ or $b=2$, $t=0$. We obtain the two cases of (3.32).

Observe that if C is the minimum of $\varphi(y,z)$ then the first case must hold.

LEMMA 15. *Let Φ be in standard form, $a>0$. Then,*

$$(3.41)\quad \text{if}\ \ a^3=\tfrac{2}{5}\Delta,\qquad \Phi/a=(x+\tfrac{1}{2}y+z)^2-\tfrac{5}{4}y^2-2z^2;$$

THEOREM 1. *We have the following possibilities*:

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(3.12) $L^3=\frac{2}{5}|\Delta|$ *and* $\Phi\sim\Phi_2=(\frac{2}{5}\Delta)^{\frac{1}{3}}(x^2-y^2-z^2+yz+2zx+xy)$;

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(3.14) $L^3=\frac{8}{25}|\Delta|$ *and* $\Phi\sim\Phi_4=(\frac{8}{25}\Delta)^{\frac{1}{3}}(x^2-y^2-\frac{3}{2}z^2+yz+2zx+xy)$;

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LEMMA 13. *If the coefficients of* Φ *are incommensurable or if* Φ *does not represent both* L *and* $-L$, *then* $L^3<.30033\,|\Delta|$.

For $\Phi_1,\cdots,\Phi_4$ have commensurable coefficients and represent both L and $-L$.

Suppose now that Φ attains its lower bound $L(\Phi)$. We may take

$$\Phi=ax^2+by^2+cz^2+2fyz+2gzx+2hxy,\qquad(3.21)$$
$$a\Phi=(ax+hy+gz)^2+\varphi(y,z),$$

$$\varphi(y,z)=Cy^2-2Fyz+Bz^2,\qquad BC-F^2=a\Delta,\qquad(3.22)$$

where $|a|=L$. Let $L>0$. The binary form φ is negative definite if $a\Delta>0$ and indefinite if $a\Delta<0$. We may suppose that C and $a\Delta$ have opposite signs and that φ is reduced. By a parallel transformation on x, we can reduce (Lemmas 2 and 10) both the binaries (a, h, b) and (a, g, c) without altering φ. Reduction here may be in the wide sense.

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$$(x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2-\tfrac{5}{4}y^2-\tfrac{1}{2}yz-\tfrac{5}{4}z^2; \tag{3.31}$$

and, if $a^3=-\frac{2}{3}\Delta$, Φ/a is equal to

$$(x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2+\tfrac{3}{4}y^2+\tfrac{3}{2}yz-\tfrac{5}{4}z^2 \quad \textit{or} \quad (x+\tfrac{1}{2}z)^2+y^2+yz-\tfrac{5}{4}z^2. \tag{3.32}$$

We may take $a=2$ so that $\Delta=\pm 12$. Since Φ is equivalent to Φ_1 of Theorem 2, the coefficients of which are integral multiples of $\frac{1}{2}L=\frac{1}{2}a=1$, the coefficients $a,\cdots,h$ and therefore the cofactors B, C, F of Φ are integers.

If $\Delta=12$, φ is a reduced negative definite binary form so that

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$$-C=5 \text{ or } -C\geqq 8;\qquad -B=5 \text{ or } -B\geqq 8. \tag{3.34}$$

From (3.33) and (3.34) we find $B=C=-5$, $F=1$. Since (a,h,b) is reduced indefinite, of minimum a, we have

$$0<\sqrt{5}-h<a\leqq -b<\sqrt{5}+h,\qquad -5=ab-h^2=2b-h^2, \tag{3.35}$$

whence $h=1$, $b=-2$. Similarly $g=1$, $c=-2$. We obtain (3.31).

If $\Delta=-12$, φ is reduced indefinite with $C>0$ so that $B<0$ and

$$0<\sqrt{12}+F<C<\sqrt{12}-F,\qquad BC-F^2=-24. \tag{3.36}$$

The indefinite binary (a,g,c) gives (3.342). The definite binary (a,h,b) of minimum a gives $C\geqq 3$. Using (3.36), we obtain $C=4$, $F=-3$, $B=-5$ or $C=4$, $F=-2$, $B=-5$.

As before, $g=1$, $c=-2$. Since (a,h,b) is reduced positive definite and $2b-t^2=C=3$ or 4, we must have $b=2$, $t=1$ or $b=2$, $t=0$. We obtain the two cases of (3.32).

Observe that if C is the minimum of $\varphi(y,z)$ then the first case must hold.

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$$\textit{if}\quad a^3=\tfrac{2}{5}\Delta,\qquad \Phi/a=(x+\tfrac{1}{2}y+z)^2-\tfrac{5}{4}y^2-2z^2; \tag{3.41}$$

(3.42) *if* $a^3 = \frac{1}{3}\Delta, \quad \Phi/a = (x+y+z)^2 - 2y^2 - 2yz - 2z^2;$

(3.43) *if* $a^3 = \frac{8}{25}\Delta, \quad \Phi/a = (x+\frac{1}{2}y+z)^2 - \frac{5}{4}y^2 - \frac{5}{2}z^2;$

if $a^3 = -\frac{2}{5}\Delta$, Φ/a *is equal to*

(3.51)
$$(x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2 + \tfrac{3}{4}y^2 + \tfrac{5}{2}yz - \tfrac{5}{4}z^2$$
$$\text{or } (x+\tfrac{1}{2}y+z)^2 + \tfrac{3}{4}y^2 + 2yz - 2z^2;$$

(3.52) *if* $a^3 = -\frac{1}{3}\Delta, \quad \Phi/a = (x+z)^2 + y^2 + 2yz - 2z^2;$

(3.53) *if* $a^3 = -\frac{8}{25}\Delta, \quad \Phi/a = (x+\frac{1}{2}y+\frac{1}{2}z)^2 + \frac{5}{4}y^2 + \frac{5}{2}yz - \frac{5}{4}z^2.$

In (3.41) and (3.43) we may interchange y and z. The proof of these results is similar to the proof of Lemma 14 and may therefore be omitted.

LEMMA 16. *Let $\Delta > 0$ and suppose that Φ represents L but not $-L$. Then $L^3 \leqq \frac{64}{243}\Delta$.*

We may take $a = L$ and suppose that φ is reduced negative definite so that $BC \leqq \frac{4}{3}a\Delta$.

The binary forms (a, h, b), (a, g, c) have minimum a, are indefinite and do not represent $-a$. Therefore by Lemma 12, $a^2 \leqq -4B/9$, $a^2 \leqq -4C/9$. These inequalities yield Lemma 16.

Lemma 16 holds also if Φ does not represent numbers arbitrarily close to $-L$.

3.6. *Definite ternary quadratic forms.* Let Φ be a positive definite ternary quadratic form. We may suppose that a is the minimum of Φ and that C is the minimum of the positive definite binary form $\varphi(y, z)$. Then[12]

(3.61) $$a^2 \leqq \tfrac{4}{3}C, \qquad aC \leqq \tfrac{3}{2}\Delta, \qquad a^3 \leqq 2\Delta.$$

4. **Indefinite quaternary quadratic forms. Notation and standard form.** Let

(4.1) $$\psi = \psi(x, y, z, t) = (a, b, c, d, f, g, h, u, v, w)(x, y, z, t)^2$$

be an indefinite quaternary quadratic form of absolute lower bound $m \geqq 0$. We shall suppose that $m > 0$, so that necessarily $\Delta(\psi) \neq 0$, and that m or $-m$ is attained. Considering if necessary $-\psi$ we may therefore suppose that m is represented and plainly properly represented by ψ. We may take $d = m$ and write

(4.21) $$d\psi = (ux + vy + wz + dt)^2 + \Phi(x, y, z),$$

[12] Korkine and Zolotareff, 1.

$$\Phi = (\alpha, \beta, \gamma, \varrho, \sigma, \tau)\,(x, y, z)^2, \tag{4.22}$$
$$\alpha = ad - u^2, \cdots, \varrho = fd - vw, \cdots,$$

$$n = L(\Phi), \qquad n > 0.\text{[13]} \tag{4.3}$$

We suppose that n is attained[14] by $|\Phi|$ and we take

$$\alpha = -n \text{ if } \Phi \text{ represents } -n, \tag{4.31}$$

$$\alpha = n \text{ if } \Phi \text{ represents } n \text{ but not } -n. \tag{4.32}$$

Write now

$$\alpha\Phi = (\alpha x + \tau y + \sigma z)^2 + d\varphi(yz), \qquad \varphi = Cy^2 - 2Fyz + Bz^2, \tag{4.33}$$

where $B, C, \cdots$ are the cofactors of $b, c, \cdots$ in Δ. It is easily verified that

$$\Delta_1 = \Delta(\Phi) = d^2\Delta, \qquad \Delta_2 = \Delta(\varphi) = BC - F^2 = \alpha\Delta. \tag{4.4}$$

A proper transformation on y and z and a parallel transformation on x puts the ternary form Φ into standard form (§ 3).

By a parallel transformation on t, which plainly does not disturb Φ, we can reduce the three binary forms obtained from Ψ by putting two of x, y, z equal to zero, in virtue of Lemmas 2 and 10 since d is evidently the minimum of any section of Ψ when that section represents d.

In the case of φ the reduction is in the strict sense. For other binary sections, when definite, the reduction must be interpreted occasionally in the wide sense.

When this reduction has been performed we shall say that Ψ is in *standard form*. The standard form is not necessarily unique.

Signature. Let S denote the signature of Ψ. Our discussion falls into two essentially distinct parts:

$$S = 0 \text{ so that } \Delta > 0,\ \Phi \text{ is indefinite of signature } -1: \tag{4.5}$$

$$\begin{cases} S = -2,\ \Delta < 0,\ \Phi \text{ is negative definite;} \\ S = +2,\ \Delta < 0,\ \Phi \text{ is indefinite of signature } +1. \end{cases} \tag{4.6}$$

5. The case $S = 0$, $\Delta > 0$, Φ indefinite of signature -1.

(5.1) $\alpha = -n$. The binary (a, u, d) has positive discriminant $-4\alpha = 4n$ and minimum d so that by Theorem 1,

$$d^2 = \tfrac{4}{5}n \quad \text{or} \quad d^2 \leq \tfrac{1}{2}n. \tag{5.12}$$

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[13] If $n = 0$, then we can make $|d\Psi| < \frac{1}{4}d^2 + e$ for any $e > 0$. But $d \neq 0$ and is the lower bound of $|\Psi|$.

[14] This restriction will be removed later.

Since Φ is indefinite of minimum n and hessian $d^2\Delta$ by (4.4), Theorem 2 yields

$$(5.14) \qquad n^3/d^2\Delta = \tfrac{2}{3},\ \tfrac{2}{5} \quad \text{or} \quad \leqq \tfrac{1}{3}.$$

Case (5.121) *with* (5.141) *or* (5.142). Here, by Theorem 2, we may take

$$\Phi(x, y, z) = n(-x^2 - y^2 + z^2 + yz + \lambda zx + \mu xy),$$

the precise values of λ and μ being irrelevant, and so with an obvious change of notation,

$$\Psi/d = (ux + vy + wz + t)^2 + \tfrac{5}{4}(-x^2 - y^2 + z^2 + yz + \lambda zx + \mu xy).$$

Herein take $x = 1, y = z = 0$. Since $|\Psi/d| \geqq 1$, $|(u+t)^2 - \frac{5}{4}| \geqq 1$ for any integer t, so that $2u$ is an odd integer. We may take $2u = 1$. Similarly, by taking $x = z = 0, y = 1$, we obtain $2v = 1$. Now put $x = 0, y = -1, z = 1$. We find that $2w - 1$ must be an odd integer and so $w = 0$. But now $\Psi(0, 3, 1, 1) = 0$, whereas Ψ has minimum $d \neq 0$.

From (5.121) and (5.143) and from (5.122) and (5.14) we obtain respectively

$$d^6 \leqq (\tfrac{4}{5})^3 \tfrac{1}{3} d^2\Delta, \quad d^6 \leqq \tfrac{1}{2^3}\tfrac{2}{3} d^2\Delta.$$

We conclude that, *if* $S = 0$ *and* Φ *represents* $-n$, then

$$(5.15) \qquad d^4 \leqq \tfrac{64}{375}\Delta.^{15}$$

(5.2) *Case* $\alpha = n$. Here Φ is indefinite and does not represent $-n$. We suppose that Ψ is in standard form. Then φ is negative definite reduced so that we may suppose

$$(5.23) \quad 0 < -C \leqq -B, \quad \tfrac{3}{4}BC \leqq -C(-B + \tfrac{1}{4}C) \leqq \alpha\Delta \quad \text{by (2.122)}.$$

The binary forms (α, τ, β), (α, σ, γ) have hessians dC, dB and minimum α. They do not represent $-\alpha$. By Lemma 12 therefore

$$(5.24) \qquad \alpha^2 \leqq -\tfrac{4}{9}dC, \quad \alpha^2 \leqq -\tfrac{4}{9}dB.$$

Consider now the ternary form $\Psi_{y=0}$. It has hessian $B < 0$ and minimum $d > 0$. Therefore it is indefinite and Theorem 2 gives

$$(5.26) \qquad -d^3/B = \tfrac{2}{3},\ \tfrac{2}{5},\ \tfrac{1}{3},\ \tfrac{8}{25} \quad \text{or} \quad < .30033,$$

[15] Improvement is possible but not worth while.

and similarly from $\psi_{z=0}$

$$-d^3/C = \tfrac{2}{3},\ \tfrac{2}{5},\ \tfrac{1}{3},\ \tfrac{8}{25} \quad \text{or} < .30033. \tag{5.27}$$

Number the cases of (5.26) and (5.27) $B_1, \cdots, B_5, C_1, \cdots, C_5$, respectively.

Case B_5C_5. Employing (5.23) and (5.24) we obtain

$$d^{18} < (\tfrac{8}{25})^6\, (\tfrac{4}{3})^3\, \tfrac{64}{243}\, d^2\Delta^4, \quad d^4 < \tfrac{2^7\, 2}{3^2\, 5^3}\Delta < \tfrac{64}{375}\Delta. \tag{B_5C_5}$$

Case B_5C_i $(i = 1, 2, 3, 4)$. Since ψ is in standard form, the ternary section $\psi_{y=0}$ is also in standard form. The second parts of Lemmas 14 and 15 are applicable and yield in the respective cases[16]

$$\alpha/d^2 = \tfrac{3}{4}(C_1, C_2), \quad \alpha/d^2 = 1(C_3), \quad \alpha/d^2 = \tfrac{5}{4}(C_4). \tag{5.28}$$

From (5.26), (5.27), (5.23) and (5.28), we obtain

$$d^4/\Delta < \tfrac{64}{375}(B_1, B_2), \quad d^4/\Delta < \tfrac{32}{225}(B_3), \quad d^4/\Delta < \tfrac{64}{375}(B_4)$$

and consequently

$$d^4 < \tfrac{64}{375}\Delta. \tag{B_5C_i}$$

Case B_iC_j $(i, j = 1, 2, 3, 4)$. In addition to (5.28) we have

$$\alpha/d^2 = \tfrac{3}{4}(B_1, B_2), \quad \alpha/d^2 = 1(B_3), \quad \alpha/d^2 = \tfrac{5}{4}(B_4). \tag{5.29}$$

Inspection of (5.26) and (5.27) shows that the only possible combinations are $B_i C_j$ $(i, j = 1, 2)$, B_3C_3 and B_4C_4. For B_3C_3 and B_4C_4 we have respectively

$$d^4/\Delta \leqq \tfrac{4}{27}, \quad d^4/\Delta \leqq \tfrac{64}{375}.$$

Case B_1C_1. By the second part of Lemma 14 and the footnote above, we have

$$\psi_{y=0} = d(t^2+x^2-z^2+2xz+zt+xt), \tag{5.31}$$

$$\psi_{z=0} = d(t^2+x^2-y^2+2xy+yt+xt), \tag{5.32}$$

whence

$$\psi/d = t^2+x^2-y^2-z^2+xt+yt+zt+2xy+2xz+2fyz, \tag{5.33}$$

and

$$\Delta = \tfrac{3}{4}d^4[4-(f-1)^2]. \tag{5.34}$$

But

$$\tfrac{9}{4}d^6 = BC \leqq \tfrac{4}{3}\alpha\Delta = d^2\Delta, \quad d^4 \leqq \tfrac{4}{9}\Delta, \tag{5.35}$$

[16] The second case in Lemma 14 is excluded since $y^2+yz-\frac{5}{4}z^2$ represents $\frac{3}{4}$ for $y = z = 1$. But by hypothesis α is the minimum of (α, σ, γ).

so that

$$0 \leqq f \leqq 2. \tag{5.36}$$

For $x = 0$, $t = -1$, $y = z = 1$, we have $\psi/d = 2f-3$ so that either $2f-3 \geqq 1$, $f \geqq 2$ (and so $f = 2$) or else $2f-3 \leqq -1$, $f \leqq 1$. In the latter case take $x = t = 0$, $y = z = 1$. Then $\psi/d = 2f-2$ so that $2f-2 \leqq -1$, $2f \leqq 1$.

Put $x = 0$, $t = y$ in ψ/d. We obtain the binary form $y^2+(2f+1)yz-z^2$ indefinite, of minimum unity and discriminant $(2f-1)^2+4 \leqq 8$. By Theorem 1 therefore, this discriminant must be either 8 or 5 whence $f = \frac{1}{2}$ or 0. The former value is excluded since $\psi(1, -1, 1, 1) = (1-2f)\,d$.

We are left with $f = 0$ and $f = 2$. The form in the latter case transforms improperly into the form from $f = 0$ when we replace z by $-z$ and x by $x+2z$. But interchange of y and z, an improper transformation, does not alter (5.33). It follows that $(B_1 C_1)$ yields the form

$$\psi_1 = d(t^2+x^2-y^2-z^2+2zx+2xy+xt+yt+zt), \quad d^4 = \tfrac{4}{9}\Delta \tag{5.37}$$

by (5.34). This is the first form in Theorem A.

It remains to show that this form actually has minimum d. But

$$4\,\psi_1/d = (2x+2y+2z+t)^2+3t^2-2(2y+z)^2-6x^2,$$

and the form $X^2+3T^2-2Y^2-6Z^2$ is not *null*, as is easily shown by descent.

Case $B_2 C_1$. We have (5.32), and by Lemma 15, $\psi_{y=0}/d$ is equal to

$$(t+\tfrac{1}{2}x+z)^2+\tfrac{3}{4}x^2+2xz-2z^2 \quad \text{or} \quad (t+\tfrac{1}{2}x+\tfrac{1}{2}z)^2+\tfrac{3}{4}x^2+\tfrac{5}{2}xz-\tfrac{5}{4}z^2$$

so that either

$$\psi/d = (t+\tfrac{1}{2}x+\tfrac{1}{2}y+z)^2+\tfrac{3}{4}x^2-\tfrac{5}{4}y^2-2z^2+\tfrac{3}{2}xy+2xz+2\varrho yz \tag{5.42}$$

or

$$\psi/d = (t+\tfrac{1}{2}x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2+\tfrac{3}{4}x^2-\tfrac{5}{4}y^2-\tfrac{5}{4}z^2+\tfrac{3}{2}xy+\tfrac{5}{2}xz+2\varrho yz. \tag{5.43}$$

Replace in (5.43) x by $-x-2y-3z$ and t by $-t$, a proper transformation. We obtain (5.42) with $\varrho-\frac{1}{4}$ in place of ϱ. Our transformation leaves the binary form $\varphi = (C, -F, B)$ unaltered and so still reduced negative definite. We may thus confine ourselves to the case (5.42). Now apart from a multiplier, φ is $(24, 12-12\varrho, 40)$ so that

$$-24 < 24-24\varrho \leqq 24, \quad 0 \leqq \varrho < 2, \quad 1 \leqq 2f < 5 \quad \text{if} \quad 2f = 2\varrho+1. \tag{5.44}$$

Since $\psi(0, 0, 1, 1) = (2f-2)\,d$, we have either $2f-2 \leqq -1$, $2f \leqq 1$ and so $2f = 1$, or else $2f-2 \geqq 1$, $2f \geqq 3$. Now $\psi(0, 1, 1, -1)$

$= (2f-4)d$. If $2f-4 \geqq 1$, then $2f \geqq 5$ contradicting (5.44). Therefore $2f-4 \leqq -1$, $2f \leqq 3$ and so $2f = 3$. But $\Psi(-1, 1, 1, 1) = (2f-3)d$ so that $2f \neq 3$.

The only remaining possibility, $2f = 1$, yields the form

$$(5.45) \quad \Psi_2 = d(x^2 - y^2 - z^2 + t^2 + xt + yt + 2zt + 3zx + 2xy + xz),$$

the second form in Theorem A.

To show that Ψ_2 has minimum d, it is sufficient to show that Ψ_2/d is not null. Now $12\Psi_2/d$ is equal to

$$(5.46) \quad 3(2t+x+y+2z)^2 + (3x+3y+4z)^2 - 6(2y+z)^2 - 34z^2,$$

and the form $3T^2 + X^2 - 6Y^2 - 34Z^2$ is not null. If it is null, we may suppose that $3T^2 + X^2 = 6Y^2 + 34Z^2$ where X, Y, Z, T are not all even. Since $3T^2 + X^2$ cannot be congruent to 2 or 6 modulo 8, we see that Y and Z are both even or both odd. If Y and Z are even, then $3T^2 + X^2 \equiv 0 \pmod 8$, X and T are even, whereas X, Y, Z, T are not all even. Hence Y and Z are odd, $3T^2 + X^2 \equiv 8 \pmod{16}$ which is impossible. Our assertion follows.

Case $B_1 C_2$. We obtain $B_1 C_2$ from $B_2 C_1$ by interchanging Y and Z and so obtain (5.45) with Y and Z interchanged, an improper transformation. But from (5.46) the improper transformation $x = -X$, $y = -Y$, $z = -Z$, $t = T + X + Y + 2Z$ leaves Ψ_2 unaltered. Thus Ψ_2 is improperly equivalent to itself and the cases $B_1 C_2$, $B_2 C_1$ both yield the form (5.45).

Case $B_2 C_2$. Here we have

$$(5.51) \quad d^6 = \tfrac{4}{25} BC, \qquad BC \leqq \tfrac{4}{3}\alpha\Delta = d^2\Delta, \qquad d^4 \leqq \tfrac{4}{25}\Delta.$$

Actually by using Lemma 15 and the arguments of $B_1 C_1$, $B_2 C_1$, we can show that $B_2 C_2$ leads to the form

$$(5.52) \quad \Psi_3 = d[x^2 - y^2 - z^2 + t^2 + 3xy + 3xz + 4yz + xt + yt + zt], \quad d^4 = 4\Delta/33.$$

Summing up the various cases we obtain Theorem A in the case when $|\Phi|$ represents n.

(5.6) $|\Phi|$ *does not represent* n. Then for an infinity of values α of Φ we have either

$$(5.61) \quad -n - \varepsilon n < \alpha < -n \quad \text{or} \quad n < \alpha < n + \varepsilon n$$

where ε is an arbitrarily small positive number. If (5.611) is satisfied for arbitrarily many α's, then Theorem 1 and Lemma 12 applied to the indefinite binary form (a, u, d), give for some α,

$$d^2 \leqq -(\tfrac{4}{9} + \delta)\alpha, \quad \delta > 0.$$

Also by Theorem 2 we have certainly $n^3 < 8d^2\Delta/25$. We obtain

$$d^4 < (\tfrac{4}{9}+\delta)^3\,\tfrac{8}{25}(1+\varepsilon)^3\Delta, \quad d^4 \leqq \tfrac{4^3}{9^3}\cdot\tfrac{8}{25}\Delta, \tag{5.62}$$

since ε and δ are arbitrary.

If however (5.611) is not satisfied for an infinity of α's, then (5.612) must hold for arbitrarily many α's. There exists therefore an α satisfying (5.612) and such that both

$$-d^3/B < \tfrac{8}{25} \quad \text{and} \quad -d^3/C < \tfrac{8}{25}. \tag{5.63}$$

Also (corresponding to (5.24)) we have

$$n^2 \leqq -\tfrac{4}{9}dB, \quad n^2 \leqq -\tfrac{4}{9}dC. \tag{5.64}$$

From (5.23), (5.612), (5.63) and (5.64) we obtain

$$d^{18} < (\tfrac{8}{25})^6 B^3C^3 \leqq (\tfrac{8}{25})^6(\tfrac{4}{3})^3\alpha^3\Delta^3 < (\tfrac{8}{25})^6(\tfrac{4}{3})^3(1+\varepsilon)^4(\tfrac{4}{9})^2\tfrac{4}{3}d^2\Delta^4,$$

for any positive ε, so that $d^4/\Delta < \frac{64}{375}$.

Summing up, we obtain Theorem A on the sole assumption that $|\Psi|$ represents m. As in the case of indefinite ternary forms,[17] this restriction may be removed.

6. **The case $S = -2$; Φ negative definite, $\Delta < 0$.** We may take $\Delta = -1$.

Lemma 17. *Ψ represents both d and $-d$ if*

$$d^4 > \tfrac{128}{729}. \tag{6.01}$$

For if Ψ does not represent both d and $-d$, the indefinite binary form (a, u, d) of minimum d and hessian α, does not represent $-d$ so that by Lemma 12, $d^2 \leqq -4\alpha/9$.

But the ternary form $-\Phi$ is definite of hessian d^2 and minimum $-\alpha$ so that by (3.613),

$$-\alpha^3 \leqq 2d^2. \tag{6.02}$$

Hence $d^6 \leqq (\frac{128}{729})d^2$, $d^4 \leqq \frac{128}{729}$, contradicting (6.01).

Lemma 17 follows. Assuming (6.01), we may turn the discussion of $S = -2$ into the case $S = +2$, since $-\Psi$ has signature 2 and represents d. However, a preliminary discussion of $S = -2$ will be of service in the case $S = 2$.

[17] Dickson, 2.

LEMMA 18. *If* $d^4 > \frac{1}{4}$, *then* $d^2 = -4\alpha/5$.

For if $d^2 \neq -4\alpha/5$, then $d^2 \leqq -\frac{1}{2}\alpha$ by Theorem 1. From (6.02) we obtain $d^4 \leqq \frac{1}{4}$.

THEOREM 3. *If* $S(\Psi) = -2$ *and* Ψ *represents* m, *the minimum of* $|\Psi|$, *then* $m^4 \leqq -4\Delta/7$. *If equality holds, then*

$$\Psi \sim \Psi_1 = m(t^2-x^2-y^2-z^2+xt+yt+zt).^{18}$$

If Ψ *is not equivalent to* Ψ_1, *then* $m^4 \leqq -10\Delta/33$.

In proving Theorem 3, we may suppose that

$$d^4 = m^4 > \tfrac{10}{33} > \tfrac{1}{4}, \tag{6.11}$$

so that by Lemma 18,

$$d^2 = -\tfrac{4}{5}\alpha. \tag{6.12}$$

The binary form $\varphi = (C, -F, B)$ is positive definite reduced so that we may take

$$0 < C \leqq B, \ C(B-\tfrac{1}{4}C) \leqq -\alpha = \tfrac{5}{4}d^2 \tag{6.13}$$

by (2.122).

By Theorem 2, Lemmas 14 and 15, and (6.12) applied to the indefinite ternary forms $\Psi_{y=0}$, $\Psi_{z=0}$, we have

$$d^3/C = \tfrac{2}{3},\ \tfrac{2}{5},\ \tfrac{8}{25} \text{ or } < \cdot 30033, \tag{6.14}$$

$$d^3/B = \tfrac{2}{3},\ \tfrac{2}{5},\ \tfrac{8}{25} \text{ or } < \cdot 30033. \tag{6.15}$$

Denote the cases of (6.14), (6.15) by $C_1, \cdots, C_4, B_1, \cdots, B_4$.

Case $B_4 C_j$ $(j \geqq 2)$. Here by (6.13),

$$d^4 \leqq \tfrac{64}{875} (j \geqq 3);\ \tfrac{5}{2}(\tfrac{25}{8}-\tfrac{5}{8})d^6 < \tfrac{5}{4}d^2,\ d^4 < \tfrac{1}{5} (j = 2). \tag{6.21}$$

Case $B_i C_1$ $(i \geqq 3)$. Here

$$\tfrac{3}{2}(\tfrac{25}{8}-\tfrac{3}{8})d^6 \leqq \tfrac{5}{4}d^2,\ d^4 \leqq \tfrac{10}{33}. \tag{6.22}$$

Case $B_1 C_1$. By Lemma 14, we have

$$\Psi_{z=0}/d = (t+\tfrac{1}{2}x+\tfrac{1}{2}y)^2 - \tfrac{5}{4}x^2 - \tfrac{1}{2}xy - \tfrac{5}{4}y^2, \tag{6.31}$$

and a similar expression for $\Psi_{y=0}$ on interchanging y and z. Hence

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[18] Ψ_1 is improperly equivalent to itself by interchange of y and z.

(6.32) $\psi/d = (t+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z)^2-\frac{5}{4}x^2-\frac{5}{4}y^2-\frac{5}{4}z^2-\frac{1}{2}xy-\frac{1}{2}xz+2\varrho yz,$

the coefficient of yz being $2f = 2\varrho+\frac{1}{2}$.

Apart from a multiplier, the reduced binary form φ is $(24, 4-20f, 24)$ so that

(6.33) $$-24 < 8-40f \leqq 24, \quad -2 \leqq 5f < 4.$$

In (6.32) put $x = t = 0$. We obtain the form $-y^2+2fyz-z^2$, definite by (6.33), and of absolute ninimum 1 so that

(6.34) $$3 \leqq 4-4f^2, \quad -1 \leqq 2f \leqq 1.$$

Put now $x = 0$, $t = y$ in (6.32). We obtain the indefinite binary form $y^2+(2f+1)yz-z^2$ of minimum 1 and discriminant $(2f+1)^2+4 \leqq 8$ by (6.34). Theorem 1 yields therefore

(6.35) $$(2f+1)^2+4 = 8 \text{ or } 5, \qquad f = \tfrac{1}{2} \text{ or } 0.$$

But $\psi(-1, 1, 1, 1) = (2f-1)d$ which rules out $f = \frac{1}{2}$. The only possibility is $f = 0$ and we obtain the form ψ_1 of Theorem 3.

It remains to show that ψ_1 has the minimum d. Since $4\psi_1/d = 7t^2-(2x-t)^2-(2y-t)^2-(2z-t)^2$ and $7t^2$ where $t \neq 0$, is not the sum of three squares, ψ_1 is not null and so has minimum d. The form ψ_1 is the first form of Theorem B.

Case B_2C_1. In addition to (6.31) we have by Lemma 15,

(6.41) $$\psi_{y=0}/d = (t+\tfrac{1}{2}x+z)^2-\tfrac{5}{4}x^2-2z^2,$$

and therefore

(6.42) $\psi/d = (t+\frac{1}{2}x+\frac{1}{2}y+z)^2-\frac{5}{4}x^2-\frac{5}{4}y^2-2z^2-\frac{1}{2}xy+2\varrho yz,$

the coefficient of yz being $2f = 2\varrho+1$.

Apart from a multiplier, the reduced binary form φ is $(24, 10-20f, 24)$ so that

(6.43) $$-24 < 20-40f \leqq 24, \qquad -1 \leqq 10f < 11.$$

In (6.42) put $x = t = 0$. We obtain the form $-y^2+2fyz-z^2$ of absolute minimum 1. If $f^2 = 1$, the form is null. If $f^2 > 1$, the form is indefinite and therefore (Theorem 1), $5 \leqq 4f^2-4$, $9 \leqq 4f^2$, which contradicts (6.43). Hence $f^2 < 1$, the form is definite and

(6.44) $$3 \leqq 4-4f^2, \qquad -1 \leqq 2f \leqq 1.$$

Put now in (6.42) $x = 0$, $t = -y$. We obtain the form $y^2+2(1-f)yz+z^2 = \psi$, of minimum 1. If $(1-f)^2 \geqq 1$, then $f \leqq 0$, the form is indefinite and

$5 \leqq (2-2f)^2-4$, which contradicts (6.43). Hence $(1-f)^2 < 1$, ψ is definite and

$$3 \leqq 4-4(1-f)^2, \qquad 1 \leqq 2f \leqq 3, \qquad 2f = 1,$$

by (6.44). Since $\Psi(-1, 1, 1, 1) = (2f-1)d$, $2f \neq 1$.

Hence no forms can arise in B_2C_1. The combination B_2C_1 is not possible.

Case B_2C_2[19]. We have (6.41) and the result obtained by interchanging y and z so that

(6.51) $\Psi/d = (t+\frac{1}{2}x+y+z)^2-\frac{5}{4}x^2-2y^2-2z^2+2\varrho yz, \quad 2f = 2\varrho+2.$

From the reduced definite form φ,

$$(6.52) \qquad -2 < 2-2f \leqq 2, \qquad 0 \leqq f < 2.$$

Since $\Psi(1, 1, 1, 0) = (2f-1)d$ and $|\Psi/d| \geqq 1$, we have $2f-1 \leqq -1$ and so $f = 0$, or else $2f-1 \geqq 1, f \geqq 1$. But $\Psi(1, 1, -1, 2) = (3-2f)d$. If $3-2f \leqq -1$, then $f \geqq 2$, which contradicts (6.52). Hence $3-2f \geqq 1$, $f \leqq 1$ and so $f = 1$. But $\Psi(2, 1, 1, 0) = (2f-2)d$, which rules out $f = 1$. We are left with $f = 0$ and the form

(6.53) $\Psi_2 = d(t^2-x^2-y^2-z^2+xt+2yt+2zt+xy+xz), \quad d^4 = 4/15.$

It remains to show that Ψ_2 has the minimum d. Now $4\Psi_2/d$ is equal to $(2t+2y+2z+x)^2-5x^2-2(2y+z)^2-6z^2$ and the form $T^2-5X^2-2Y^2-6Z^2$ is not null. For if $T^2-5X^2 = 2Y^2+6Z^2$, we may suppose that T, X, Y, Z are not all even. Since T^2-5X^2 cannot be congruent to 2 or 6 modulo 8, Y and Z are both even or both odd. If Y and Z are even, then $T^2-5X^2 \equiv 0 \pmod 8$, whence T and X are even. Hence Y and Z are odd, and $T^2-5X^2 \equiv 8 \pmod{16}$, which is impossible. Our assertion follows.

Case B_3C_2.[20] We have (6.41) with y and z interchanged and, by Lemma 15,

$$(6.61) \qquad \Psi_{y=0}/d = (t+\tfrac{1}{2}x+z)^2-\tfrac{5}{4}x^2-\tfrac{5}{2}z^2,$$

so that

$$(6.62) \qquad \Psi/d = (t+\tfrac{1}{2}x+y+z)^2-\tfrac{5}{4}x^2-2y^2-\tfrac{5}{2}z^2+2\varrho yz$$

where

$$(6.63) \qquad -1 \leqq \varrho < 1$$

from the negative definite reduced binary form $-\varphi$. We may plainly suppose that $\varrho \geqq 0$.

[19] We have immediately here $d^6 = (4Bc/25)$, $d^4 \leqq \frac{4}{15} < \frac{18}{58}$. The present discussion is required in the sequel.

[20] We have immediately here $d^4 \leqq \frac{1}{5}$ by (6.13).

Now

(6.64) $\Psi(4, -1, 2, 3) = (4-4\varrho)d, \quad \Psi(1, -1, 1, 2) = (\frac{1}{2}-2\varrho)d.$

Since $|\Psi/d| \geqq 1$, (6.64) and $0 \leqq \varrho \leqq 1$ yield $4\varrho = 3$. But $\Psi(0, 1, 2, 0) = (4\varrho-3)d$. The case $B_3 C_2$ cannot therefore arise.

From the results of (6.2)-(6.6) we derive Theorem 3. The discussion of § 7 will enable us to improve (6.22) and also to discuss the case $\alpha = -2d^2$. We shall in fact derive two more minima.

7. The case $S=2$; Φ **indefinite,** $\Delta < 0$. We may take $\Delta = -1$.

LEMMA 19. *If* $d^4 > \frac{64}{375}$, *then* Φ *cannot represent* $-n$.

Let Φ represent $-n$. Then $\alpha = -n$. Theorem 1 applied to the indefinite binary form (a, u, d) of minimum d and hessian α yields

(7.01) $$d^2 = \tfrac{4}{5}n \quad \text{or} \quad d^2 \leqq \tfrac{1}{2}n.$$

Theorem 2 applied to the indefinite ternary form Φ of minimum n and hessian $-d^2$ yields

(7.02) $$n^3/d^2 = \tfrac{2}{3}, \tfrac{2}{5} \quad \text{or} \quad \leqq \tfrac{1}{3}.$$

If (7.011) and (7.021) or (7.022), then, by Theorem 2, we may take

(7.03) $$\Psi/d = (ux+vy+wz+t)^2 + \tfrac{5}{4}(-x^2+y^2+z^2-xy-\varepsilon xz-\delta yz),$$

the precise values of ε and δ being irrelevant.

We observe that $|\Psi/\alpha| \geqq 1$. Take $x=1$, $y=z=0$. We see that $2u$ is an odd integer. Take $x=y=1$, $z=0$. We see that $2v-2u$ is an odd integer. But now Ψ vanishes for $x=3$, $y=-1$, $z=0$, $t=v-3u+\frac{5}{2}$, which is an integer. The cases under discussion cannot therefore arise.

The remaining cases of (7.01) and (7.02) yield

$$d^6 \leqq \tfrac{1}{8}\cdot\tfrac{2}{3}d^2; \qquad d^6 \leqq (\tfrac{4}{5})^3\tfrac{1}{3}d^2; \qquad d^4 \leqq \tfrac{64}{375},$$

contradicting the hypothesis of the lemma.

Lemma 19 may be much improved, but the improvement does not serve us.

Let now $\alpha = n$. Then Φ does not represent $-n$, and Theorem 2 gives

(7.11) $$\alpha^3 = \nu d^2, \qquad \nu < \cdot 30033.$$

The form φ is now indefinite reduced. We may identify it with any member, φ_{2i}, of the chain of equivalent reduced forms to which it belongs. In the notation of § 3, we have

(7.12) $$C = A_{2i} > 0, \qquad -B = A_{2i+1} > 0, \qquad R = 2\sqrt{\alpha}.$$

The form (α, τ, β) is definite, of minimum α and hessian dC. The form (α, σ, γ) is indefinite, of minimum α and hessian dB, and does not represent $-\alpha$. Hence

(7.13) $$\alpha^2 \leqq \tfrac{4}{3}\, d A_{2i}; \qquad \alpha^2 \leqq \tfrac{4}{9}\, d\, A_{2i+1} \qquad \text{(all } i\text{)}.$$

The ternary form $\Psi_{z=0}$ is definite and satisfies the conditions of (3.6). Hence

(7.14) $$d^2 = \mu\,\alpha, \quad \mu \leqq \tfrac{4}{3}; \qquad \alpha d \leqq \tfrac{3}{2} A_{2i} \qquad \text{(all } i\text{)}.$$

For the indefinite ternary form $\Psi_{y=0}$ of minimum d and hessian $-A_{2i+1}$, Theorem 2 yields the following possibilities:

(7.15) $$d^3/A_{2i+1} = \lambda_i, \qquad \lambda_i = \tfrac{2}{3}, \tfrac{2}{5}, \tfrac{1}{3}, \tfrac{8}{25} \text{ or } < \cdot 30033.$$

From (7.11) and (7. 141) we deduce

(7.16) $$d^4 = \mu^3 \nu \qquad (\mu \leqq \tfrac{4}{3},\ \nu < \cdot 30033).$$

Since $K_i = R/A_{i+1}$, (7.14) gives

(7.17) $$d^4 \leqq 9\mu/K_{2i-1}^2 \qquad \text{(all } i\text{)},$$

while (7.141) and (7.15) give

(7.18) $$d^4 \leqq 4\lambda_i^2/\mu K_{2i}^2 \qquad \text{(all } i\text{)}.$$

From (7.15) and (7.132)

(7.19) $$d^4 \leqq 8\lambda_i^{\frac{3}{2}}/3K_{2i}^2 \qquad \text{(all } i\text{)}.$$

We observe that μ and ν are independent of i, while λ_i depends on i. Let us note that for

(7.22) $$\lambda = \lambda_i = \tfrac{2}{3}, \tfrac{2}{5}, \tfrac{1}{3} \text{ or } \tfrac{8}{25},$$

we have by Lemmas 14 and 15 in the respective cases,

(7.25) $$\mu = d^2/\alpha = \tfrac{4}{3}, \tfrac{4}{3}, 1 \text{ or } \tfrac{4}{5}.$$

If therefore, for some i, $\lambda_i = \frac{8}{25}$, then $\mu = \frac{4}{5}$ and (7.16) gives

(7.24) $$d^4 < \tfrac{64}{125} \cdot \tfrac{8}{25} < \tfrac{1}{5}.$$

We *assume from now on that*

(7.25) $$d^4 \geqq \tfrac{1}{5} > \tfrac{64}{375}.$$

From (7.22)-(7.25) we obtain the three cases below:

$$\begin{array}{lll} \text{I.} & \lambda_i = \frac{1}{3} \quad \text{for some } i; & \mu = 1: \\ \text{II.} & \lambda_i = \frac{2}{3} \text{ or } \frac{2}{5} \text{ for some } i; & \mu = \frac{4}{3}: \\ \text{III.} & \lambda_i < \cdot 30033 \text{ for all } i; & \mu \leqq \frac{4}{3}. \end{array}$$

These cases are exhaustive and do not overlap. We observe that in I. we cannot have $\lambda_i = \frac{2}{3}$ or $\frac{2}{5}$ nor, in II. $\lambda_i = \frac{1}{3}$.

(7.3) *Case* I. divides into

$$\begin{array}{ll} \text{I}_1. & \text{For some } j,\ \lambda_j < .30033; \\ \text{I}_2. & \text{For all } \ j,\ \lambda_j = \frac{1}{3}. \end{array}$$

Case I_1. From (7.17) and (7.25),

$$K_{2i-1} \leqq 3\sqrt{5} < 7 \quad (\text{all } i). \tag{7.311}$$

From (7.18), (7.25)

$$K_{2i} \leqq \tfrac{2}{3}\sqrt{5} \quad (\lambda_i = \tfrac{1}{3}), \quad K_{2j} < 2\lambda_j\sqrt{5} < 1 + \tfrac{1}{5} + \tfrac{1}{6}. \tag{7.312}$$

From (7.312) and (7.311) we deduce

$$g_{2i} = 1 \ (\text{all } i), \quad g_{2i-1} \leqq 5 \quad (\text{all } i), \tag{7.313}$$

since $K_i > g_i$ and $(6, 1, \cdots) + (0, 1, \cdots) > (6, 2) + (0, 2) = 7$.

The useful inequality

$$K_i > g_i + \frac{1}{1 + g_{i-1}} + \frac{1}{1 + g_{i+1}} \quad (\text{all } i) \tag{7.32}$$

applied to (7.3122) yields, in virtue of (7.313), $g_{2j-1} = g_{2j+1} = 5$.

But now $K_{2j-1} > (5, 2) + (0, 2) = 6$ and (7.131) gives, on squaring,

$$\alpha^3 < \tfrac{64}{9} \cdot \tfrac{d^2}{36}, \quad \nu < \tfrac{16}{81} < \tfrac{1}{5}, \ d^4 = \mu^3 \nu < \tfrac{1}{5}, \tag{7.321}$$

from (7.16), which contradicts (7.25).

Remains therefore only

$$\mathrm{I}_2. \ \lambda_i = \tfrac{1}{3}, \quad K_{2i} = K_0 \quad (\text{all } i).$$

LEMMA 20. *If the K_{2i} are all equal, then*

$$g_{2i} = g_0, \quad g_{4i+1} = g_1, \quad g_{4i+3} = g_3 \quad (\text{all } i). \tag{7.33}$$

Write $X=(g_0, g_{-1}, g_{-2},\ldots)$, $Y=(g_2, g_3,\ldots)$. The equation $F_0+S_0=K_0$ $=K_2=F_2+S_2$ may be written

$$(0, g_1, Y)+X = (0, g_1, X)+Y.$$

Since X and Y are positive, we deduce $X=Y$ and so

$$g_0 = g_2, \quad g_{-1} = g_3, \quad g_{-2} = g_4, \ldots$$

The lemma follows since K_0 and K_2 may be any two consecutive K_{2i}, K_{2i+2}.

In I_2, therefore, the sequence of g's is $(1^*, g_1, 1, g_3^*)$. An easy calculation gives

(7.34) $$\varphi_0 = 2\sqrt{\alpha/\xi}\,(A_0 y^2+B_0 yz-A_1 z^2)$$

where

(7.341) $$\begin{aligned}\xi &= (g_1 g_3+2g_1+2g_3+2)^2-4\\ &= (g_1+2)(g_3+2)(g_1 g_3+2g_1+2g_3),\end{aligned}$$

(7.342) $$A_0 = g_1+2, \qquad B_0 = g_3(g_1+2), \qquad A_1 = g_1 g_3+g_1+g_3,$$

and

(7.343) $$K_0^2 = \xi/A_1^2, \quad K_{-1}^2 = \xi/A_0^2.$$

We may suppose that $1 \leqq g_1 \leqq g_3 \leqq 5$ (7.313). If $g_1=1$, then (7.32) gives $K_0 > 1+\frac{1}{2} > \frac{2}{3}\sqrt{5}$, contradicting (7.3121). Thus $g_1 \geqq 2$. If $g_3=5$, then $K_3>6$ and we obtain (7.321). Thus $g_3 \leqq 4$.

Now from (7.18), $d^4 = (4/9K_0^2)$. The condition $d^4 \geqq \frac{1}{5}$ leaves us the two cases

(7.35) $$d^4 = \tfrac{2}{9}\ (g_1=g_3=4), \quad d^4 = \tfrac{1}{5}\cdot\tfrac{861}{851}\ (g_1=3,\ g_3=4).$$

In (7.351) we find from (7.34) and $\mu=1$ that $\alpha = d^2 = \sqrt{2}/3$, and

$$\Phi/\alpha = (x+\tau' y+\sigma' z)^2+\tfrac{3}{4}(y^2+4yz-4z^2),$$

the form Φ/α having minimum 1. Take $y=0$, $z=1$. Then $|(x+\sigma')^2-3| \geqq 1$, whence σ' is an integer. Taking $z=0$, $y=1$, we see that $2\tau'$ is an odd integer. We may take $\sigma'=0$, $\tau'=\frac{1}{2}$, and we obtain

$$\Psi/d = (t+ux+vy+wz)^2+(x+\tfrac{1}{2}y)^2+\tfrac{3}{4}(y^2+4yz-4z^2), \quad |\Psi/d| \geqq 1.$$

We find that w is an integer by taking $x=y=0$, $z=1$; that u is an integer by taking $y=0$, $x=z=1$; and that v is an integer by taking $x=2$, $y=1$, $z=2$. We may take u, v, w all zero. Replacing y by $y-2z$ and x by $x+z$, we obtain the equivalent form

(7.36) $$\Psi_3 = d(t^2-6z^2+x^2+xy+y^2), \quad d^4 = \tfrac{2}{9}.$$

This form has minimum d. For $4\Psi_3/d$ is equal to

$$(2t)^2-6(2z)^2+(2x+y)^2+3y^2$$

which cannot be zero, by congruential arguments, unless x, y, z, t are all zero.

Further, Ψ_3 is improperly equivalent to itself, since it is unaltered by interchange of x and y.

In (7.352) however we obtain $\alpha = d^2 = (19/3\sqrt{195})$,

$$\Phi/\alpha = (x+\tau' y+\sigma' z)^2+\tfrac{3}{19}(5y^2+20yz-19z^2), \quad |\Phi/\alpha| \geqq 1.$$

Take $y=0$, $z=1$. We obtain σ' an integer, $\sigma'=0$. Take $y=-4$, $z=1$. We see that $4\tau'$ is an integer. We may suppose that $-\frac{1}{2}<\tau'\leqq\frac{1}{2}$. Take $x=0$, $z=0$, $y=1$. Then $\tau'^2+\frac{15}{19}\geqq 1$. Hence $\tau'=\frac{1}{2}$. But for $x=2$, $y=1$, $z=-1$, we have

$$0<\Phi/\alpha = \tfrac{25}{4}-\tfrac{102}{19}<1,$$

a contradiction. Thus (7.352) cannot arise.

Hence (7.25) in Case I. leads only to the form Ψ_3 of (7.36) with $d^4=\frac{2}{9}$.

(7.4) *Case* II. Here $\mu=\frac{4}{3}$, $\alpha = 3d^2/4$. From (7.17) and (7.25)

(7.41) $$K_{2i-1} \leqq \sqrt{60} < 7\cdot 75, \quad g_{2i-1} \leqq 7 \quad (\text{all } i).$$

But now (7.32) gives $K_{2j}>1+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{4}$, and (7.18) yields $d^4<48\lambda_j^2/25<\frac{1}{5}$ if $\lambda_j<\sqrt{15}/12$. Since $\sqrt{15}/12>.30033$, we have in Case II.

(7.42) $$\lambda_i = \tfrac{2}{3} \text{ or } \tfrac{2}{5} \quad (\text{all } i).$$

The discussion of (7.42) falls into two parts:

$$\text{II}_1.\ \lambda_i = \tfrac{2}{3} \quad (\text{all } i),$$
$$\text{II}_2.\ \lambda_i = \tfrac{2}{5} \quad \text{for some } i.$$

(7.5) *Case* II_1. From (7.18) and (7.25)

(7.51) $$d^4 = (4/3K_{2i}^2), \quad K_{2i}^2 \leqq 20/3, \quad K_{2i} \leqq \tfrac{2}{3}\sqrt{15}<\tfrac{8}{3}, \quad g_{2i}=1 \text{ or } 2.$$

By Lemma 20,

(7.52) $$g_{2i} = 1 \ (\text{all } i) \quad \text{or} \quad g_{2i} = 2 \ (\text{all } i).$$

The sequence of g's is (g_0^*, g_1, g_0, g_3^*). If $g_0 = 1$, then (7.34)-(7.343) give with (7.51)

(7.53) $\frac{3}{4} d^4 = (g_3 g_1 + g_3 + g_1)^2 / \{(g_3 + 2)(g_1 + 2)(g_3 g_1 + 2g_3 + 2g_1)\}$,

and we may suppose that $g_1 \leqq g_3$.

From (7.17), (7.18), (7.343) we obtain

$$A_1 \leqq 3A_0, \quad g_1 g_3 + g_1 + g_3 \leqq 3g_1 + 6 \tag{7.54}$$

by (7.342). *A fortiori* $g_1^2 + 2g_1 \leqq 3g_1 + 6$, $g_1 \leqq 3$; and (7.54) gives

$$g_1 = 1,\ g_3 \leqq 4; \quad g_1 = 2 \leqq g_3 \leqq 3; \quad g_1 = 3 = g_3. \tag{7.55}$$

From (7.53) we obtain the following set of values for d^4,

$g_1 = 1$; $\frac{4}{15}\,(g_3 = 1)$, $\frac{25}{72}\,(g_3 = 2)$, $\frac{4}{3} \cdot \frac{49}{165}\,(g_3 = 3)$, $\frac{3}{7}\,(g_3 = 4)$:
$g_1 = 2$; $\frac{4}{9}\,(g_3 = 2)$, $\frac{121}{240}\,(g_3 = 3)$:
$g_1 = 3$; $\frac{4}{7}\,(g_3 = 3)$.

With the exception of the first and last of these numbers, all these values of d^4 exceed $\frac{10}{33}$ and are less than $\frac{4}{7}$. Also in each of these cases Ψ represents $-d$. Hence, in each case, $-\Psi$ has signature -2, represents d and has

$$\tfrac{10}{33} < d^4 < \tfrac{4}{7}.$$

But this contradicts Theorem 3.

In the case $g_1 = g_3 = 3$, $d^4 = \frac{4}{7}$, it is easily verified that we arrive at the form

$$d[(t + \tfrac{1}{2}x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 + \tfrac{3}{4}(x + \tfrac{1}{3}y + z)^2 + \tfrac{2}{3}(y^2 + 3yz - 3z^2)],$$

which is equivalent to $-\Psi_1$ of Theorem 3.

In the case of $g_1 = g_3 = 1$, $d^4 = \frac{4}{15}$, we arrive at the form

$$\Psi_2' = d[(t + \tfrac{1}{2}x + \tfrac{1}{2}z)^2 + \tfrac{3}{4}(x + z)^2 + 2(y^2 + yz - z^2)]$$

which is *not* equivalent to $-\Psi_2$ of (6.53). For plainly,

$$\Psi_2'/d \sim t^2 + tx + x^2 + 2(y^2 + yz - z^2)$$

and the adjoint of $2\Psi_2'/d$ is

$$-20(2t^2 - 2tx + 2x^2) + 3(-4y^2 - 4yz + 4z^2) \equiv -10(2x - t)^2 \pmod 3$$

and is therefore never congruent to $+1$ modulo 3 but is congruent to 0 or -1.

On the other hand, replacing t by $t-y-z$ in (6.53) we see that

$$-\Psi_2/d \sim -t^2-tx+x^2+2(y^2+yz+z^2),$$

and the adjoint of $-2\Psi_2/d$ is

$$12(t^2+tx-x^2)-5(4y^2-4yz+4z^2) \equiv -5(2z-y)^2 \pmod 3$$

and therefore never congruent to -1 modulo 3 but is congruent to 0 or $+1$.

It follows that the adjoints of Ψ_2' and $-\Psi_2$ are not equivalent, whence Ψ_2' and $-\Psi_2$ cannot be equivalent. They are the forms in (B 3) of Theorem B.

If $g_{2i}=2$, we find that

(7.561) $12d^4=(2g_1g_3+g_3+g_1)^2/\{(g_1+1)(g_3+1)(g_1g_3+g_1+g_3)\}, \quad \alpha=\frac{3}{4}d^2,$

(7.562) $\Phi/\alpha=(x+\tau'y+\sigma'z)^2+\{8/3(2g_1g_3+g_1+g_3)\}$
$\times\{4(g_1+1)y^2+4g_3(g_1+1)yz-(2g_1g_3+g_1+g_3)z^2\},$

(7.563) $\Psi/d=(t+u'x+v'y+w'z)^2+(3\Phi/4\alpha).$

The first condition of (7.54) gives

(7.564) $$2g_1g_3+g_1+g_3 \leqq 12(g_1+1).$$

We may suppose that $g_1 \leqq g_3$. The condition $d^4 \geqq \frac{1}{5}$ rules out $g_1=1$ and $g_1=2,\ g_3 \leqq 5$.

From (7.564) we have therefore the following possibilities,

(7.565) $$g_1=2,\quad g_3=6; \qquad 3 \leqq g_1 \leqq g_3 \leqq 6.$$

In (7.563) put $y=0$. We obtain an indefinite ternary form of minimum 1 and hessian $-\frac{3}{2}$. By Theorem 2, therefore, $2u'$, $u'^2+\frac{3}{4}$, $2w'u'$, $w'+\frac{3}{2}\sigma'$ are integers, whence we may take $u'=\frac{1}{2}$, $w'=0$, $\sigma'=0$ by means of appropriate transformations of the type $t=t'+\lambda x'+\mu z$, $x=x'+\nu z$.

In (7.563) put $y=-g_3z$. We obtain an indefinite ternary form of minimum 1 and hessian $-\frac{3}{2}$. We find that $2v'g_3$ and $\tau'g_3$ are integers, both even or both odd.

Similarly, taking $y=-g_1Y$, $z=-(1+2g_1)Y$, we obtain an indefinite ternary form of minimum 1 and hessian $-\frac{3}{2}$, whence $2vg_1$ and $\tau'g_1$ are integers, both even or both odd.

Let δ be the greatest common divisor of g_1 and g_3. Then $2v'\delta$ and $r'\delta$ are both integers.

The only case in (7.565) which leads to a form with the properties desired is given by $g_1 = g_3 = 4$ and the form arising is equivalent to the form Ψ_3 of (7.36). The other cases of (7.565) lead to contradictions.

The case $g_1 = g_3 = 4$. Here

$$\Phi/\alpha = (x+r'y)^2+\tfrac{4}{3}(y^2+4yz-2z^2)$$

and $4r'$ is an integer. We may suppose that $r' = 0, \pm\frac{1}{4}$ or $\frac{1}{2}$. Take $y=1$, $z=-1$. Take $x=2$ if $r'=\frac{1}{2}$; $x=\mp 3$ if $r'=\pm\frac{1}{4}$. We obtain a number numerically less than 1, whereas $|\Phi/\alpha|\geq 1$. Hence $r'=0$ and

$$\Psi/d = (t+\tfrac{1}{2}x+v'y)^2+\tfrac{3}{4}x^2+y^2+4yz-2z^2, \quad v' = 0, \pm\tfrac{1}{4} \text{ or } \tfrac{1}{2}.$$

Take $x=2X$, $y=X$, $z=-X$. We obtain the indefinite binary form $(t+X+v'X)^2-2X^2$ of minimum 1 and discriminant 8, whence (Theorem 1) v' is an integer and so $v'=0$. We obtain the form

$$\Psi_3 = d(t^2+tx+x^2+y^2+4yz-2z^2), \qquad d^4 = \tfrac{2}{9},$$

which is equivalent to Ψ_3 of (7.36) on replacing y by $y-2z$.

The remaining cases of (7.565). We treat in detail the case $g_1 = 3$, $g_3 = 6$, the analysis in the other cases being of the same nature. For $g_1 = 3$, $g_3 = 6$, we obtain

$$\Phi/\alpha = (x+r'y)^2+\tfrac{8}{63}(16y^2+96yz-21z^2), \quad r' = 0, \pm\tfrac{1}{3}.$$

Take $y=1$, $z=-1$. If $r'=\pm\frac{1}{3}$ take $x=\mp 4$. We obtain $|\Phi/\alpha|<1$ whereas $|\Phi/\alpha|\geq 1$. Hence $r'=0$ and so

$$\Psi/d = (t+\tfrac{1}{2}x+v'y)^2+\tfrac{3}{4}x^2+\tfrac{2}{21}(16y^2+96yz-21z^2), \quad v' = 0, \pm\tfrac{1}{3}.$$

If $v'=\pm\frac{1}{3}$, take $x=2$, $y=1$, $z=-1$ and $t=-4$ or 2 respectively. We obtain $|\Psi/d|<1$ whereas $|\Psi/d|\geq 1$. Hence $v'=0$, which is ruled out by $\Psi(0,1,-1,3)=-\frac{2}{3}$.

The discussion of Case II_1. is complete.

(7.6) *Case* II_2. For some i, $\lambda_i = \frac{2}{5}$, $d^3 = -\frac{2}{5}B$. By a proper transformation on x, z, t we can, by Theorem 2, take

$$\Psi_{y=0} = d(x^2-z^2+t^2-xt-2zx-zt).$$

Take now, in Ψ, z in place of t as the leading variable and write

$$c\Psi = (\,)^2 + \Phi_1(x, y, t), \qquad c = -d.$$

The form Φ_1 is negative definite, for the signature of $c\Psi$ is -2. The coefficient of x^2 in Φ_1 is $\alpha_1 = -5d^2/4$ and $-\alpha_1$ is the minimum of $-\Phi_1$. For if the minimum of $-\Phi_1$ is $n_1 < 5d^2/4$ we can make $|c\Psi| < d^2$ whereas $|\Psi| \geqq d$.

We obtain now, as in § 6, from Φ_1 the negative definite binary form

$$\varphi(y, t) = Dy^2 - 2Vyt + Bt^2.$$

Let us observe that we can find indefinite ternary sections of Ψ having minimum d and for hessian *any* value represented by φ_1.

Let $(D', -V', B')$ be the equivalent reduced form to φ with $0 < -B' \leqq -D'$. Then $-B' \leqq -B$.

If $-B > -B'$, then $d^3 > -\frac{2}{5}B'$. By the preceding paragraph and Theorem 2 we must have $d^3 = -\frac{2}{3}B'$. But now $B = \frac{5}{3}B'$ so that, by Lemma 4, $-B \geqq -D'$. By the preceding paragraph and Theorem 2 we have either $d^3 = -\frac{2}{3}D'$ or $d^3 = -\frac{2}{5}D'$. The first case falls under B_1C_1 of § 6 and leads to the form $-\Psi_1$ of Theorem 3. The second case falls under B_2C_1 of § 6, there shown to be impossible.

Remains the case $B = B'$, so that $-B \leqq -D'$,

$$d^3 = -\tfrac{2}{5}B'; \; d^3 = -\tfrac{2}{5}D' \text{ or } -\tfrac{8}{25}D' \text{ or} < -\cdot 30033\,D',$$

by Theorem 2, the value $d^3 = -\frac{1}{3}D'$ being incompatible with $\alpha_1 = -5d^2/4$.

The first case leads to B_2C_2 and the form $-\Psi_2$ of § 6. The second case falls under B_3C_2 of § 6 and is therefore ruled out. Finally the third case falls under B_4C_2 of § 6 and leads to $d^4 < \frac{1}{5}$. The discussion of II_2. is complete.

(7.7) *Case* III. For all i,

(7.71) $$\lambda_i < .30033, \qquad \mu \leqq \tfrac{4}{3}.$$

From (7.16) and (7.25),

(7.72) $$\mu^3\nu \geqq \tfrac{1}{5}, \qquad \nu < .30033.$$

From (7.18), (7.25) and (7.72),

(7.73) $$K_{2i}^2 \leqq 20\lambda_i^2/\mu < 4, \qquad g_{2i} = 1 \qquad \text{(all } i\text{)}.$$

From (7.17) and (7.25),

(7.74) $$K_{2i-1}^2 \leqq 60, \qquad g_{2i-1} \leqq 7 \qquad \text{(all } i\text{)}.$$

If $g_{2i-1} = 7$ for some i, then (7.32) gives $K_{2i-1} > 8$, contradicting (7.73). Hence $g_{2i-1} \leqq 6$ (all i).

If $g_{2i-1} = 6$ for some i, then $K_{2i-1} > 7$, and (7.13) yields

$$\alpha^3 \leqq \tfrac{64}{9} \frac{d^2}{K_{2i-1}^2}, \qquad \nu \leqq \tfrac{64}{441}, \tag{7.75}$$

whence a stronger inequality for μ in (7.72). But now (7.731) gives

$$K_{2i} < 1.28 < 1 + \tfrac{1}{7} + \tfrac{1}{7}.$$

But by (7.32) since $g_{2i-1} \leqq 6$ (all i), $K_{2i} > 1 + \tfrac{1}{7} + \tfrac{1}{7}$. Hence $g_{2i-1} \leqq 5$ for all i.

If $g_{2i-1} = 5$ for some i, then $K_{2i-1} > 6$ and (7.751) yields $\nu < \tfrac{16}{81}$; and (7.72) gives a stronger inequality for μ.

From (7.73) and (7.72) we deduce therefore

$$K_{2i} < 3/\sqrt{5} < 1 + \tfrac{1}{5} + \tfrac{1}{6}. \tag{7.76}$$

The conditions $g_{2i-1} \leqq 5$ (all i), (7.32) and (7.76) show that $g_{2i-1} = 5$ for all i. But now $K_{2i} = 3/\sqrt{5}$ contradicting (7.76). Hence $g_{2i-1} \leqq 4$ for all i.

Now (7.72) and (7.731) give

$$K_{2i} < 1 + \tfrac{1}{4} + \tfrac{1}{5}. \tag{7.77}$$

If for some i, $g_{2i-1} \leqq 3$, then (7.32) and $g_{2j-1} \leqq 4$ (all j) yield $K_{2i} > 1 + \tfrac{1}{4} + \tfrac{1}{5}$, contradicting (7.77). Hence $g_{2i-1} = 4$ for all i. We obtain

$$K_{2i} = (1, 4, \cdots) + (0, 4, 1, \cdots) = \sqrt{2}, \quad K_{2i-1} = 4\sqrt{2} \quad (\text{all } i). \tag{7.78}$$

From (7.75) and (7.781)

$$\alpha^3 \leqq \tfrac{64}{9} \tfrac{d^2}{32}, \qquad \nu \leqq \tfrac{2}{9}.$$

This value of ν increases the value of μ in (7.72) and we deduce from (7.731) that $K_{2i} < 1.4$, which contradicts (7.781).

Thus, if $d^4 \geqq \tfrac{1}{5}$, no forms arise in Case III.

Summing up the results of §§ 6 and 7, we obtain Theorem B, provided that $|\Psi|$ represents m and that $|\Phi|$ represents n. The case "$|\Phi|$ does not represent n" may be discussed as in (5.6).

Finally we may remove the restriction that $|\Psi|$ represents m as in the case of indefinite ternary forms[21].

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ON THE REPRESENTATION OF INTEGERS AS SUMS OF AN EVEN NUMBER OF SQUARES OR OF TRIANGULAR NUMBERS.[1]

By R. D. Carmichael.

1. **Introduction.** Ramanujan (*Collected Papers*, pp. 179–199) has employed the remarkable trigonometric sums

$$(1.1)\qquad c_q(n) = \sum_\lambda \cos\frac{2\pi\lambda n}{q} = \sum_\lambda e^{2\pi i\lambda n/q}, \qquad q = 1, 2, 3, \cdots,$$

$$(1.2)\qquad s_q(n) = \sum_\lambda (-1)^{\frac{1}{2}(\lambda-1)} \sin\frac{2\pi\lambda n}{q}, \qquad q = 1, 2, 3, \cdots,$$

where the sums are taken for the $\varphi(q)$ positive integers which are prime to q and do not exceed q. It is convenient to employ also (with Ramanujan, l. c., p. 190) a combination of these sums, namely,

$$(1.3)\quad \gamma_q(n) = c_q(n)\cos\frac{1}{2}\pi s(q-1) - s_q(n)\sin\frac{1}{2}\pi s(q-1), \quad q = 1, 2, 3, \cdots,$$

where s is any positive integer. For a given pair of (integral) values of q and s the function $\gamma_q(n)$ becomes $\pm c_q(n)$ or $\pm s_q(n)$.

If s and n are positive integers and $r_{2s}(n)$ denotes the number of representations of n as a sum of $2s$ squares then the principal result here given relating to squares is contained in the following formula:

$$(1.4)\qquad \frac{1}{m}\sum_{n=1}^{m} \gamma_\varrho(n)\, n^{1-s}\, r_{2s}(n) = \frac{\pi^s \varepsilon_\varrho(s)\varphi(\varrho)}{(s-1)!\,\varrho^s} + O\left(\frac{1}{m}\right), \quad s > 3,$$

where ϱ is any positive integer and $\varepsilon_\varrho(s)$ is 1, 0 or 2^s according as ϱ is odd, $\varrho \equiv 2 \bmod 4$ or $\varrho \equiv 0 \bmod 4$. For $\varrho = 1$ this says that the function $n^{1-s} r_{2s}(n)$ is in the mean (on the average) equal to $\pi^s/(s-1)!$ if $s > 3$. For other values of ϱ the formula implies a weighted asymptotic average of the same function, the weight factors being $\gamma_\varrho(n)$ for varying n. These are therefore suitable functions to smooth out on the average the irregularities of the function $n^{1-s} r_{2s}(n)$. What is proved is that the irregularity is smoothed out in the limit as m becomes infinite; some empirical evidence (not recorded in the paper) indicates that the smoothing effect probably is generally in evidence for rather small values of m.

The paper also contains results similar to (1.4) concerning representations of integers as sums of an even number of triangular numbers and also

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concerning Ramanujan's functions $\sum_{r,s}(n)$ which he (l. c., pp. 136–162) associates with the problem of representing integers as sums of squares.

2. **Properties of Ramanujan's Trigonometric Sums.** We need certain properties of the trigonometric sums $c_q(n)$ and $s_q(n)$ which are proved or implied in an unpublished paper of the author. These are readily verified and are as follows:

$$(2.1)\quad \sum_{n=1}^{q\varrho} c_q(n)\,c_\varrho(n) = 0 \text{ if } \varrho \neq q,$$

$$\sum_{n=1}^{q} c_q^2(n) = q\,\varphi(q);$$

$$(2.2)\quad \sum_{n=1}^{q\varrho} s_q(n)\,s_\varrho(n) = 0 \text{ if } \varrho \neq q,$$

$$\sum_{n=1}^{q} s_q^2(n) = \frac{1}{4}\,q\sum_{\lambda}\left\{(-1)^{\frac{1}{2}(\lambda-1)} - (-1)^{\frac{1}{2}(q-\lambda-1)}\right\}^2;$$

$$(2.3)\quad \sum_{n=1}^{q\varrho} c_q(n)\,s_\varrho(n) = 0 \text{ if } \varrho \neq q.$$

As a special case of the second equation in (2.2), we have

$$(2.4)\qquad \sum_{n=1}^{4q} s_{4q}^2(n) = 4q\,\varphi(4q).$$

By means of these relations it is easy to show that

$$(2.5)\qquad \sum_{n=1}^{q\varrho} \gamma_q(n)\,\gamma_\varrho(n) = 0 \quad \text{if} \quad \varrho \neq q$$

and that

$$(2.6)\qquad \sum_{n=1}^{q} \gamma_q^2(n) = \begin{cases} q\,\varphi(q) & \text{if } s \text{ is even or } q \text{ is odd,} \\ q\,\varphi(q) & \text{if } s \text{ is odd and } q \equiv 0 \bmod 4, \\ 0 & \text{if } s \text{ is odd and } q \equiv 2 \bmod 4. \end{cases}$$

From the definitions of $c_q(n)$ and $s_q(n)$ it follows that neither of them exceeds $\varphi(q)$ in absolute value. Since q and s are integers it follows readily that $|\gamma_q(n)| \leqq \varphi(q)$. Now $\varphi(q) \leqq q$, the sign of equality holding only when $q = 1$. Then, by aid of (2.5), we readily show that

$$(2.7)\qquad \left|\sum_{n=1}^{m} \gamma_q(n)\,\gamma_\varrho(n)\right| < q^2\varrho^2 \quad \text{if} \quad \varrho \neq q.$$

3. **The functions $r_{2s}(n)$ and $\delta_{2s}(n)$.** Let s be a positive integer and let $r_{2s}(n)$ denote the number of ways of representing the positive integer n as a sum of $2s$ squares. If $\delta_{2s}(n)$ denotes the function

$$(3.1)\qquad \delta_{2s}(n) = \frac{\pi^s\, n^{s-1}}{(s-1)!}\, g_s(n)$$

where

$$
(3.2)\qquad \begin{aligned} g_s(n) = \frac{\gamma_1(n)}{1^s} + \frac{\gamma_4(n)}{2^s} + \frac{\gamma_3(n)}{3^s} + \frac{\gamma_8(n)}{4^s} \\ + \frac{\gamma_5(n)}{5^s} + \frac{\gamma_{12}(n)}{6^s} + \frac{\gamma_7(n)}{7^s} + \cdots, \qquad s > 1, \end{aligned}
$$

and $g_1(n)$ denotes half of what the second member of (3.2) becomes for $s = 1$, then Ramanujan (l. c., pp. 187–190, 194, 159) has shown that $r_{2s}(n) = \delta_{2s}(n)$ for $s = 1, 2, 3, 4$ and in general that

$$
(3.3)\qquad r_{2s}(n) = \delta_{2s}(n) + O\left(n^{\frac{1}{2}s}\right).
$$

4. **Asymptotic relations involving $\delta_{2s}(n)$.** In deriving asymptotic relations involving $g_s(n)$ it is convenient to treat three cases separately.

In the first place, let ϱ be any odd positive integer. Multiply both members of (3.2) by $\gamma_\varrho(n)$, sum as to n from 1 to m, divide by m and so obtain the relation

$$
(4.1)\qquad \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\, g_s(n) = \varrho^{-s}\frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho^2(n) + S + T, \qquad s > 1,
$$

where

$$
S = \sum{}' \frac{1}{(2\lambda-1)^s}\, \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\,\gamma_{2\lambda-1}(n),
$$

$$
T = \sum \frac{1}{(2\lambda)^s}\, \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\,\gamma_{4\lambda}(n),
$$

and where the sums are taken for λ ranging over the set $1, 2, 3, \cdots$ except that in the former case $2\lambda - 1$ must avoid the value ϱ. From (2.6) it follows that

$$
\varrho^{-s}\frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho^2(n) = \varrho^{-s}\varphi(\varrho) + O\left(\frac{1}{m}\right).
$$

Using (2.7) we see that

$$
|S| \leq \frac{1}{m}\sum{}'(2\lambda-1)^{-s}\varrho^2(2\lambda-1)^2.
$$

Hence

$$
S = O\left(\frac{1}{m}\right) \quad \text{if} \quad s > 3.
$$

Likewise it may be shown that

$$
T = O\left(\frac{1}{m}\right) \quad \text{if} \quad s > 3.
$$

Combining these results we have the case when ϱ is odd of the following general relation:

$$(4.2)\qquad \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\, g_s(n) = \varepsilon_\varrho(s)\,\varrho^{-s}\,\varphi(\varrho)+O\left(\frac{1}{m}\right),\qquad s>3,$$

where $\varepsilon_\varrho(s)$ is 1,0 or 2^s according as ϱ is odd, $\varrho\equiv 2 \bmod 4$ or $\varrho\equiv 0 \bmod 4$.

In completing the proof of this formula it is convenient to treat separately the two cases when $\varrho\equiv 2 \bmod 4$ and $\varrho\equiv 0 \bmod 4$. Since no novelty of argument is involved the proofs are omitted.

The result contained in (4.2) may be expressed in terms of $\delta_{2s}(n)$ in the form

$$(4.3)\qquad \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\, n^{1-s}\,\delta_{2s}(n) = \frac{\pi^s\,\varepsilon_\varrho(s)\,\varphi(\varrho)}{(s-1)!\,\varrho^s}+O\left(\frac{1}{m}\right),\qquad s>3.$$

5. **Asymptotic relations involving $r_{2s}(n)$.** If we employ (3.3) and the fact that $r_{2s}(n)=\delta_{2s}(n)$ for $s=1,2,3,4$, we readily show that

$$\frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\, n^{1-s}\, r_{2s}(n) = \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\, n^{1-s}\,\delta_{2s}(n)+O\left(\frac{1}{m}\right).$$

Thence from (4.3) it follows that we have

$$(5.1)\qquad \frac{1}{m}\sum_{n=1}^{m}\gamma_\varrho(n)\, n^{1-s}\, r_{2s}(n) = \frac{\pi^s\,\varepsilon_\varrho(s)\,\varphi(\varrho)}{(s-1)!\,\varrho^s}+O\left(\frac{1}{m}\right),\qquad s>3.$$

The weight factors $\gamma_\varrho(n)$ are suitable for smoothing out (in the limit) the irregularities of the function $n^{1-s}\,r_{2s}(n)$. In order to give an indication of the character of this irregularity let us consider the case when $s=4$. It is well known that $r_8(n)$ is equal to sixteen times the sum of the cubes of the divisors of n when n is odd and is equal to sixteen times the difference of the sum of the cubes of the even divisors of n and that of the odd divisors when n is even. Then, if $n=2p$ where p is an odd prime, we have

$$n^{-3}\,r_8(n) = 16\left(1-\frac{1}{8}+\frac{1}{p^3}-\frac{1}{(2p)^3}\right) = 14+\frac{14}{p^3}.$$

As p increases this comes as near 14 as we please. We have $2^{-3}\,r_8(2)=14$. To get values of $n^{-3}\,r_8(n)$ greater than the average, let n be a multiple of $t!$ where t is large. Then $n^{-3}\,r_8(n)$ may be made as nearly equal to

$$16\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\right)>19$$

as we please by taking t sufficiently large. Thus $n^{-3}\,r_8(n)$ oscillates rather widely. But if we take $\varrho=1$ in (5.1) we have the average of the m values of the function for $n=1,2,\cdots,m$; and this average approaches the limit $\pi^4/6\ (=16\cdot 23\cdots)$.

By taking $\varrho = 1$ in (5.1) and (4.3) we have the following general theorem: *The functions* $n^{1-s} r_{2s}(n)$ *and* $n^{1-s} \delta_{2s}(n)$ *are each in the mean (on the average) equal to* $\pi^s/(s-1)!$ *if* $s > 3$, *the error term of the average of the values for* $n = 1, 2, \cdots, m$ *being* $O(1/m)$.

6. **On the representation of integers as sums of triangular numbers.** Let $R_{2s}(n)$ denote the number of ways of representing n as a sum of $2s$ triangular numbers, where s is any positive integer, and write

$$R_{2s}(n) = D_{2s}(n) + E_{2s}(n) \tag{6.1}$$

where

$$D_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}\left(n+\frac{1}{4}s\right)^{s-1} \sum_{\mu=1}^{\infty} (2\mu-1)^{-s}(-1)^{(\mu-1)s} c_{2\mu-1}\left(\frac{4n+s}{\eta(s)}\right), \quad s>1, \tag{6.2}$$

where $\eta(s)$ is 1, 2 or 4 according as s is odd, is twice an odd number or is a multiple of 4, while $D_2(n)$ is half of what the second member of (6.2) becomes when $s = 1$. Then Ramanujan (l. c., pp. 190–192) has shown that $E_{2s}(n) = 0$ for $s = 1, 2, 3, 4$ while in general

$$R_{2s}(n) \sim D_{2s}(n). \tag{6.3}$$

By a slight modification of the methods of the preceding section we shall prove that

$$\begin{aligned} &\frac{1}{m}\sum_{n=1}^{m} c_{2\lambda-1}\left(\frac{4n+s}{\eta(s)}\right)\left(n+\frac{1}{4}s\right)^{1-s} D_{2s}(n) \\ &\qquad = \frac{(-1)^{(\lambda-1)s}(\frac{1}{2}\pi)^s \varphi(2\lambda-1)}{(s-1)!\,(2\lambda-1)^s} + O\left(\frac{1}{m}\right), \quad s>3, \end{aligned} \tag{6.4}$$

and

$$\frac{1}{m}\sum_{n=1}^{m} c_{2\lambda}\left(\frac{4n+s}{\eta(s)}\right)\left(n+\frac{1}{4}s\right)^{1-s} D_{2s}(n) = O\left(\frac{1}{m}\right), \quad s>3,\ s\equiv 0 \bmod 4, \tag{6.5}$$

where in each case λ ranges over the set $1, 2, 3, \cdots$.

When $s \equiv 0 \bmod 4$ no essential modification is needed for proving either (6.4) or (6.5). Let us next consider the case when s is twice an odd number. Then the infinite series in the second member of (6.2) defines the function $h_s(n)$ where

$$h_s(n) = \sum_{\mu=1}^{\infty} \frac{c_{2\mu-1}(2n+\frac{1}{2}s)}{(2\mu-1)^s}. \tag{6.6}$$

Therefore we have

$$\frac{1}{m}\sum_{n=1}^{m} c_{2\lambda-1}\left(2n+\frac{1}{2}s\right) h_s(n) = (2\lambda-1)^{-s}\frac{1}{m}\sum_{n=1}^{m} c_{2\lambda-1}^2\left(2n+\frac{1}{2}s\right) + S,$$

where

$$S = \sum{}' \frac{1}{(2\mu-1)^s}\,\frac{1}{m}\sum_{n=1}^{m} c_{2\mu-1}\left(2n+\frac{1}{2}s\right) c_{2\lambda-1}\left(2n+\frac{1}{2}s\right),$$

the prime in $\sum'$ denoting that the sum is taken for μ ranging over the set 1, 2, 3, $\cdots$ exclusive of the value $\mu = \lambda$.

Now if $t(2\lambda-1) \leqq m < (t+1)(2\lambda-1)$, we have

$$\begin{aligned}\frac{1}{m}\sum_{n=1}^{m} c_{2\lambda-1}^2\left(2n+\frac{1}{2}s\right) &= \frac{1}{t(2\lambda-1)}\sum_{n=1}^{t(2\lambda-1)} c_{2\lambda-1}^2\left(2n+\frac{1}{2}s\right)+O\left(\frac{1}{m}\right)\\ &= \frac{1}{2\lambda-1}\sum_{n=1}^{2\lambda-1} c_{2\lambda-1}^2\left(2n+\frac{1}{2}s\right)+O\left(\frac{1}{m}\right)\\ &= \varphi(2\lambda-1)+O\left(\frac{1}{m}\right)\end{aligned}$$

on account of the second equation in (2.1) and the fact that $c_{2\lambda-1}(n)$ has the period $2\lambda-1$ and the fact that $2n+\frac{1}{2}s$ runs over a complete set of residues modulo $2\lambda-1$ when n runs over such a set.

A similar argument will show that $S = O(1/m)$ when $s > 3$.

Then we have

$$\frac{1}{m}\sum_{n=1}^{m} c_{2\lambda-1}\left(2n+\frac{1}{2}s\right)h_s(n) = (2\lambda-1)^{-s}\varphi(2\lambda-1)+O\left(\frac{1}{m}\right), \qquad s>3.$$

From this result and (6.2) we conclude to (6.4) for the case when s is twice an odd number.

The case when s is odd may be treated similarly; thus the proof of (6.4) is completed.

By means of more precise information concerning the asymptotic character of $E_{2s}(n)$ it would be possible to translate relations (6.4) and (6.5) into corresponding relations involving $R_{2s}(n)$; but these results will not now be given.

7. **On Ramanujan's function $\Sigma_{r,s}(n)$.** Let $\sigma_s(n)$ denote the sum of the sth powers of the divisors of the positive integer n and let $\sigma_s(0) = \frac{1}{2}\zeta(-s)$ where $\zeta(s)$ is the Riemann Zeta-function, and form the function

$$(7.1)\quad \Sigma_{r,s}(n) = \sigma_r(0)\sigma_s(n)+\sigma_r(1)\sigma_s(n-1)+\sigma_r(2)\sigma_s(n-2)+\cdots+\sigma_r(n)\sigma_s(0).$$

Then Ramanujan (l. c., p. 136) has shown that

$$(7.2)\quad \begin{aligned}\Sigma_{r,s}(n) &= \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\,\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\,\sigma_{r+s+1}(n)\\ &\quad+\frac{\zeta(1-r)+\zeta(1-s)}{r+s}\,n\,\sigma_{r+s-1}(n)+O\left\{n^{\frac{2}{3}(r+s+1)}\right\}\end{aligned}$$

whenever r and s are positive odd integers; moreover he has shown that the error term is absent when $r=1$, $s=1, 3, 5, 7, 11$; $r=3$, $s=3, 5, 9$; $r=5$, $s=7$. He has also shown (l. c., p. 184) that

$$(7.3)\qquad \sigma_s(n) = n^s \zeta(s+1)\left\{\frac{c_1(n)}{1^{s+1}}+\frac{c_2(n)}{2^{s+1}}+\frac{c_3(n)}{3^{s+1}}+\cdots\right\},\quad s>0.$$

By the methods employed in the preceding sections of this paper it may be shown that

$$(7.4)\qquad \frac{1}{m}\sum_{n=1}^{m} c_\varrho(n)\, n^{-s}\sigma_s(n) = \zeta(s+1)\varrho^{-s-1}\varphi(\varrho)+O\left(\frac{1}{m}\right),\quad s>2.$$

Similarly we see that

$$(7.5)\qquad \frac{1}{m}\sum_{n=1}^{m} c_\varrho(n)\, n^{-s-1}\sigma_s(n) = \zeta(s+1)\frac{1}{m}\sum_{n=1}^{m}\frac{c_\varrho^2(n)}{n}+O\left(\frac{1}{m}\right),\quad s>2.$$

Since $|c_\varrho(n)| \leqq \varrho$ it follows that

$$\frac{1}{m}\sum_{n=1}^{m}\frac{c_\varrho^2(n)}{n} \leqq \varrho^2\cdot\frac{1}{m}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}-\log m\right)+\varrho^2\frac{\log m}{m}.$$

Therefore

$$\frac{1}{m}\sum_{n=1}^{m}\frac{c_\varrho^2(n)}{n} = O\left(\frac{\log m}{m}\right).$$

Then from (7.5) it follows that

$$(7.6)\qquad \frac{1}{m}\sum_{n=1}^{m} c_\varrho(n)\, n^{-s-1}\sigma_s(n) = O\left(\frac{\log m}{m}\right).$$

From (7.2), (7.4) and (7.6) we find that

$$(7.7)\qquad \begin{aligned}&\frac{1}{m}\sum_{n=1}^{m} c_\varrho(n)\, n^{-r-s-1}\Sigma_{r,s}(n)\\ &= \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\,\frac{\varphi(\varrho)\zeta(r+1)\zeta(s+1)}{\varrho^{r+s+2}}+O\left(\frac{\log m}{m}\right)\end{aligned}$$

whenever r and s are positive odd integers.

ON CH. JORDAN'S SERIES FOR PROBABILITY.[1]

BY J. V. USPENSKY.

An interesting article by Ch. Jordan, "Sur le théorème de Bernoulli et son inverse",[2] contains a remarkable series capable of representing a given infinite sequence of numbers under rather general conditions. The formal structure of this series is obviously suggested by Tshebysheff's least squares interpolation method, as one sees from Ch. Jordan's analysis. But questions pertaining to the conditions of validity of that series are not considered by Ch. Jordan himself, and we hope that necessary complements contained in the present paper will be welcomed by those students of probability and statistics who care for rigorous and solid foundations for their methods.

Let ν be an integer $\geqq 0$, x a positive parameter, and $\psi(x, \nu)$ a function defined by

$$\psi(x, \nu) = \frac{x^\nu e^{-x}}{\nu!}.$$

The derivatives of this function can be presented thus:

$$\frac{d^k \psi(x, \nu)}{d x^k} = G_k(\nu)\, \psi(x, \nu),$$

where $G_k(\nu)$ is a polynomial in ν and $1/x$ whose expression is

$$G_k(\nu) = k!\, x^{-k} \sum_{l=0}^{k} (-1)^l \frac{C_\nu^{k-l}}{l!} x^l.$$

Now, if

$$y_0, y_1, y_2, \cdots$$

is a given sequence of numbers, we can define constants a_k by the series

$$(1) \qquad a_k = \frac{x^k}{k!} \sum_{\nu=0}^{\infty} y_\nu\, G_k(\nu)$$

and form another series

$$(2) \qquad a_0\, \psi(x, \nu) + a_1 \frac{d\psi(x, \nu)}{dx} + a_2 \frac{d^2 \psi(x, \nu)}{dx^2} + \cdots$$

which was suggested by Ch. Jordan as fitted to represent for $\nu = 0, 1, 2, \cdots$ the numbers $y_0, y_1, y_2, \cdots$ of a given sequence. However, it is important to investigate the conditions which secure the validity of this heuristically

[1] Received August 8, 1930.

[2] Bulletin de la Société Mathématique de France, vol. LIV, fasc. 12, p. 101–137, 1926.

obtained conclusion. The following analysis is based on the assumption that the power series

$$y_0 + y_1 z + y_2 z^2 + \cdots$$

possesses the convergence radius $R > 2$. Under such circumstances it is possible to prove: i. that all the series (1) are convergent and represent definite constants a_k, ii. that the series (2) is convergent for every positive value of x and for $\nu = 0, 1, 2, \cdots$ represents $y_0, y_1, y_2, \cdots$.

1. **Discussion of the series** $\sum_{k=0}^{\infty} \xi^{-k} \frac{d^k}{dx^k} \frac{x^\nu e^{-x}}{\nu!}$. As $x^\nu/\nu!$ is obviously the residue of the function $e^{xs} s^{-\nu-1}$ with respect to the pole $s = 0$ we have

$$\frac{x^\nu e^{-x}}{\nu!} = \frac{1}{2\pi i}\int_C e^{x(s-1)} s^{-\nu-1}\, ds$$

the path of integration being a closed curve surrounding the point $s = 0$. Hence

$$\frac{d^k}{dx^k}\frac{x^\nu e^{-x}}{\nu!} = \frac{1}{2\pi i}\int_C e^{x(s-1)} (s-1)^k s^{-\nu-1}\, ds. \tag{3}$$

This expression will be of great use in the discussion of the infinite series

$$\sum_{k=0}^{\infty} \xi^{-k} \frac{d^k}{dx^k}\frac{x^\nu e^{-x}}{\nu!},$$

which, as we shall presently see, is convergent for $|\xi| > 1$. To prove this let C be a circle having its center at the point 1, containing 0 inside and leaving the point $1 + \xi$ outside. As on this circle

$$|(s-1)\xi^{-1}|$$

remains constant and < 1 the series

$$\sum_{k=0}^{\infty} (s-1)^k \xi^{-k} = \frac{\xi}{\xi + 1 - s}$$

is uniformly convergent, and, being multiplied by $e^{x(s-1)} s^{-\nu-1}$ can be integrated term by term. By virtue of (3) we obtain

$$\sum_{k=0}^{\infty} \xi^{-k} \frac{d^k}{dx^k}\frac{x^\nu e^{-x}}{\nu!} = -\frac{\xi e^{-x}}{2\pi i}\int_C \frac{e^{sx} s^{-\nu-1}}{s - \xi - 1}\, ds.$$

Now let C' be an arbitrary circle concentric to C and containing the point $1 + \xi$ inside. By Cauchy's residue theorem applied to the ring between C and C' we have

$$\int_{C'} \frac{e^{sx} s^{-\nu-1}}{s-\xi-1}\, ds - \int_C \frac{e^{sx} s^{-\nu-1}}{s-\xi-1}\, ds = 2\pi i\, e^{x(1+\xi)} (1+\xi)^{-\nu-1},$$

whence

$$\sum_{k=0}^{\infty} \xi^{-k} \frac{d^k}{dx^k} \frac{x^\nu e^{-x}}{\nu!} = \xi e^{x\xi}(1+\xi)^{-\nu-1} - \frac{\xi e^{-x}}{2\pi i}\int_{C'} \frac{e^{sx} s^{-\nu-1}}{s-\xi-1}\,ds.$$

As the radius of C' can be taken arbitrarily large it is clear that

$$\frac{\xi e^{-x}}{2\pi i}\int_{C'} \frac{e^{sx} s^{-\nu-1}}{s-\xi-1}\,ds = \xi\varphi(\xi)$$

is an *entire* function of ξ. Thus finally

$$\sum_{k=0}^{\infty} \xi^{-k} \frac{d^k}{dx^k} \frac{x^\nu e^{-x}}{\nu!} = \xi e^{x\xi}(1+\xi)^{-\nu-1} - \xi\varphi(\xi). \tag{4}$$

The right-hand member gives the analytical continuation of the left-hand member represented by a series which is convergent only for $|\xi|>1$.

2. **Convergence of series (1).** By Cauchy's formula

$$\frac{d^k x^\nu e^{-x}}{dx^k} = \frac{k!}{2\pi i}\int_{\Gamma} e^{-z} z^\nu (z-x)^{-k-1}\,dz,$$

the path of integration being a closed curve surrounding the point $z=x$ and otherwise arbitrary. Hence

$$\frac{x^k}{k!} G_k(\nu) = \frac{e^x}{2\pi i}\int_{\Gamma} \frac{e^{-z}(zx^{-1})^\nu}{z-x}\left(\frac{x}{z-x}\right)^k dz. \tag{5}$$

We assume that the radius of convergence of the series

$$f(z) = y_0 + y_1 z + y_2 z^2 + \cdots$$

is $R>2$. We can take for Γ a circle of radius

$$r = cx, \quad \text{where} \quad 1<c<R-1,$$

described with the point x as center. This circle will contain the point 0 inside. Moreover if z is any point of Γ we have

$$|zx^{-1}| \leqq \frac{(r+x)}{x} = c+1<R$$

and consequently the series

$$\sum_{\nu=0}^{\infty} y_\nu (zx^{-1})^\nu = f\left(\frac{z}{x}\right)$$

is uniformly convergent on Γ so that by (5)

$$a_k = \frac{e^x}{2\pi i}\int_{\Gamma} \frac{e^{-z} f\left(\frac{z}{x}\right)}{z-x}\left(\frac{x}{z-x}\right)^k dz. \tag{6}$$

This proves the convergence of the series (1) and at the same time gives an appropriate analytic expression of a_k.

3. **Sum of the series** (2). Multiplying the preceding expression of a_k by

$$\frac{d^k \psi(x, \nu)}{d x^k}$$

and taking into account that

$$\sum_{k=0}^{\infty} \left(\frac{x}{z-x}\right)^k \frac{d^k \psi(x, \nu)}{d x^k}$$

converges uniformly since

$$\left|\frac{z-x}{x}\right| = c > 1$$

we find

$$a_0 \psi(x, \nu) + a_1 \frac{d \psi(x, \nu)}{d x} + a_2 \frac{d^2 \psi(x, \nu)}{d x^2} + \cdots$$

$$= \frac{e^x}{2\pi i} \int_\Gamma \frac{e^{-z} f\left(\frac{z}{x}\right)}{z-x} \sum_{k=0}^{\infty} \left(\frac{z-x}{x}\right)^{-k} \frac{d^k \psi(x, \nu)}{d x^k} dz$$

or, using (4)

$$a_0 \psi(x, \nu) + a_1 \frac{d \psi(x, \nu)}{d x} + a_2 \frac{d^2 \psi(x, \nu)}{d x^2} + \cdots$$

$$= \frac{x^\nu}{2\pi i} \int_\Gamma f\left(\frac{z}{x}\right) z^{-\nu-1} dz - \frac{x^{-1} e^x}{2\pi i} \int_\Gamma e^{-z} f\left(\frac{z}{x}\right) \varphi\left(\frac{z-x}{x}\right) dz.$$

Since

$$f\left(\frac{z}{x}\right) \varphi\left(\frac{z-x}{x}\right)$$

is regular in Γ the second integral vanishes and the first integral reduces to

$$\frac{y_\nu}{2\pi i} \int_\Gamma \frac{dz}{z} = y_\nu.$$

Thus

$$y_\nu = a_0 \psi(x, \nu) + a_1 \frac{d \psi(x, \nu)}{d x} + a_2 \frac{d^2 \psi(x, \nu)}{d x^2} + \cdots$$

for every $\nu \geqq 0$ as we intended to prove.

If we set $\psi(x, \nu) = 0$ when ν is a negative integer it is easy to see that the preceding series can also be exhibited in the form

$$y_\nu = a_0 \psi(x, \nu) - a_1 \Delta \psi(x, \nu-1) + a_2 \Delta^2 \psi(x, \nu-2) - \cdots.$$

It is remarkable that x can be left entirely arbitrary and we possess an advantage to dispose of it so as to obtain better convergence. The

second form of Ch. Jordan's series shows clearly that an analogous series can represent the sum

$$\sum_{\nu=0}^{m} y_\nu .$$

In fact we have

$$\sum_{\nu=0}^{m} y_\nu = \frac{a_0}{m!}\int_x^\infty e^{-z} z^m\, dz - a_1\, \psi(x, m) + a_2\, \Delta\, \psi(x, m-1) - \cdots.$$

4. **Application to Probability.** If the sequence $y_0, y_1, y_2, \cdots$ represents the probabilities of $0, 1, 2, \cdots$ successes in n independent trials with constant probability p, so that in general

$$y_\nu = C_n^\nu p^\nu q^{n-\nu}, \quad q = 1 - p$$

the generating function of y_ν will be

$$f(z) = (pz + q)^n.$$

The corresponding expression for a_k in this case is

$$(7) \qquad a_k = x^{-n} \frac{e^x}{2\pi i} \int_\Gamma \frac{(pz + qx)^n}{z - x} \left(\frac{x}{z - x}\right)^k e^{-z}\, dz.$$

Expression of a_k in a finite form can be obtained by evaluating the residue of the integrand at the pole $z = x$. Taking $x = np$ we find

$$\begin{aligned}
a_0 &= 1, & a_4 &= (n-2)\frac{x^4}{8n^3}, \\
a_1 &= 0, & a^5 &= -(5n-6)\frac{x^5}{30n^4}, \\
a_2 &= -\frac{x^2}{2n}, & a_6 &= -(3n^2 - 26n + 24)\frac{x^6}{144n^5}, \\
a_3 &= \frac{x^3}{3n^2} & & \cdots\cdots\cdots\cdots\cdots.
\end{aligned}$$

With these values adopted for $a_0, a_1, a_2, \cdots$ the probability Q of the number of successes not exceeding a given limit m can be expressed by the series

$$Q = 1 - \frac{1}{m!}\int_0^x e^{-z} z^m\, dz + a_2\, \Delta\, \psi(x, m-1) - a_3\, \Delta^2\, \psi(x, m-2) + \cdots.$$

This series is surely convergent, but for practical use it is not enough to know that the series is convergent, and it is of the utmost importance to have an idea of the rapidity of the convergence. To this end we shall estimate the upper limit of $|a_k|$. Taking in (7)

$$z = x + Re^{i\varphi},$$

and leaving R arbitrary at first, we have

$$|a_k| \leqq \frac{x^{-n}}{2\pi} \int_0^{2\pi} |x + p\,R e^{i\varphi}|^n \left(\frac{x}{R}\right)^k e^{-R\cos\varphi}\, d\varphi,$$

or, if we substitute $x = np$ and set $R = n\xi$,

$$(8) \qquad |a_k| \leqq \frac{1}{2\pi} \left(\frac{p}{\xi}\right)^k \int_0^{2\pi} |1 + \xi e^{i\varphi}|^n e^{-n\xi\cos\varphi}\, d\varphi.$$

Now, setting

$$u = |1 + \xi e^{i\varphi}|\, e^{-\xi\cos\varphi},$$

we have

$$\frac{d\log u}{d\varphi} = \frac{\xi^2 \sin\varphi}{1 + 2\xi\cos\varphi + \xi^2}(2\cos\varphi + \xi).$$

Hence, if $\xi \leqq 2$ the maximum of u is attained when

$$\cos\varphi = -\frac{\xi}{2} \text{ with } u = e^{\xi^2/2}.$$

It follows from (8)

$$(9) \qquad |a_k| < \left(\frac{p}{\xi}\right)^k e^{n\xi^2/2}$$

provided $\xi \leqq 2$. From now on we distinguish two cases.

Case 1. $k \leqq 4n$. The right-hand member of (9) attains its minimum when

$$\xi^2 = \frac{k}{n} \leqq 4.$$

Hence we have

$$|a_k| < \left(\frac{enp}{k}\right)^{k/2} p^{k/2},\ k \leqq 4n.$$

Case 2. $k > 4n$. In this case we simply take $\xi = 2$ and then it follows from (8)

$$|a_k| < (\tfrac{1}{2} p\, e^{2n/k})^k < \left(p\,\frac{e^{1/2}}{2}\right)^k.$$

As to

$$\frac{d^k \psi(x, \nu)}{dx^k}$$

we confine ourselves not to the best but to the simplest estimations of its upper limit. To this end we resort to the expression (3). Taking for C the unit circle $|s| = 1$ we find

$$\left|\frac{d^k \psi(x, \nu)}{dx^k}\right| \leqq \frac{e^{-x}}{2\pi} \int_0^{2\pi} e^{x\cos\varphi} (2 - 2\cos\varphi)^{k/2} d\varphi.$$

If $k \leqq 4x$ the maximum of the integrand corresponds to

$$\cos\varphi = 1 - \frac{k}{2x}$$

and is

$$e^x \left(\frac{k}{x}\right)^{k/2} e^{-k/2}$$

so that

$$\left|\frac{d^k \psi(x,\nu)}{dx^k}\right| < \left(\frac{k}{x}\right)^{k/2} e^{-k/2}$$

if $k \leqq 4x$. If, on the contrary, $k > 4x$ that maximum is

$$2^k e^{-x}$$

and correspondingly

$$\left|\frac{d^k \psi(x,\nu)}{dx^k}\right| < 2^k e^{-2x}.$$

Combining this with the limits obtained for $|a_k|$ we have

$$|a_k \Delta^{k-1} \psi(x, m-k+1)| < \left(\frac{enp}{k}\right)^{1/2} p^{k/2} \text{ if } k \leqq 4np,$$

$$|a_k \Delta^{k-1} \psi(x, m-k+1)| < \tfrac{1}{2}(ep)^{k/2} e^{-2np} \text{ if } 4np < k < 4n,$$

$$|a_k \Delta^{k-1} \psi(x, m-k+1)| < \tfrac{1}{2}(e^{1/2}p)^k e^{-2np} \text{ if } k \geqq 4n.$$

Hence, we see that the convergence of Ch. Jordan's series for probability is very rapid if p is small. If n is large and p so small that np is a moderate number, the sum of the series differs but little from its first term representing the so-called "Poisson's approximate formula". The same series may be used even if p is not very small, but then the convergence is naturally slow, and the use of ordinary approximate formulas is preferable.

A FUNCTIONAL EQUATION IN DIFFERENTIAL GEOMETRY.[1]

By T. H. Gronwall.

1. **Introduction.** It is proposed to determine all surfaces of constant curvature having an equation of the form

$$X(x)+Y(y)+Z(z) = 0;$$

the functional equation corresponding to this geometrical problem is obtained as follows. Writing $X(x) = u$, $Y(y) = v$, $Z(z) = w$, so that

$$u+v+w = 0, \tag{1}$$

we may express x, y and z as functions of u, v and w in the form

$$x = \int \frac{du}{\sqrt{f(u)}}, \qquad y = \int \frac{dv}{\sqrt{g(v)}}, \qquad z = \int \frac{dw}{\sqrt{h(w)}}, \tag{2}$$

and the condition for constant curvature $= K/4 \neq 0$ is readily seen to be

$$\begin{aligned} f(u)\,g'(v)\,h'(w)+g(v)\,h'(w)f'(u)+h(w)f'(u)\,g'(v)& \\ -K[f(u)+g(v)+h(w)]^2 &= 0. \end{aligned} \tag{3}$$

It will be shown that this functional equation (where the three variables are connected by (1)) has no solutions other than the trivial ones obtained in paragraph 2 and corresponding to surfaces of revolution about one of the coördinate axes. Although the result is thus essentially negative, the function theoretic method of proof may claim some interest, since it is connected with the process for determining all functions having an algebraic addition theorem, as well as with the theory of entire functions of finite order.[2]

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[2] The elementary method of differentiation and elimination leads to inextricable calculations. However, the corresponding equation for minimal surfaces,

$$(g+h)f'+(h+f)g'+(f+g)h' = 0$$

yields readily to this method, as was shown by J. Weingarten, Ueber die durch eine Gleichung von der Form $X+Y+Z=0$ darstellbaren Minimalflächen, Göttinger Nachrichten, 1887, p. 272–275. He obtains as solutions the surfaces of revolution and also a class of surfaces discovered by H. A. Schwarz, Fortgesetzte Untersuchungen über specielle Minimalflächen, Sitzungsber. Akad. Berlin, 1872, p. 3–27 = Gesammelte Mathematische Abhandlungen, Berlin 1892, vol. 1, p. 126–148.

2. The exceptional case, leading to surfaces of revolution. Differentiate (3) with respect to u and v, and subtract the results, whence

$$(4)\quad (gf''-fg'')h'+(g'f''-f'g'')h-2K(f+g+h)(f'-g')=0.$$

Eliminating h' between this and (3), we find

$$(5)\quad \begin{aligned}&(gf''-fg'')\,[hf'g'-K(f+g+h)^2]\\ &-(fg'+gf')\,[(g'f''-f'g'')h-2K(f+g+h)(f'-g')]=0,\end{aligned}$$

and the exceptional case in h arises when this equation reduces to an identity in h, which requires that $gf''-fg''=0$ and either $fg'+gf'=0$ or $f'-g'=0$. In the first case, $f'/f=-g'/g=\text{const.}=\alpha$, so that $f=e^{\alpha(u-u_0)}$, $g=e^{-\alpha(v-v_0)}$, or by an obvious change of variables, $f=e^{\alpha u}$, $g=e^{-\alpha v}$. Substituting in (4), we find

$$-\frac{\alpha^2 h}{K}=\frac{(e^{\alpha(u+v)}+1)^2}{e^{\alpha(u+v)}+Ke^{\alpha v}(e^{\alpha(u+v)}+1)},$$

which is not a function of $w=-u-v$ alone unless $K=0$ and $h=0$, which is impossible. The assumption $\alpha=0$ is seen from (3) also to give $K=0$ contrary to our hypothesis. In the second case, $f'=g'=\text{const.}=\alpha$, so that $f=\alpha(u-u_0)$, $g=\alpha(v-v_0)$, or by the same change of variables as before, $f=\alpha u$, $g=\alpha v$, and substituting in (3), we find $\alpha^2(u+v)h'+\alpha^2h-K(\alpha u+\alpha v+h)^2=0$ which becomes by (1),

$$\frac{\alpha^2(h-\alpha w)'}{\alpha^2(h-\alpha w)-K(h-\alpha w)^2}=\frac{1}{w},$$

with the general solution

$$h=\frac{\alpha w(1+\alpha c+Kcw)}{1+Kcw},$$

c being the integration constant. The surface is one of revolution about an axis parallel to the z-axis, since by (2) $x-x_0=2(u/\alpha)^{1/2}$, $y-y_0=2(v/\alpha)^{1/2}$, so that (1) gives $-4w/\alpha=(x-x_0)^2+(y-y_0)^2$.

Equation (1) is symmetrical in u, v, w, and (3) in f, g, h; by cyclic permutation, we obtain from (5) two equations which, when identical in f and g respectively, constitute exceptional cases in which the solutions give surfaces of revolution about the x- and y-axis.

From now on, we shall deal with the non-exceptional case, where none of the three equations of the type (5) is an identity in h, f and g respectively. Assuming these functions to possess third derivatives, they are analytic, for differentiate (5) with respect to u and v, subtract and

eliminate the quadratic term in $f+g+h$ between the equation thus obtained and (5); suppressing the factor $fg'+gf'$, we obtain

$$\begin{aligned}(gf'''+fg'''-g'f''-f'g'')\,[(g'f''-f'g'')h-2K(f+g+h)(f'-g')]\\ -(gf''-fg'')\,[(g'f'''+f'g'''-2f''g'')h-2K(f+g+h)(f''+g'')]\\ =0.\end{aligned}$$

Eliminating h between this and (5), we find an equation involving $f, g, \cdots, f''', g'''$ and giving v a constant value, we have a third order differential equation in f, the solution of which is obviously analytic. By symmetry, g and h are also analytic.[3]

We shall now investigate the properties of these analytic functions with the aid of (3) and (4).

3. **The functions f, g, h are meromorphic, and their poles of the first order.** Let f, g and h be holomorphic at u_0, v_0 and w_0 respectively, where $u_0+v_0+w_0=0$ according to (1). Then there exists an $r>0$ such that the three functions have algebraic singularities only for $|u-u_0|<r$, $|v-v_0|<r$, $|w-w_0|<r$. Make $u-u_0=v-v_0=-(w-w_0)/2$, then (5) becomes an algebraic equation in h with coefficients which have algebraic singularities only for $|w-w_0|<2r$, so that h has only such singularities in this circle. By symmetry, it follows that f, g and h have algebraic singularities only for $|u-u_0|<2r$, $|v-v_0|<2r$, $|w-w_0|<2r$. Repeating this process, we may replace r successively by $2r, 4r, \cdots, 2^n r, \cdots$, so that our three functions have none but algebraic singularities at finite distance. Let $u=a$ be an algebraic singular point of $f(u)$; replacing u by $u+a$, we may evidently assume $a=0$, and in the vicinity of $u=0$, we have the expansion

$$(6)\qquad f(u)-f_0 = c_0u^{p/q}+c_1u^{(p+1)/q}+\cdots;\quad c_0\neq 0,$$

where we have placed the constant term f_0 to the left. By (1) we also have, for any v where $g(v)$ and $h(-v)$ are holomorphic,

$$(7)\qquad \begin{aligned} h(w) &= h(-u-v) = h(-v)-h'(-v)u+\frac{h''(-v)}{2!}u^2+\cdots,\\ h'(w) &= h'(-v)-h''(-v)u+\frac{h'''(-v)}{2!}u^2+\cdots.\end{aligned}$$

We now substitute (6) and (7) in (3) and distinguish several cases:

Case I, $p/q<-1$. Then the term of lowest order in u occurs in Kf^2, and we must have $K=0$.

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[3] When the equation in $f, \cdots, g'''$ reduces to an identity in f and its derivatives, we get one or more third order differential equations in g, and arrive at the same conclusion.

Case II, $p/q = -1$. The terms containing u^{-2} in (3) give

$$g(v)h'(-v)+h(-v)g'(v)+Kc_0 = 0. \tag{8}$$

Let $c_\lambda u^{(p+\lambda)/q}$ be the lowest term with non-integer exponent in (7), then the lowest terms with such an exponent in (3) give

$$\frac{p+\lambda}{q}[g(v)h'(-v)+h(-v)g'(v)]-2Kc_0 = 0,$$

and by comparison with the preceding equation, $(p+\lambda)/q = -2$ contrary to hypothesis. Hence in case II, f has a pole of the first order.

Case III, $-1<p/q<1$. Then $(gh'+hg')f'$ contains the term of lowest order $p/q-1$, and setting its coefficient equal to zero, we find $g(v)h'(v)+h(-v)g'(v) = 0$, whence $h(-v) = cg(v)$, $c =$ constant,

$$g'h' = -c[g'(v)]^2+\cdots,$$
$$gh'+hg' = c[g'(v)^2-g(v)g''(v)]u+\cdots.$$

Case IIIa, $-1<p/q<0$. The term of next lowest order in (3) is $-Kc_0^2 u^{2p/q}$, so that $K=0$ contrary to hypothesis.

Case IIIb, $0<p/q<1$. (the case $p/q=0$ cannot occur, since the constant term stands to the left in (7)). The terms of orders 0 and p/q in (3) give

$$-cf_0 g'(v)^2-K[f_0+(1+c)g(v)]^2 = 0,$$
$$-cg'(v)^2+\frac{p}{q}c[g'(v)^2-g(v)g''(v)]-2K[f_0+(1+c)g(v)] = 0.$$

When g is constant, then $h = cg$ is constant, and also f by (3); our surface is a plane, and $K = 0$ contrary to hypothesis. When g is not a constant, the differentiation of the first equation above gives

$$cf_0g''(v)+K(1+c)\ [f_0+(1+c)g(v)] = 0,$$

and the elimination of g'^2 and g'' from the second equation yields

$$[f_0+(1+c)g(v)]\ [(1+c)g(v)-(1+p/q)f_0] = 0$$

which again implies $g(v) =$ const., unless $c = -1$ and $f_0 = 0$. In this case, the second equation above reduces to $(1-q/p)g'^2 = gg''$ with the general solution $g =$ const.$(v-v_0)^{p/q}$.

Let u be such that $f(u)$ and $h(-v_0-u)$ are holomorphic, and let v turn around v_0; then $g(v)$ is multiplied by a constant factor $\lambda^2 \neq 1$, while f and h are unchanged, and (3) becomes

$$\lambda^2[fg'h'+gh'f'+hf'g']-K[f+\lambda^2 g+h]^2 = 0,$$

whence by (3), with a proper determination of the sign of λ, $f+\lambda^2 g+h = \lambda[f+g+h]$. Turning once more about v_0, this becomes $f+\lambda^4 g+h = \lambda[f+\lambda^2 g+h]$, whence $(\lambda^2-1)(\lambda^2-\lambda)g = 0$ or $\lambda^2 = 1$, contrary to our assumption.

Case IV, $p/q \geqq 1$. Let $c_\lambda u^{(p+\lambda)/q}$ be the lowest term with non-integer exponent, the lowest such term in (3) occurs in $(gh'+hg')f'$ and yields $h(-v) = \text{const.}\, g(v)$ as in Case III. The further discussion is similar to that in Case IIIb, and leads to the same conclusion.

Thus f is uniform, its only singularities at finite distance are poles of the first order, and by symmetry, this is also true of g and h.

4. **The functions f, g and h have no zeros.** Suppose that f has a zero, which we may assume to be at $u = 0$, so that for $|u|$ sufficiently small

$$f(u) = c_n u^n + c_{n+1} u^{n+1} + \cdots, \qquad c_n \neq 0 \text{ and } n \geqq 1,$$

and for any v which is not a pole of $g(v)$ or $h(-v)$, we have the expansion (7). First assume $n \geqq 2$, then the constant term in (3) gives

$$h(-v) = -g(v),$$

and substituting this in (7) the expansion of (3) is

$$\begin{gathered}(c_n u^n + c_{n+1} u^{n+1} + \cdots)\,(g'^2 + g' g'' u + \cdots) \\ +(n c_n u^{n-1} + (n+1) c_{n+1} u^n + \cdots)\,((g g'' - g'^2) u + \tfrac{1}{2}(g g''' - g' g'') u^2 + \cdots) \\ -K(-g' u - \tfrac{1}{2} g'' u^2 + c_n u^n + \cdots)^2 = 0.\end{gathered}$$

When $n \geqq 3$, the term in u^2 gives $g' = 0$, so that g, and consequently also h, is a constant, contrary to the assumption. When $n = 2$, the terms in u^2 and u^3 give

$$\begin{aligned} 2c_2 g g'' - (c_2 + K) g'^2 &= 0, \\ c_2 g g''' + 3 c_3 g g'' - 2 c_3 g'^2 + K g' (2 c_2 - g'') &= 0, \end{aligned}$$

whence by differentiation and elimination

$$c_3 (3K - c_2) g' + 4 c_2^2 K = 0,$$

so that $g(v)$ is a linear function $= \alpha(v - v_0)$, and $h(w) = -g(-w) = \alpha(w + v_0)$, and we are in the exceptional case of paragraph 2.

Thus there remains only the case $n = 1$; since we shall have to consider all the zeros of f, g and h, let $u = a$ be a zero of f, so that in its vicinity

(9) $$f(u) = f'(a)\,(u-a) + \tfrac{1}{2} f''(a)\,(u-a)^2 + \cdots, \quad f'(a) \neq 0.$$

Replacing u by $u-a$ and $-v$ by $-v-a$ in (7), introducing in (3) and equating the coefficients of $(u-a)^n$ to zero for $n=0$ and 1, we find

$$(10)\quad \begin{aligned} &f'(a)\,[g(v)\,h'(-v-a)+h(-v-a)\,g'(v)]-K[g(v)+h(-v-a)]^2=0,\\ &-f'(a)\,g(v)\,h''(-v-a)+f''(a)[g(v)\,h'(-v-a)+h(-v-a)\,g'(v)]\\ &\qquad -2K[f'(a)-h'(-v-a)]\,[g(v)+h(-v-a)]=0. \end{aligned}$$

When $g(v)+h(-v-a)=0$, then $h''(-v-a)=0$, so that we are again in the exceptional case.

The first equation (10) may be written

$$\frac{d}{dv}\frac{h(-v-a)}{g(v)}+\frac{K}{f'(a)}\left[1+\frac{h(-v-a)}{g(v)}\right]^2=0,$$

and assuming $g(v)+h(-v-a)\neq 0$, this is integrated at once in the form

$$(11)\qquad \frac{h(-v-a)}{g(v)}=\frac{v-\alpha}{\beta-v},$$

where

$$(12)\qquad K(\alpha-\beta)=f'(a).$$

Using (11) and (12), the second equation (10) reduces to

$$(13)\quad \begin{aligned} &\frac{d^2h(-v-a)}{dv^2}+\frac{2}{v-\beta}\frac{dh(-v-a)}{dv}\\ &+\frac{f''(a)}{f'(a)}\left(\frac{1}{v-\alpha}-\frac{1}{v-\beta}\right)h(-v-a)+\frac{2f'(a)}{v-\beta}=0. \end{aligned}$$

When $f''(a)=0$, (13) gives

$$h(-v-a)=\frac{C}{v-\beta}-f'(a)(v-\beta)+C',$$

and (11)

$$g(v)=-\frac{C+C'(\alpha-\beta)}{v-\alpha}+f'(a)(v-\alpha)-C',$$

so that $g(v)$ and $h(w)$ have zeros $v=b$ and $w=c$. If $g''(b)=h''(c)=0$, then $C=C'=0$, and we have the exceptional case, so that, by a suitable permutation of f, g and h, we may assume that $f''(a)\neq 0$. Then (13) shows immediately that $h(-v-a)$ has no pole at $v=\beta$, and vanishes at $v=\alpha$. Since h is uniform, and $v=\infty$ is a regular singular point of (13), it follows that h is a polynomial. Let the degree of this polynomial be n; when $n=1$, (11) shows that we have the exceptional case, and when $n>1$, (13) gives

$$n(n-1)+2n+(\alpha-\beta)f''(a)/f'(a)=0,$$

or by (12)

$$f''(a)=-n(n+1)K.$$

Since $h(-\alpha-a) = 0$, it follows that $g(v)$ is also a polynomial of degree n.

Permuting f, g and h, the preceeding discussion shows that, except in the case of paragraph 2, f, g and h are polynomials of degree $n > 1$, and $f''(u)+n(n+1)K = 0$ at every zero of $f(u)$, and similarly for $g(v)$ and $h(w)$. Since f, g and h have no double zeros, it follows that $f''(u)+n(n+1)K$ etc. vanish identically, so that f, g and h are polynomials of the second degree, the coefficient of the highest term in each being $-3K$. On account of $K \neq 0$, the limit as $v \to \infty$ to the left in (11) is therefore $+1$, while the right hand member tends to -1. This contradiction shows that, except in the case of paragraph 2, f, g and h have no zeros.

5. **Proof that g and h cannot be exponentials.** To simplify some details in subsequent paragraphs, it will now be shown that there is no solution of (3) in which

$$g(v) = Be^{\beta v}, \qquad h(w) = Ce^{\gamma w},$$

except the case where f, g and h are constants, which would require the surface to be a plane and $K = 0$. Permuting f and h in (4), and substituting the expressions above, we find

$$\begin{gathered} BC(\gamma-\beta)[(\beta+\gamma)f'+\beta\gamma f]\, e^{(\beta-\gamma)v-\gamma u} \\ -2K(f+Be^{\beta v}+Ce^{-\gamma v-\gamma u})(C\gamma e^{-\gamma v-\gamma u}-B\beta e^{\beta v}) = 0. \end{gathered}$$

Multiplying out, we see that each term contains a factor $e^{\lambda v}$, where λ has the following values

$$0,\ \beta,\ -\gamma,\ \beta-\gamma,\ 2\beta,\ -2\gamma,$$

and in order that the expression above shall vanish, two or more of these exponents must be equal. It is seen at once that the following cases only are possible:

$$\begin{gathered} \gamma = 0, \qquad e^{2\beta v}\cdot 2KB^2\beta = 0, \qquad \beta = 0; \\ -\gamma = \beta, \qquad e^{\beta v}\cdot 2Kf\beta(Ce^{\beta u}+B) = 0, \qquad \beta = 0; \\ -\gamma = \beta-\gamma, \qquad \beta = 0, \qquad e^{-2\gamma(v+u)}\cdot 2KC^2\gamma = 0, \qquad \gamma = 0; \\ -\gamma = 2\beta, \qquad e^{4\beta(v+u)}\cdot KC^2\gamma = 0, \qquad \gamma = 0. \end{gathered}$$

Hence g and h are constants, and substitution in (3) gives $f = -g-h = \text{const.}$ when $K \neq 0$.

6. **Distribution of the poles of f, g and h.** Let $u = a$ be a pole of $f(u)$ (which is simple according to paragraph 3) with the residue a_0; then the equation corresponding to (8) is

$$(14) \qquad g(v)\,h'(-v-a)+h(-v-a)\,g'(v)+K a_0 = 0.$$

This may be rewritten in the two forms

$$(15) \quad \frac{d}{dv}\,\frac{h(-v-a)}{g(v)} = \frac{K a_0}{g(v)^2}, \quad \frac{d}{dv}\,\frac{g(v)}{h(-v-a)} = -\frac{K a_0}{h(-v-a)^2},$$

and since $g(v)$ and $h(-v-a)$ have no zeros, it follows that $h(-v-a)/g(v)$ and its reciprocal are entire functions, so that

$$(16) \qquad \frac{h(-v-a)}{g(v)} = e^{\Gamma(v)},$$

where $\Gamma(v)$ is an entire function. From (16) it is seen that either $g(v)$ and $h(w)$ both lack poles, or $g(v)$ has a pole $v=b$, $h(w)$ a pole $w=c$, where

$$(17) \qquad a+b+c = 0.$$

It is therefore possible that f, g and h may have one pole each, so that $(u-a)f(u)$, $(v-b)\,g(v)$, $(w-c)\,h(w)$ have neither zeros nor poles.

Next, assume that one of f, g, h has more than one pole; let, for instance, f have the poles a and a', the latter with the residue a_0'. Then, in analogy to (15),

$$(18) \quad \frac{d}{dv}\,\frac{h(-v-a')}{g(v)} = \frac{K a_0'}{g(v)^2}, \quad \frac{d}{dv}\,\frac{g(v)}{h(-v-a')} = -\frac{K a_0'}{h(-v-a')^2},$$

and

$$(19) \qquad \frac{h(-v-a')}{g(v)} = e^{\Gamma_1(v)}.$$

Multiplying (15) by a_0', (18) by a_0, subtracting and integrating, we find

$$(20) \qquad a_0' e^{\Gamma(v)} - a_0\, e^{\Gamma_1(v)} = A.$$

When the constant A does not vanish, (20) shows that the entire function $e^{\Gamma(v)}$ takes neither of the values 0 and A/a_0', hence it reduces to a constant by Picard's theorem, and (15) gives $a_0 = 0$ contrary to hypothesis. Therefore $A = 0$, and (16), (19) and (20) give $a_0' h(-v-a) - a_0\, h(-v-a') = 0$, or replacing $-v-a'$ by w

$$h(w+a'-a) = \lambda\, h(w), \quad \text{where } \lambda = a_0/a_0'.$$

Permuting g and h in (14), we have similarly

$$g(v+a'-a) = \lambda\, g(v).$$

By (1) we may replace v and w in (3) by $v+a'-a$ and $w+a-a'$, thus changing $g(v)$ into $\lambda g(v)$ and $h(w)$ into $\lambda^{-1}h(w)$ so that

$$[f(u)+\lambda g(v)+\lambda^{-1}h(w)]^2 = [f(u)+g(v)+h(w)]^2.$$

Hence either

$$\lambda g(v)+\lambda^{-1}h(w) = g(v)+h(w),$$

so that $\lambda = 1$, or $2f(u)+(\lambda+1)g(v)+(\lambda^{-1}+1)h(w)=0$,

where obviously $\lambda+1 \neq 0$, and differentiating with respect to v,

$$(\lambda+1)g'(v)-(\lambda^{-1}+1)h'(w) = 0,$$

which would make g' and h' constants and lead to the exceptional case. Therefore $\lambda=1$, so that

$$(21) \qquad g(v+a'-a) = g(v), \qquad h(w+a'-a) = h(w).$$

Since we are not in the exceptional case, the quadratic equation in $f(u)$ obtained by permuting f and h in (3) is not an identity; by (21) it is satisfied by $f(u)$, $f(u+a'-a)$ and $f(u+2(a'-a))$, two of which must consequently be equal, so that

$$(22) \qquad f(u+2(a'-a)) = f(u).$$

The case where both g and h are constants having been settled in paragraph 5, we may assume that g is not constant. Being meromorphic, and having the period $a'-a$, g is either simply or doubly periodic. First suppose that g (and hence also h, by (16)) has no poles; then the doubly periodic case would require g to be a constant, contrary to our assumption. In the simply periodic case, let a' be chosen so that there is no other pole a'' of f such that $|a''-a|<|a'-a|$. Write $a'-a=\omega$, then $a''-a$ is an integral multiple of ω for any pole a'' of f, and consequently $f(u)\sin(\pi(u-a)/\omega)$ is an entire function (which may have zeros). Finally assume that g has a pole b (and consequently h a pole c satisfying (17)), and therefore more than one (adding a period to b). Let b' be any pole of $g(v)$ different from b; then the argument leading to (21) shows that $b'-b$ is a period of f and h; so that $a''=a+b'-b$ is a pole of f. By (21), $a''-a=b'-b$ is a period of g, and consequently f, g and h all have the same periods, a primitive period parallelogram (or strip) containing exactly one pole of each. Here again the doubly periodic case is ruled out, since such a function with simple poles only must have at least two in a primitive period

parallelogram. Passing to the simply periodic case, it is now seen at once that

$$f(u)\sin\frac{\pi(u-a)}{\omega}, \qquad g(v)\sin\frac{\pi(v-b)}{\omega},$$
$$h(w)\sin\frac{\pi(w-c)}{\omega}$$

are entire functions without zeros, and f, g and h have the period ω.

On account of (17), we may take $\pi(u-a)/\omega$ etc. as new variables in (3), whereby K is replaced by $K\pi/\omega$. Then the preceding discussion shows that the following four cases are the only ones possible (except for a permutation of f, g and h in cases I and II).

Case I: $uf(u)$, $g(v)$, $h(w)$ are entire functions, g and h having no zeros.

Case II: $f(u)\sin u$, $g(v)$, $h(w)$ are entire functions of period 2π, g and h have no zeros.

Case III: $uf(u)$, $vg(v)$, $wh(w)$ are entire functions without zeros.

Case IV: $f(u)\sin u$, $g(v)\sin v$, $h(w)\sin w$ are entire functions of period 2π without zeros.

7. Upper bounds for the absolute values of the entire functions in cases I to IV. In cases I and III, write

$$F_1(u) = uf(u),\quad G_1(v) = vg(v),\quad H_1(w) = wh(w), \tag{23}$$

and in Cases II and IV

$$F_2(u) = f(u)\sin u,\quad G_2(v) = g(v)\sin v,\quad H_2(w) = h(w)\sin w. \tag{24}$$

Introducing these in (3), and making $v = w = -u/2$, we obtain

$$\begin{aligned}
&\frac{K}{4}\left[F_1(u)-2\left(G_1\left(-\frac{u}{2}\right)+H_1\left(-\frac{u}{2}\right)\right)\right]^2 \\
&= F_1'(u)\left[\frac{H_1\left(-\frac{u}{2}\right)G_1'\left(-\frac{u}{2}\right)+G_1\left(-\frac{u}{2}\right)H_1'\left(-\frac{u}{2}\right)}{u}\right. \\
&\qquad\left.+\frac{4G_1\left(-\frac{u}{2}\right)H_1\left(-\frac{u}{2}\right)}{u^2}\right] \\
&+F_1(u)\left[\frac{G_1'\left(-\frac{u}{2}\right)H_1'\left(-\frac{u}{2}\right)}{u}\right. \\
&\qquad\left.+\frac{H_1\left(-\frac{u}{2}\right)G_1'\left(-\frac{u}{2}\right)+G_1\left(-\frac{u}{2}\right)H_1'\left(-\frac{u}{2}\right)}{u^2}\right],
\end{aligned} \tag{25}$$

and

$$K\left[F_2(u)-2\cos\frac{u}{2}\left(G_2\left(-\frac{u}{2}\right)+H_2\left(-\frac{u}{2}\right)\right)\right]^2$$

$$= 2F_2'(u)\left[\left(H_2\left(-\frac{u}{2}\right)G_2'\left(-\frac{u}{2}\right)+G_2\left(-\frac{u}{2}\right)H_2'\left(-\frac{u}{2}\right)\right)\cot\frac{u}{2}\right.$$

$$\left.+2G_2\left(-\frac{u}{2}\right)H_2\left(-\frac{u}{2}\right)\cot^2\frac{u}{2}\right]$$

$$(26)\qquad +\,2F_2(u)\left[G_2'\left(-\frac{u}{2}\right)H_2'\left(-\frac{u}{2}\right)\cot\frac{u}{2}\right.$$

$$+\frac{H_2\left(-\frac{u}{2}\right)G_2'\left(-\frac{u}{2}\right)+G_2\left(-\frac{u}{2}\right)H_2'\left(-\frac{u}{2}\right)}{2\sin^2\frac{u}{2}}$$

$$\left.+G_2\left(-\frac{u}{2}\right)H_2\left(-\frac{u}{2}\right)\cot^2\frac{u}{2}\right].$$

We shall now use some well known properties of $M(r)$, the maximum absolute value of an entire function $F(u)$ for $|u| = r$. This $M(r)$ is continuous and non-decreasing as r increases, and is either analytic or composed of analytic pieces joined at the points $r_1, r_2, \cdots$, where $r_n \to \infty$ as $n \to \infty$, so that $M'(r) = dM(r)/dr$ exists everywhere except possibly at $r_1, r_2, \cdots$.[4]

Moreover, when $|F(u')| = M(r)$, where $|u'| = r \neq r_1, r_2, \cdots$, the Cauchy-Riemann equations in polar coördinates show that

$$(27)\qquad |F'(u')| = M'(r).$$

Let $M_1(r)$, $M_2(r)$, $M_3(r)$ be the M-functions for the three entire functions $F(u)$, $G(u)$, $H(u)$, and write

$$(28)\qquad M(r) = \max.\,(M_1(r),\ M_2(r),\ M_3(r)).$$

Then $M(r)$ is evidently continuous and non-decreasing, and the only points where its derivative may not exist are the points $r_1, r_2, \cdots$ belonging to either of M_1, M_2, and M_3, as well as the points where two of M_1, M_2, M_3 are equal. Since the values of r for which $M_1(r) = M_2(r)$ cannot have a limit point on an interval where M_1 and M_2 are both analytic unless $M_1(r) = M_2(r)$ identically on the interval, the values r_n for which $M'(r)$ does not exist are such that $r_n \to \infty$ as $n \to \infty$. For $r \neq r_n$, (27) holds

[4] O. Blumenthal, Sur le mode de croissance des fonctions entières, Bulletin de la société mathématique de France vol. 35 (1907) p. 97–109.

for at least one of F, G and H. We note finally that by Cauchy's formula

$$F'(u) = \frac{1}{2\pi i} \int_{|v-u|=1} \frac{F(v)\,dv}{(v-u)^2},$$

and since $|F(v)| \leqq M(|u|+1)$,

$$|F'(u)| \leqq M(|u|+1). \tag{29}$$

Now let us apply this to the functions (23) for $r \neq r_n$; by a permutation of F_1, G_1, H_1, we may assume that it is F_1 that satisfies (27), and for $u = u'$, the absolute value of the expression to the left in (25) is not less than

$$\frac{|K|}{4}\left[|F_1(u')| - 2\left|G_1\left(-\frac{u'}{2}\right) + H_1\left(-\frac{u'}{2}\right)\right|\right]^2$$
$$\geqq \frac{|K|}{4}\left[M(r)^2 - 8M(r)\,M\left(\frac{r}{2}+1\right)\right].$$

The right hand member in (25) has an absolute value not greater than

$$M'(r)\left(\frac{2}{r}+\frac{4}{r^2}\right)M\left(\frac{r}{2}+1\right)^2 + M(r)\left(\frac{1}{r}+\frac{2}{r^2}\right)M\left(\frac{r}{2}+1\right)^2,$$

and consequently we have, for $r \neq r_n$ and sufficiently large,

$$M(r)^2 < e^{2r}\,[M'(r) + M(r)]\,M\left(\frac{r}{2}+1\right)^2. \tag{30}$$

Turning to (24), we observe that when the imaginary part of u' is bounded as $r \to \infty$, then, since the functions (24) have the period 2π, $M(r)$ is bounded and (30) holds. When, however, the imaginary part of u' becomes infinite with r, then

$$\cot\frac{u'}{2} \quad \text{and} \quad \frac{1}{\sin\frac{u'}{2}}$$

are bounded as $r \to \infty$; since $\left|\cos\frac{u'}{2}\right| < e^{r/2}$ we find, treating (26) in the same manner as (25), that (30) is valid in this case also.

Now write

$$\mu(r) = e^r M(r) \tag{31}$$

which transforms (30) into

$$\frac{\mu'(r)}{\mu(r)^2} > \frac{e^{2-2r}}{\mu\left(\frac{r}{2}+1\right)^2},$$

and integrate[5] from r to $2\alpha r-2$, where

(32) $$\frac{1}{2}<\alpha<\frac{1}{\sqrt{2}}.$$

Then, for all r sufficiently large,

$$\frac{1}{\mu(r)}-\frac{1}{\mu(2\alpha r-2)}>\int_r^{2\alpha r-2}\frac{e^{2-2\varrho}\,d\varrho}{\mu\left(\frac{\varrho}{2}+1\right)^2}$$

$$>\frac{1}{\mu(\alpha r)^2}\int_r^{2\alpha r-2}e^{2-2\varrho}\,d\varrho>\frac{e^{-2r}}{\mu(\alpha r)^2},$$

whence

$$\mu(r)<e^{2r}\,\mu(\alpha r)^2.$$

Writing

(33) $$\nu(r)=e^{2r/(2\alpha-1)}\,\mu(r),$$

the preceding inequality becomes

$$\nu(r)<\nu(\alpha r)^2,$$

whence by iteration

(34) $$\nu(r)<\nu(\alpha^n r)^{2^n}.$$

For $r>r_0$ sufficiently large, we may determine n so that

$$\alpha^{n+1}r<r_0\leqq\alpha^n r;$$

consequently, by (32), $\alpha^n r<r_0/\alpha<2r_0$ and $n\log\alpha+\log r\geqq\log r_0>0$,

$$n<\frac{\log r}{\log\frac{1}{\alpha}},\qquad 2^n<r^\beta,\qquad 1<\beta=\frac{\log 2}{\log\frac{1}{\alpha}}<2,$$

and by (34)

$$\nu(r)<\nu(2r_0)^{r^\beta}.$$

For any fixed δ between β and 2, (31) and (33) now give

(35) $$M(r)<e^{r^\delta},\qquad \delta<2,$$

for all r sufficiently large.

8. **Conclusion of the proof.** In cases I and II, it follows that, for $|u|=r$, since $|\sin u|$ is greater than some constant c for an indefinitely increasing sequence of values of r,

$$|g(u)|\text{ and }|h(u)|<\frac{1}{c}e^{r^\delta}$$

[5] The presence of some of the points r_n on the interval does not disturb the integration of the expression to the left, as is seen by subdividing the interval at these points and observing that $\mu(r)$ is continuous.

for these r, and since g and h are entire functions without zeros, and $\delta < 2$, we have

$$g(u) = Be^{\beta u}, \qquad h(u) = Ce^{\gamma u},$$

but this gives no solution of (3) as shown in paragraph 5. In case III, the same argument gives

$$g(v) = \frac{Be^{\beta v}}{v}, \qquad h(w) = \frac{Ce^{\gamma w}}{w},$$

and in case IV

$$g(v) = \frac{Be^{\beta v}}{\sin v}, \qquad h(w) = \frac{Ce^{\gamma w}}{\sin w}.$$

Substitution in (14) with $a = 0$ yields

$$BC(\beta + \gamma)\, e^{(\beta-\gamma)v}/v \quad + K a_0 = 0$$

and

$$BC(\beta + \gamma)\, e^{(\beta-\gamma)v}/\sin v + K a_0 = 0$$

respectively, so that in both cases $\beta + \gamma = 0$, and $a_0 = 0$ contrary to hypothesis.

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THE REPRESENTATION OF PROJECTIVE SPACES.[1]

BY J. H. C. WHITEHEAD.

Introduction. The projective geometry of paths originated in a paper by H. Weyl,[2] who showed that any two affine connections whose components are related by equations of the form[3]

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j \tag{0.1}$$

give the same paths. The change over from Γ to $\bar{\Gamma}$ is called a *projective change of connection*, and the study of those properties which are invariant under such changes of connection, *the projective geometry of paths.*

The problem of determining the conditions under which two affine connections give the same paths was considered independently (a copy of Weyl's paper not being available) by L. P. Eisenhart,[4] who obtained the conditions (0.1), and gave their significance in terms of a change of parameter on the paths.

The relation between changes of parameter and projective changes of parameter was discussed by O. Veblen[5] in the light of Weyl's work. These two papers, by Eisenhart and Veblen, show how the projective geometry of paths may be formulated in terms of the parametrization. This point of view was adopted some years later by J. Douglas,[6] who took for the paths a system of parameterized curves, and showed that they determine an affine connection, whose components are, in general, functions of positions and direction. Any change of parameter, assuming analyticity throughout, gives rise to a projective change of connection of the form (0.1), where ψ_i are functions of x^i and dx^i, homogeneous of degree zero in the latter. The projective geometry of paths is now seen to consist of those theorems about a system of paths which are independent of the parameterization. Douglas renames it, on this account, the descriptive geometry of paths, and in § 10 of this paper, we have followed him in this respect.

[1] Received June 9, 1930.

[2] Zur Infinitesimalgeometrie; Einordnung der projektiven und der konformen Auffassung. Gött. Nach. (1921), pp. 99-112.

[3] Levi-Civita — Annali di Math. (1896), in connection with a dynamical problem, established these conditions for Riemannian spaces with corresponding geodesics.

[4] Spaces with corresponding paths, Proc. Nat. Acad. of Sciences (1922); vol. 8, pp. 233-238.

[5] Projective and affine geometry of paths. Proc. Nat. Acad. 1922, vol. 8, pp. 347-350.

[6] The general geometry of paths. Annals of Math. (1928), vol. 29, pp. 143-168.

The idea of a projective connection is due to E. Cartan,[7] who associated with each point of his original manifold a flat projective tangent space. He defined a projective connection by means of the differential equations relating the tangent spaces at different points. He also discussed the problem of attaching a projective connection to the system of curves in a space of two dimensions, given by a differential equation of the second degree. There does not, however, seem to be any way of building up a complete theory without reinforcing the foundations laid down in this paper, as the relation between the tangent spaces and the underlying manifold was too indefinite. This point was discussed by Weyl in a paper (1929) to which we refer below.

This was closely followed by a paper by T. Y. Thomas[8] who defined a projective connection using the methods of tensor analysis. The components of the projective connection associated with the paths defined by an affine connection Γ are given by

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{n+1}(\delta^i_j \Gamma^a_{ak} + \delta^i_k \Gamma^a_{aj}), \tag{0.2}$$

and are unaltered by projective changes of the affine connection. The transformation law of the projective connection Π is easily deduced from that of Γ, and to T. Y. Thomas is due the formulation of the projective geometry of paths as the theory of an invariant with a given transformation law.

In a paper which appeared simultaneously with that of T. Y. Thomas, O. Veblen and J. M. Thomas[9] discussed non-homogeneous projective normal coördinates. A projective normal coördinate system, z, is uniquely defined by a given point q, a given coördinate system x, and the relation

$$P^i_{jk} z^j z^k = 0, \tag{0.3}$$

where P^i_{jk} are the components of the projective connection in z. A necessary convergence proof was, however, lacking in their argument that such coördinates exist.

J. A. Schouten[10] showed how Cartan's ideas can be developed with the

[7] Sur les variétés à connexion projective, Bulletin de la Soc. Math. de France (1924), vol. 52, pp. 205–241.

[8] On the projective and equi-projective geometry of paths. Proc. Nat. Acad. of Sciences (1925), vol. 11, pp. 199–203.

[9] Projective normal coördinates for the geometry of paths, Proc. Nat. Acad. (1925), vol. 11, pp. 204–207.

[10] On the place of conformal and projective geometry in the theory of linear displacement, Proc. Akad. van Wetenschappen te Amsterdam (1924), vol. 27, pp. 405–429. See also Erlanger Programm und Übertragungslehre, Rend. di. Palermo (1926), vol. 50, pp. 142–169.

help of König connections. These papers are valuable for their bearing on the foundations of differential geometry, and for their presentation of the projective and conformal geometries as special cases of the more general theory of König connections. But the König connection, in its present stage of development, does not seem to be a suitable weapon for attacking many problems of projective geometry—the equivalence problem, for instance.

The idea of introducing the affine theory with one extra dimension is due to T. Y. Thomas[11] who defined an affine connection for an associated manifold of $(n+1)$ dimensions[12] by means of the relations

$$
\left\{
\begin{aligned}
{}^*\Gamma^i_{jk} &= \Pi^i_{jk}, \\
{}^*\Gamma^\alpha_{0\beta} &= -\frac{1}{n+1}\,\delta^\alpha_\beta, \\
{}^*\Gamma^0_{jk} &= \frac{1+n}{1-n}\,\Pi_{jk},
\end{aligned}
\right. \tag{0.4}
$$

where Π^i_{jk} are the components of a projective connection, and

$$
\Pi_{jk} = \frac{\partial \Pi^a_{jk}}{\partial x^a} - \frac{\partial \Pi^a_{ja}}{\partial x^k} + \Pi^b_{jk}\,\Pi^a_{ba} - \Pi^b_{ja}\,\Pi^a_{kb}.
$$

Thomas defined as projective normal coördinates for Π the affine normal coördinates for ${}^*\Gamma$, and showed that the necessary and sufficient conditions for two projective connections to be equivalent can be expressed in terms of the normal tensors for ${}^*\Gamma$, which constitute a complete set of invariants. In proving this he showed that a transformation of the form

$$
\bar{x}^\alpha = \bar{x}^\alpha(x^0, \cdots, x^n), \tag{0.5}
$$

which carries ${}^*\Gamma$ into the affine connection ${}^*\bar{\Gamma}$, associated with a projective connection $\bar{\Pi}$, is necessarily of the form

$$
\left\{
\begin{aligned}
\bar{x}^0 &= x^0 + \log\left|\frac{\partial \bar{x}}{\partial x}\right|, \\
\bar{x}^i &= \bar{x}^i(x^1, \cdots, x^n),
\end{aligned}
\right. \tag{0.6}
$$

provided that $\left(\dfrac{\partial \bar{x}^\alpha}{\partial x^0}\right)_q = \delta^\alpha_0$, when evaluated at some particular point q.

In a later paper, T. Y. Thomas[13] considered the extent to which the form of the transformation (0.5) is determined by the condition that it

[11] A projective theory of affinely connected manifolds, Math. Zeit. (1926), vol. 25, pp. 723–733.

[12] Greek letters, used as indices, will take the values $0, \cdots, n$, and Roman indices $1, \cdots, n$.

[13] Concerning the $\mathfrak{G}$ group of transformations, Proc. Nat. Acad. (1928), vol. 14, pp. 728–734.

carries $^*\Gamma$ into $^*\bar{\Gamma}$. He showed that when $n = 2$ and $^*\Gamma$ is not flat, then (0.5) is necessarily of the form (0.6), and when $n \geqq 3$, it must be of this form unless Π is subject to certain restrictions.

In the same year O. Veblen and J. M. Thomas[14] gave an alternative solution of the equivalence problem. If this paper is compared with that of T. Y. Thomas and with a later paper by O. Veblen,[15] it is seen that their method gives a solution in terms of the successive projective derivatives of the projective curvature tensor.

In this last paper Veblen gave a systematic account of projective invariants, which brought the projective nature of the theory into evidence. The transformations of a projective tensor were seen to give a representation of the sub-group of projective transformations which leave a given point invariant.

In a lecture to the London Mathematical Society Veblen[16] reformulated the subject in a way which extended its range, and at the same time brought it closer to classical projective geometry. He gave an account of projective displacement in which the relation between the tangent spaces and the underlying manifold was sharply defined. The scheme put forward by Cartan could thus be included in the theory of a projective connection.

Weyl,[17] on the other hand, gave a review of the subject in which he took Cartan's scheme as fundamental, and by adding three postulates, relating the tangent spaces to the underlying manifold, showed how this and the theory of T. Y. Thomas could be fitted together.

It seems to us that the simplest approach is obtained by emphasizing the invariant theory, and regarding infinitesimal displacement as a consequence. This point of view underlies the present paper.

A detailed discussion of the tangent spaces was given by Veblen[18] in a paper concerned with projective connections of an essentially different nature from those introduced by T. Y. Thomas. These were derived from a quadratic differential form, and may be regarded as the basis of a generalized non-Euclidean geometry.

In this paper we shall confine our attention to projective connections

[14] Projective invariants of the affine geometry of paths, Annals of Math. (1926), vol. 27, pp. 279–296.

[15] Projective tensors and connections, Proc. Nat. Acad. (1928), vol. 14, pp. 154–166.

[16] Generalized Projective Geometry, Journal of the London Math. Soc. (1929), vol. 4, pp. 140–160. This paper will be referred to as G. P.

[17] On the foundations of general infinitesimal geometry, Bulletin American Math. Soc. (1929), pp. 716–725.

[18] A generalization of the quadratic differential form, Quarterly Journal, Oxford series (1930), vol. 1, pp. 60–76.

for which the conditions (1.1) of the text are satisfied, and it is to be understood that they are satisfied when we use the term projective connection. Projective connections, in this sense, differ from the associated affine connections of T. Y. Thomas in that, writing $\Pi^{\alpha}_{\beta\gamma}$ for $*\Gamma^{\alpha}_{\beta\gamma}$, Π^{0}_{jk} are arbitrary, and not one, but an infinity of components are determined in each coordinate system.

In a recent paper[19] we studied flat projective connections by means of the ray representation, in which points of n-dimensional flat projective space are represented by means of straight lines through some fixed point in a flat affine space of $(n+1)$ dimensions. In the present paper we shall extend this method to spaces which are not flat, and shall be able to make points in the projective space, P_n, correspond to paths through a fixed point in the affine space, A_{n+1}. This gives a clear geometrical explanation of the treatment originated by T. Y. Thomas.

Relying only on the analysis involved in the affine theory, we give, in § 2, a geometrical method of finding non-homogeneous projective normal coördinates, and in § 4 we show how these are related to the affine normal coördinates for the affine space, A_{n+1}. In § 8 we show that if two projective connections are equivalent under transformations of the form (0.5), then they are equivalent under transformations of the form (1.3) of the text, which take the place of the transformations (0.6) used by T. Y. Thomas; and we give the geometrical reason for this.

In § 9 we make a generalization, suggested by Veblen in G. P., which is useful in § 10, where we show how projective connections may be normalized to obtain projective connections in the sense of T. Y. Thomas. The latter we call normalized projective connections, and in § 11 we show that the non-homogeneous projective normal coördinates for a normalized projective connection are those defined by Veblen and J. M. Thomas. The missing convergence proof is thus indirectly supplied. There is no essential difference in the equivalence problem, and our methods may be applied without modification to normalized projective connections. The question raised by T. Y. Thomas as to how far the transformation (0.5) is characterized by the fact that it carries $*\Gamma$ into $*\bar{\Gamma}$ is completely answered in §§ 8 and 11. This transformation must be of the form (0.6) if the connection admits only one radius vector (see § 1), but otherwise need not be of this form.

When we refer to a function without further qualification, we shall mean an analytic function regular at all points under consideration. We

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[19] On a class of projectively flat affine connections. Proc. London Math. Society (1931), vol. 32, pp. 93–114. We shall use the same notation and terminology as in this paper, which will be referred to as P. F.

shall be consistent in using A_{n+1} and P_n to denote, respectively, an affine space of $(n+1)$ dimensions and a projective space of n dimensions. We shall use a comma and a semi colon to denote, respectively, ordinary partial differentiation and covariant differentiation, thus

$$\begin{cases} T_{\beta,\gamma} = \dfrac{\partial T_\beta}{\partial x^\gamma} \\ T_{\beta;\gamma} = T_{\beta,\gamma} - T_\alpha \Pi^\alpha_{\beta\gamma}. \end{cases}$$

I owe a great deal to Professor Veblen and wish to thank him for his criticism and advice.

1. **The representation.** The projective connections which we are considering are symmetric, and satisfy the following conditions

$$(1.1) \qquad \begin{matrix} (a) \\ (b) \end{matrix} \begin{cases} \Pi^\alpha_{0\beta} = \delta^\alpha_\beta \\ \Pi^\alpha_{\beta\gamma,0} = 0. \end{cases}$$

Let $R^\alpha_{\beta\gamma\delta}$ be the components of the projective curvature tensor corresponding to $\Pi^\alpha_{\beta\gamma}$ (see G. P.). Then,

$$\begin{aligned} R^\alpha_{\beta\gamma 0} &= \Pi^\alpha_{\beta\gamma,0} - \Pi^\alpha_{\beta 0,\gamma} + \Pi^\lambda_{\beta\gamma}\Pi^\alpha_{\lambda 0} - \Pi^\lambda_{\beta 0}\Pi^\alpha_{\lambda\gamma} \\ &= \Pi^\alpha_{\beta\gamma,0}, \text{ in virtue of (1.1a).} \end{aligned}$$

Hence (1.1), taken together, are equivalent to

$$(1.2) \qquad \begin{matrix} (a) \\ (b) \end{matrix} \begin{cases} \Pi^\alpha_{0\beta} = \delta^\alpha_\beta, \\ R^\alpha_{\beta\gamma 0} = 0. \end{cases}$$

Projective geometry was defined in G. P. as the invariant theory of Π under transformations of the form

$$(1.3) \qquad \begin{cases} \bar{x}^0 = x^0 + \log \varrho(x^1, \cdots, x^n) \\ \bar{x}^i = \bar{x}^i(x^1, \cdots, x^n). \end{cases}$$

The conditions (1.2) are invariant under these transformations.

Let Π be an affine connection for an affine space A_{n+1}. If there exists a coördinate system in which the components, $\Pi^\alpha_{\beta\gamma}$, of this connection satisfy the conditions (1.2), then A_{n+1}, referred to this coördinate system, may be taken to represent a projective space P_n. We shall use K_{n+1} to denote any coördinate system for A_{n+1}. A K_{n+1} which is a representation for P_n, will be denoted by R_n. In an R_n, $x + x^0$, a point of P_n is

determined by the coördinates $(x^1 \cdots x^n)$ and any value of the factor x^0. The points of P_n are, therefore, represented in an R_n by the curves of parameter x^0. These curves will be called *rays*. Any K_{n+1} obtained from an R_n by a transformation of the form (1.3) is itself an R_n. The question arises whether two K_{n+1}'s, in which the conditions (1.2) are satisfied, are necessarily related by a transformation of the form (1.3). We shall see that the answer to this question is in the negative, but that A_{n+1}, referred to either K_{n+1}, may be taken to represent the same P_n. An R_n, therefore, will mean any K_{n+1} in which the conditions (1.2) are satisfied.

What are the special properties, enjoyed by an A_{n+1}, in virtue of which it may be taken to represent a P_n? To answer this question we shall translate (1.2) into invariant conditions and find their geometrical significance.

Let $x + x^0$ be any R_n. Then the conditions (1.2) may be written

$$(1.4) \qquad \begin{array}{l} \text{(a)} \\ \text{(b)} \end{array} \left\{ \begin{aligned} \delta^\alpha_{0,\beta} + \delta^\lambda_0\, \Pi^\alpha_{\lambda\beta} &= \delta^\alpha_\beta, \\ R^\alpha_{\beta\gamma\lambda}\, \delta^\lambda_0 &= 0. \end{aligned} \right.$$

Let ξ^α be the components, in any $K_{n+1}(\bar{x}^\alpha)$, of the vector which is defined in $x + x^0$ by δ^α_0. Then in $(\bar{x}^\alpha)$ the equations (1.4) will become

$$(1.5) \qquad \begin{array}{l} \text{(a)} \\ \text{(b)} \end{array} \left\{ \begin{aligned} \xi^\alpha_{;\beta} &= \delta^\alpha_\beta, \\ R^\alpha_{\beta\gamma\lambda}\, \xi^\lambda &= 0. \end{aligned} \right.$$

Conversely let Π be any affine connection, such that there exists a vector satisfying (1.5). Then we can find a coördinate system in which the components of this vector are δ^α_0, and in which, therefore, (1.5) have the form (1.2). So we have:

THEOREM I. *A necessary and sufficient condition that a given A_{n+1} may be taken to represent a P_n, is that a vector should exist which satisfies* (1.5).

It will be seen, in § 6, that the rays all pass through a fixed point in A_{n+1}, which may have to be introduced as an ideal element. The vector which defines this congruence of curves will, therefore, be called a radius vector. We can now define an R_n as a K_{n+1} in which δ^α_0 are the components of a radius vector. Notice that the form of equations (1.3) is completely characterised by the relations

$$\delta^\alpha_0 = \frac{\partial \bar{x}^\alpha}{\partial x^0} = \delta^\lambda_0 \frac{\partial \bar{x}^\alpha}{\partial x^\lambda}.$$

Two K_{n+1}'s, therefore, in which the components of the same radius vector are δ^α_0, are related by a transformation of the form (1.3).

Let $\Pi^\alpha_{\beta\gamma}$ be the components of connection in any R_n, $x + x^0$. Then, since $\Pi^\alpha_{\beta\gamma,0} = 0$, the continuous group of transformations given by

$$(1.6) \qquad \begin{cases} \bar{x}^0 = x^0 + t, \\ \bar{x}^i = x^i, \end{cases}$$

will transform Π into itself. This is the group generated by the vector δ^α_0. A radius vector, therefore, generates a continuous group of affine collineations.

This property, as Professor M. S. Knebelman pointed out to me, can easily be deduced from the invariant conditions (1.5). A necessary and sufficient condition that a vector ξ generates a continuous group of affine collineations is that[20]

$$(1.7) \qquad \xi^\alpha_{;\beta;\gamma} + R^\alpha_{\beta\gamma\lambda}\,\xi^\lambda = 0.$$

If ξ satisfies (1.5a) we have $\xi^\alpha_{;\beta;\gamma} = 0$, and so the conditions (1.5a) and (1.7) are together equivalent to (1.5). It may happen that more than one radius vector exist, in which case, as we shall see later, there will be representations which are not related by equations of the form (1.3).

Let ξ^α be the components, in any K_{n+1}, of a radius vector and let

$$\frac{dx^\alpha}{d\lambda} = \xi^\alpha.$$

Then we have

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Pi^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} - \frac{dx^\alpha}{d\lambda} = \xi^\alpha_{;\beta}\,\xi^\beta - \xi^\alpha.$$

From these equations and from (1.5a), it follows that the rays are paths.

We shall in the next section be concerned with hypersurfaces defined in any R_n, $x + x^0$, by the equations

$$(1.8) \qquad x^0 = \text{const.}$$

Since the radius vector defines a group of affine collineations, the affine properties of all the hypersurfaces defined by (1.8) will be the same. Two factors which differ by a constant, or two representations composed of two such factors and the same coördinate system, x^i, will be called equivalent. The components of connection will be unaltered by a change from any representation to an equivalent representation.

We can now explain the nature of a representation, $x + x^0$, in descriptive terms. The factor, x^0, picks out a family of hypersurfaces given by (1.8), of which, as we have just shown, only one need be considered. We may,

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[20] L. P. Eisenhart, Non-Riemannian geometry, p. 126. This book will be referred to as N. R. G. Eisenhart uses B^i_{jkl} where we use R^i_{jkl}.

therefore, suppose the constant in (1.8) to be zero. The coördinates x^i will determine a point in P_n and, at the same time, a point on $x^0 = 0$ and so, as in the case where Π is flat (P. F. §§ 4 and 5), P_n is represented on a hypersurface in A_{n+1}. The components of Π will define two invariants on $x^0 = 0$, an affine connection and a tensor, whose components are Π^i_{jk} and Π^0_{jk} respectively (P. F. § 1). Then a change of representation which leaves the factor unaltered is equivalent to a change of coördinates on $x^0 = 0$, while a different factor selects a new hypersurface for the representation of P_n. Any hypersurface which is not generated by rays may be used to represent P_n. For if, in an R_n,

$$F(x^0, \cdots, x^n) = 0$$

is the equation to a hypersurface, $\dfrac{\partial F}{\partial x^0}$ will vanish identically if and only if F is generated by rays. If this is not the case we may solve this equation to obtain

$$x^0 + \varphi(x^1, \cdots, x^n) = 0,$$

and the change of factor defined by

$$\bar{x}^0 = x^0 + \varphi(x^1, \cdots, x^n)$$

will give the required representation.

The equations to the paths in A_{n+1} are given by

$$\frac{d^2 x^\alpha}{dt^2} + \Pi^\alpha_{\beta\gamma} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} = 0, \tag{1.9}$$

or

$$\begin{cases} \text{(a)} \quad \dfrac{d^2 x^i}{dt^2} + \Pi^i_{jk} \dfrac{dx^j}{dt} \dfrac{dx^k}{dt} + 2 \dfrac{dx^0}{dt} \dfrac{dx^i}{dt} = 0, \\ \text{(b)} \quad \dfrac{d^2 x^0}{dt^2} + \Pi^0_{jk} \dfrac{dx^j}{dt} \dfrac{dx^k}{dt} + \left(\dfrac{dx^0}{dt}\right)^2 = 0. \end{cases} \tag{1.10}$$

From (1.10a) we see that the projection of any path, by means of the rays, on the hypersurface $x^0 = 0$ is one of the paths defined on the latter by Π^i_{jk}. In particular let C be any path in A_{n+1} which is not a ray, and let S_n be a hypersurface which contains C and is not generated by rays. Then, as we have remarked, we can arrange that, in some R_n, $x^0 = 0$ shall be the equation to S_n. Let Σ_2 be the surface generated by the rays which intersect C, let P and Q be any two points[21] on Σ_2, not on

[21] In this argument it will be assumed that we are dealing with a region in which one and only one path passes through any two points.

the same ray, and PQ the path joining them. We shall show that PQ is contained in Σ_2.

Let the rays through P and Q meet C in P_0 and Q_0 respectively. Then P_0Q_0, the projection of PQ onto S_n, is the path of S_n which joins P_0 and Q_0. But from (1.10), or from the general theory of sub-spaces, it follows that C is a path of S_n, and, since it joins P_0 and Q_0, coincides with P_0Q_0. The projection of PQ onto S_n, and therefore PQ itself, are contained in Σ_2. If P and Q were on the same ray, PQ would be this ray and would lie in Σ_2. We have, therefore:

THEOREM II. *The surface generated by the rays which intersect any path of A_{n+1} is a plane*[22].

It follows at once that if a plane p-space exists which does not contain a ray, then the rays which meet it generate a plane $(p+1)$-space. Plane p-spaces in P_n are, therefore, represented by plane $(p+1)$-spaces in A_{n+1}, and, in particular, paths by planes.

2. **Non-homogeneous normal coördinates.** Let q^α be any point of A_{n+1} and let S_n be the hypersurface generated by the paths which touch the hypersurface $x^0-q^0=0$ at q^α. If $x+x^0$ is any R_n the equations to S_n may, as we explained in § 1, be written

$$x^0-q^0+\varphi^0(x^1,\cdots,x^n)=0,$$

since S_n is not generated by rays. Apply the change of representation defined by

$$(2.1)\qquad \begin{cases} z^0 = x^0-q^0+\varphi^0(x^1,\cdots,x^n) \\ \bar{x}^i = x^i. \end{cases}$$

Then $x+z^0$ will be an R_n. Let $0,\ dx^1\cdots dx^n$, be the components of the tangent vector at any point of a path through q^α, and lying in the hypersurface $z^0=0$. Then, if $\bar{\Pi}^\alpha_{\beta\gamma}$ are the components of connection in $x+z^0$, we shall have, replacing $\Pi^\alpha_{\beta\gamma}$ by $\bar{\Pi}^\alpha_{\beta\gamma}$ in (1.10), and putting $dz^0=0$,

$$(2.2)\qquad \bar{\Pi}^0_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0.$$

The factor z^0 is determined by the factor x^0, and the ray q^i. We shall call it the *normal factor* corresponding to x^0 and this ray. From (2.1) we see that the normal factors for two equivalent factors, and the same

[22] A sub-space will be described as plane if any path which meets it twice is contained in it. In particular, a plane surface will be called a plane. See E. Cartan—Leçons sur la géométrie des espaces de Riemann (1928) ch. V.

ray, are themselves equivalent. If we transform from one R_n to another with the same factor, the components $\overline{\Pi}^0_{jk}$ will obey the transformation law for affine tensors. The equations (2.2) hold, therefore, in all representations with the factor z^0.

Now treat $\overline{\Pi}^i_{jk}$ as an affine connection for the hypersurface $z^0=0$, and let $(z^1\cdots z^n)$ be affine normal coördinates for $\overline{\Pi}^i_{jk}$ and the point q^i. Then, we shall have an R_n, $z+z^0$, given by

$$(2.3)\qquad \begin{cases} \text{(a)}\quad z^0 = x^0-q^0+\varphi^0(x^1,\cdots,x^n), \\ \text{(b)}\quad z^i = \varphi^i(x^1,\cdots,x^n). \end{cases}$$

Since z^i are normal coördinates for q^α, the paths through q^α which lie in $z^0=0$ will satisfy

$$\begin{cases} z^0 = 0, \\ z^i = p^i\lambda, \end{cases}$$

where p^i are constants. Hence on these paths we shall have $dz^0=0$, $dz^i=\frac{d\lambda}{\lambda}z^i$ and so, from (2.2)

$$P^0_{jk}z^jz^k=0,$$

where $P^\alpha_{\beta\gamma}$ are the components of connection $z+z^0$. We also have

$$(2.4)\qquad P^i_{jk}z^jz^k=0,$$

since z^i are normal coördinates for $\overline{\Pi}^i_{jk}$. Hence

$$(2.5)\qquad P^\alpha_{jk}z^jz^k=0,$$

and $P^\alpha_{jk}=0$ at the origin of normal coordinates. We shall call $z+z^0$ and $z^1\cdots z^n$, respectively, the *normal representation and non-homogeneous projective normal coordinates* for $x+x^0$ and the point q^α.

Since the transformation given by

$$\begin{cases} \bar z^0 = z^0+a, \\ \bar z^i = z^i \end{cases}$$

for any value of a, is an affine collineation, the equations

$$\begin{cases} z^0-a = 0, \\ \quad z^i = p^i\lambda \end{cases}$$

will define a path through the point $\bar q^\alpha$, whose coördinates in $x+x^0$ are $(q^0+a,\ q^1,\cdots,q^n)$. Hence the normal representations for q^α and $\bar q^\alpha$ are equivalent.

According to our definition, the factor, z^0, in a normal representation, has the property that the hypersurface $z^0 = 0$ is generated by the paths which touch a given hypersurface at a given point. The coördinates z^i are affine normal coördinates on this hypersurface and for this point. From these properties and the affine theory we deduced the relations (2.5). From the affine theory we know that (2.4), are sufficient to ensure that z^i shall be affine normal coördinates for P^i_{jk}. The properties given above may, therefore, be deduced from the relations (2.5) and we shall give as a definition: *A normal representation is one in which* (2.5) *are satisfied.*

3. The projective parameter. Let

$$\left\{\begin{aligned} &\frac{d^2 z^i}{dt^2} + P^i_{jk}\frac{dz^j}{dt}\frac{dz^k}{dt} + 2\frac{dz^0}{dt}\frac{dz^i}{dt} = 0 \\ &\frac{d^2 z^0}{dt^2} + P^0_{jk}\frac{dz^j}{dt}\frac{dz^k}{dt} + \left(\frac{dz^0}{dt}\right)^2 = 0 \end{aligned}\right. \tag{3.1}$$

be the equations to the paths in the normal representation for $x + x^0$ and q^α. Then for the paths through q^α which lie in $z^0 = 0$ we shall have

$$P^\alpha_{jk}\frac{dz^j}{dt}\frac{dz^k}{dt} = 0.$$

This will be true of all paths through q^α, since the projection of any one of these by the rays on the hyper-surface $z^0 = 0$ will be a path lying in the latter. For the paths through q^α, therefore, (3.1) reduce to

$$\left\{\begin{aligned} &\frac{d^2 z^i}{dt^2} + 2\frac{dz^0}{dt}\frac{dz^i}{dt} = 0, \\ &\frac{d^2 z^0}{dt^2} + \left(\frac{dz^0}{dt}\right)^2 = 0, \end{aligned}\right. \tag{3.2}$$

which give us

$$\left\{\begin{aligned} e^{z^0} z^i &= p^i t + q^i, \\ e^{z^0} &= p^0 t + q^0. \end{aligned}\right.$$

The solution such that $z^\alpha = 0$ when $t = 0$ is

$$\left\{\begin{aligned} e^{z^0} z^i &= p^i t, \\ e^{z^0} &= p^0 t + 1, \end{aligned}\right. \tag{3.3}$$

whence

$$z^i = \frac{p^i t}{p^0 t + 1}, \tag{3.4}$$

or

(3.5) $$z^i = p^i\sigma,$$
where

(3.6) $$\sigma = \frac{t}{p^0 t + 1}.$$

Since z^i are affine normal coördinates on the hypersurfaces $z^0 = \text{const.}$ they will undergo an affine transformation[23] when we change the coördinates x^i in the R_n, $x + x^0$, without changing the factor. The parameter σ defined by (3.6) is, therefore, the same for all representations which have the same factor.

We shall see in §§ 10 and 11 that the results so far obtained are immediately applicable to the theory of T. Y. Thomas. In this application $z^1 \cdots z^n$ will be the projective normal coördinates obtained by O. Veblen and J. M. Thomas, to which we have referred in the introduction, and σ will be the parameter obtained by T. Y. Thomas[24] and called by him the projective parameter. Whether Π is normalized (see § 10) or not, it would, therefore, be consistent to call σ the projective parameter for the factor x^0; but, for the following reasons, we venture to suggest a revision of the terminology.

In flat affine geometry an affine parameter is defined as a parameter for which the differential equations to the straight lines in cartesian coördinates, have the form

(3.7) $$\frac{d^2 y^i}{d\sigma^2} = 0.$$

Similarly in flat projective geometry a projective parameter is most naturally defined as a parameter for which the equations to the straight lines, in non-homogeneous projective coördinates, have the form

(3.8) $$2\,\frac{dz^i}{dt}\,\frac{d^3 z^i}{dt^3} - 3\left(\frac{d^2 z^i}{dt^2}\right)^2 = 0, \quad \text{not summed.}$$

The equations (3.7) and (3.8), respectively, are obtained by eliminating the constants from the equations

(3.9) $$\begin{cases} \text{(a)} \quad y^i = a^i\sigma + b^i, \\ \text{(b)} \quad z^i = \dfrac{p^i t + q^i}{p^0 t + q^0}. \end{cases}$$

Any parameters obtained from σ and t, by the equations

[23] For a discussion of the transformation laws see § 4.

[24] T. Y. Thomas on the projective and equiprojective geometry of paths. Loc. cit. See also L. P. Eisenhart, N. R. G. p. 106.

$$(3.10)\qquad \begin{cases} \text{(a)}\quad \bar{\sigma} = \alpha\sigma + \beta, \\ \text{(b)}\quad \bar{t} = \dfrac{\alpha t + \beta}{\gamma t + \delta}, \end{cases}$$

where α, β, γ and δ are constants, are also affine and projective parameters respectively[25].

If we use normal coördinates and confine our attention to the paths through the origin this definition may be extended to the generalized geometries. For affine parameters this definition is equivalent to that given in N. R. G., p. 57. In the generalized projective geometries we define a projective parameter as follows: *If, in a projective space, the equations to a path, in any projective normal coördinate system for a point on this path have the form* (3.9b), *then t is a projective parameter on this path.*

The parameter t which occurs in (3.4) is, according to this definition, a projective parameter. In the next section (4.6) we shall see that the form of equations (3.4) is unaltered by a change of representation.

The parameter σ which appears in (3.5) is an affine parameter defined on $z^0 = 0$ by the affine connection P^i_{jk}. This hypersurface is uniquely determined by the factor x^0 and the point q^i. We shall, therefore, call σ an *affine parameter for the factor x^0 and the point q^i.*

Let (1.9) be the equations, in any R_n, to the paths of A_{n+1}. Then it $\bar{t}$ is any parameter given by

$$\bar{t} = \bar{t}(t),$$

we shall have

$$\frac{d^2 x^\alpha}{d\bar{t}^2} + \bar{\Pi}^\alpha_{\beta\gamma}\frac{dx^\beta}{d\bar{t}}\frac{dx^\gamma}{d\bar{t}} = 0,$$

where[26]

$$(3.11)\qquad \bar{\Pi}^\alpha_{\beta\gamma} = \Pi^\alpha_{\beta\gamma} + \delta^\alpha_\beta\,\psi_\gamma + \delta^\alpha_\gamma\,\psi_\beta,$$

and

$$(3.12)\qquad 2\,\psi_\alpha\frac{dx^\alpha}{dt} = \frac{d^2\bar{t}}{dt^2}\Big/\frac{d\bar{t}}{dt}.$$

If we require that $\bar{\Pi}^\alpha_{\beta 0} = \delta^\alpha_\beta$ we must have, from (3.11)

$$\delta^\alpha_\beta = \delta^\alpha_\beta + \delta^\alpha_\beta\,\psi_0 + \delta^\alpha_0\,\psi_\beta.$$

From this it follows that $\psi_\alpha = 0$ and so, from (3.12),

$$\bar{t} = \alpha t + \beta.$$

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[25] J. Douglas, The general geometry of paths, Annals of Mathematics (1928), vol. 29, pp. 143–168.

[26] J. Douglas, Loc. cit.

At first sight, therefore, it would appear that t lacks the property of being indeterminate up to linear fractional transformations. This confusion arises, however, from the fact that in the argument we have just given we are treating Π as an affine connection for A_{n+1}, and as we see from (1.9), t is an affine parameter for the paths in A_{n+1}. The additional arbitrariness contained in the transformations of the form (3.10b) is provided by the fact that, since the equations (3.4) give the same path in P_n for all values of p^0, a single path of P_n is represented by an infinity of paths in A_{n+1}. From the point of view of P_n, therefore, the projective parameter, t, is indeterminate up to transformations of the form (3.10b), but t and $\bar{t}$ will, in general, be affine parameters on two different paths in A_{n+1}, both of which represent the same path in P_n. Without appealing to the $(n+1)$ dimensional representation this amounts to saying that the same path is given by an infinity of solutions to (1.9), any two of which, in suitable normal coördinates, are related by the equations

$$\frac{p^i t}{p^0 t+1} = \frac{p^i \bar{t}}{\bar{p}^0 \bar{t}+1},$$

whence

$$\bar{t} = \frac{t}{(p^0-\bar{p}^0)\,t+1}.$$

4. **Homogeneous normal coördinates and the laws of transformation.** By analogy with the flat case we are led to consider the coördinate system, K_{n+1}, obtained from $z+z^0$ by the equations

$$\begin{cases} Z^i = e^{z^0} z^i, \\ Z^0 = e^{z^0}, \end{cases} \tag{4.1}$$

where $z+z^0$ is the normal representation for $x+x^0$ and some point q^α. We can take Z as a homogeneous coördinate system for P_n, since there is a one to one correspondence between the sets of coördinates $z^1, \cdots, z^n$ and sets of ratios $Z^0 : \cdots : Z^n$. From (2.3) we see that

$$Z^\alpha = e^{x^0-q^0}\,\theta^\alpha\,(x^1, \cdots, x^n), \tag{4.2}$$

where $\theta^0 = e^{\varphi^0}$, $\theta^i = \theta^0 \varphi^i$. If $'Z^\alpha$ are the corresponding coördinates for the point whose coördinates (K_{n+1}) in $x+x^0$ are $('q^0, q^1, \cdots, q^n)$ we shall have

$$'Z^\alpha = e^{x^0-'q^0}\,\theta^\alpha\,(x^1, \cdots, x^n).$$

Hence from (4.2) we have

$$e^{'q^0}\,'Z^\alpha = e^{q^0} Z^\alpha. \tag{4.3}$$

If we are thinking of Z^α as homogeneous coördinates we need not distinguish between $'Z^\alpha$ and Z^α, and shall, therefore, define Z as the *homogeneous projective normal coördinate system* for $x+x^0$ and the ray, or in terms of P_n, the point, q^i. From (3.3) and (4.1) we see that the equations in Z, treated as a K_{n+1}, to the paths of A_{n+1} which pass through any point q^α on q^i, are

$$Z^\alpha - \delta_0^\alpha = p^\alpha t.$$

It follows, therefore, that

$$Y^\alpha = Z^\alpha - \delta_0^\alpha \tag{4.4}$$

are affine normal coördinates (K_{n+1}) for $x+x^0$ and q^α. We have, therefore:

THEOREM III. *The normal representation, $z+z^0$, and the affine normal coördinates Y^α, for any R_n and any point, are related by the equations* (4.1) *and* (4.4).

If $'Y^\alpha$ are the affine normal coördinates for $x+x^0$ and the point whose coördinates are $('q^0, q^1, \cdots, q^n)$, it follows from (4.3) that

$$e^{'q^0}('Y^\alpha + \delta_0^\alpha) = e^{q^0}(Y^\alpha + \delta_0^\alpha). \tag{4.5}$$

Let $\bar{Z}^\alpha$ and $\bar{Y}^\alpha$ be homogeneous projective normal coördinates and affine normal coördinates (K_{n+1}), respectively, for q^α and the representation $\bar{x}+\bar{x}^0$, where $\bar{x}+\bar{x}^0$ and $x+x^0$ are related for each other by equations of the form (1.3). Then we shall have

$$\bar{Y}^\alpha = a_\beta^\alpha Y^\beta.$$

where $a_\beta^\alpha = \left(\dfrac{\partial \bar{x}^\alpha}{\partial x^\beta}\right)_{x=q}$. Since $\dfrac{\partial \bar{x}^\alpha}{\partial x^0} = \delta_0^\alpha$ we shall also have

$$\bar{Z}^\alpha = a_\beta^\alpha Z^\beta, \tag{4.6}$$

and this will be true at all points of the ray q^i. Let $\bar{z}+\bar{z}^0$ be the normal representation for q^α and $\bar{x}+\bar{x}^0$. Then, from (4.1) we have

$$\bar{z}^i = \frac{\bar{Z}^i}{\bar{Z}^0} = \frac{a_\beta^i Z^\beta}{a_\gamma^0 Z^\gamma},$$

or

$$\bar{z}^i = \frac{a_j^i z^j}{a_k^0 z^k + 1}. \tag{4.7}$$

From (4.7) it follows that the form of the equations (3.4) is unaltered by all changes of representation. If $\bar{\sigma}$ is the affine parameter for $\bar{x}^0$, and the same point q^i, we shall have

$$\bar{z}^i = \bar{p}^i \bar{\sigma},$$

where $\bar{p}^i = a^i_j p^j$, and from (4.7) it follows that

$$\bar{\sigma} = \frac{\sigma}{\gamma\sigma + 1},$$

where $\gamma = a^0_i p^i$. We also have from (4.1)

$$\bar{z}^0 = \log \bar{Z}^0 = \log (Z^0 + a^0_i Z^i) = z^0 + \log (a^0_i z^i + 1),$$

since $Z^0 = e^{z^0}$.

Combining these results we have

$$\text{(4.8)} \qquad \begin{cases} \bar{Z}^\alpha = a^\alpha_\beta Z^\beta, \\ \bar{z}^i = \dfrac{a^i_j z^j}{a^0_k z^k + 1}, \\ \bar{z}^0 = z^0 + \log (a^0_i z^i + 1), \\ \bar{\sigma} = \dfrac{\sigma}{\gamma\sigma + 1}. \end{cases}$$

5. **Power-series expansions.** Let $x + x^0$ be any R_n and $z + z^0$ the normal representation for $x + x^0$ and any point q. We shall give the expressions for x^α as functions of z^α.

Let Y^α be affine normal coördinates (K_{n+1}) for $x + x^0$ and q. Then we shall have

$$\text{(5.1)} \qquad x^\alpha - q^\alpha = Y^\alpha - \sum_{p=2}^{\infty} \frac{1}{p!} (\Pi^\alpha_{\lambda_1 \cdots \lambda_p})_{x=q} Y^{\lambda_1} \cdots Y^{\lambda_p},$$

where[27]

$$\Pi^\alpha_{\lambda_1 \cdots \lambda_p} = \Pi^\alpha_{(\lambda_1 \cdots \lambda_{p-1}, \lambda_p)} - (p-1) \Pi^\alpha_{\sigma(\lambda_1 \cdots \lambda_{p-2}} \Pi^\sigma_{\lambda_{p-1} \lambda_p)}.$$

From (2.3) we obtain equations of the form

$$\text{(5.2)} \qquad \begin{matrix} \text{(a)} \\ \text{(b)} \end{matrix} \begin{cases} x^i - q^i = B^i (z^1, \cdots, z^n), \\ x^0 - q^0 = z^0 + B^0 (z^1, \cdots, z^n), \end{cases}$$

and from (4.1) and (4.4) it follows that

$$\text{(5.3)} \qquad \begin{cases} z^i = \dfrac{Y^i}{Y^0 + 1}, \\ z^0 = \log (Y^0 + 1). \end{cases}$$

[27] N. R. G., p. 58. We follow J. A. Schouten in writing $p!\, A_{(\lambda_1 \cdots \lambda_p)}$ to stand for the sum of all quantities obtained by permuting the subscripts in all possible ways.

Putting $Y^0 = 0$ in (5.1) we have, therefore,

$$(5.4) \qquad (x^\alpha - q^\alpha)_{Y^0=0} = \delta_i^\alpha Y^i - \sum_{p=2}^{\infty} \frac{1}{p!} (\Pi^\alpha_{k_1 \ldots k_p})_{x=q} Y^{k_1} \cdots Y^{k_p}.$$

From (5.3) we have $(z^\alpha)_{Y^0=0} = \delta_i^\alpha Y^i$, and so from (5.2) it follows that

$$(5.5) \qquad (x^\alpha - q^\alpha)_{Y^0=0} = B^\alpha (Y^1, \cdots, Y^n).$$

From (5.4) and (5.5) it follows, therefore, that

$$B^\alpha (u^1, \cdots, u^n) = \delta_i^\alpha u^i - \sum_{p=2}^{\infty} \frac{1}{p!} (\Pi^\alpha_{k_1 \ldots k_p})_{x=q} u^{k_1} \cdots u^{k_p},$$

where u^i are any n variables, and so, from (5.2), we have the relations

$$(5.6) \qquad x^\alpha - q^\alpha = z^\alpha - \sum_{p=2}^{\infty} \frac{1}{p!} (\Pi^\alpha_{k_1 \ldots k_p})_{x=q} z^{k_1} \cdots z^{k_p}.$$

The coefficients $\Pi^\alpha_{k_1 \ldots k_p}$ can be derived, by a recurrence formula, from the components Π^α_{jk} alone, that is without using the $(n+1)$-dimensional representation. We have

$$\Pi^\alpha_{\beta\gamma 0} = \Pi^\alpha_{(\beta\gamma, 0)} - 2\, \Pi^\alpha_{\lambda(\beta}\, \Pi^\lambda_{\gamma 0)} = -2\, \Pi^\alpha_{\beta\gamma}.$$

Assume $\Pi^\alpha_{\lambda_1 \ldots \lambda_{p-1} 0} = -(p-1)\, \Pi^\alpha_{\lambda_1 \ldots \lambda_{p-1}}$ where p is any positive integer greater than two. Then

$$\begin{aligned} \Pi^\alpha_{\lambda_1 \ldots \lambda_p 0} &= \Pi^\alpha_{(\lambda_1 \ldots \lambda_p, 0)} - p\, \Pi^\alpha_{\sigma(\lambda_1 \ldots \lambda_{p-1}}\, \Pi^\sigma_{\lambda_p 0)} \\ &= \frac{p}{p+1} \{\Pi^\alpha_{0(\lambda_1 \ldots \lambda_{p-1}, \lambda_p)} - (p-1)\, \Pi^\alpha_{0\sigma(\lambda_1 \ldots \lambda_{p-2}}\, \Pi^\sigma_{\lambda_{p-1}\lambda_p)} - 2\, \Pi^\alpha_{\lambda_1 \ldots \lambda_p}\} \\ &= -\left\{\frac{p(p-1)}{p+1} + \frac{2p}{p+1}\right\} \Pi^\alpha_{\lambda_1 \ldots \lambda_p}, \end{aligned}$$

using the assumption and the formula for $\Pi^\alpha_{\lambda_1 \ldots \lambda_p}$. So we have

$$(5.7) \qquad \Pi^\alpha_{\lambda_1 \ldots \lambda_p 0} = -p\, \Pi^\alpha_{\lambda_1 \ldots \lambda_p}.$$

Since the assumption is verified for $p = 2$ the relations (5.7) hold for all positive integral values of $p > 2$. We have therefore

$$(5.8) \qquad \begin{aligned} \Pi^\alpha_{k_1 \ldots k_p} &= \Pi^\alpha_{(k_1 \ldots k_{p-1}, k_p)} \\ &- (p-1) \{\Pi^\alpha_{a(k_1 \ldots k_{p-2}}\, \Pi^a_{k_{p-1} k_p)} - (p-2)\, \Pi^\alpha_{(k_1 \ldots k_{p-2}}\, \Pi^0_{k_{p-1} k_p)}\}. \end{aligned}$$

From (5.7) we have

$$(5.9) \qquad \Pi^\alpha_{0 \ldots 0 k_s \ldots k_p} = (-1)^{s-1} \frac{(p-1)!}{(p-s)!}\, \Pi^\alpha_{k_s \ldots k_p},$$

where the subscripts on the left hand side contain $(s-1)$ zeros.

We shall indicate a method of deriving (5.6) from the relations (5.9) by a straight-forward calculation. If we express (5.1) in the form

$$x^\alpha - q^\alpha = \sum_{p=0}^{\infty} Y_p^\alpha \theta^p(Y^0),$$

where $Y_0^\alpha = \delta_0^\alpha$, $Y_1^\alpha = \delta_i^\alpha Y^i$ and, for $m > 1$, $Y_m^\alpha = \frac{1}{m!}(\Pi_{k_1 \cdots k_m}^\alpha)_q Y^{k_1} \cdots Y^{k_m}$, it will be found that

$$\left\{\begin{aligned} \theta^0 &= \log(Y^0 + 1), \\ \theta^p &= (Y^0 + 1)^{-p} \quad \text{for} \quad p > 0, \end{aligned}\right. \tag{5.10}$$

from which, and the relation $z^i = \frac{Y^i}{Y^0 + 1}$, we have the equations (5.6).

6. **Properties of the normal coördinate system.** Let $x + x^0$ be any R_n. The components of the radius vector in the affine normal coördinate system (K_{n+1}), Y, for $x + x^0$ and any point q, will be given by

$$\delta_0^\beta \frac{\partial Y^\alpha}{\partial x^\beta} = Z^\alpha \tag{6.1}$$

from (4.2) and (4.4). The equations to the rays will, therefore, be

$$\frac{d Y^\alpha}{d \lambda} = Z^\alpha, \tag{6.2}$$

which give

$$Y^\alpha + \delta_0^\alpha = p^\alpha e^\lambda, \tag{6.3}$$

where p^α are constant on each ray. If we put $p^0 = 1$, $p^i = z^i$ and $\lambda = z^0$ the equations (6.3) may be identified with (4.1), since $Z^\alpha = Y^\alpha + \delta_0^\alpha$. Let a point, which will be called *the centre*, be defined in A_{n+1} by means of the equations

$$Y^\alpha = -\delta_0^\alpha.$$

The centre is not represented in any R_n, and so, unless we have further knowledge about A_{n+1} than is provided by its representation in an R_n, must be introduced as an ideal element. From (6.3) we see that the centre is contained, possibly as an ideal point, in each ray and we have:

THEOREM IV. *The rays all meet in a single point, which may be an ideal element in A_{n+1}.*

Let $C_{\beta\gamma}^\alpha$ be the components in Y of the connection Π. From (6.1) and the invariant conditions (1.5a) we have

$$\frac{\partial Z^\alpha}{\partial Y^\beta} + Z^\sigma C_{\sigma\beta}^\alpha = \delta_\beta^\alpha,$$

which reduce to

(6.4) $$C^{\alpha}_{\sigma\beta} Z^{\sigma} = 0.$$

Since Y are normal coördinates we have the following conditions:

(6.5) $$\begin{cases} \text{(a)} & C^{\alpha}_{\lambda\mu} Y^{\lambda} Y^{\mu} = 0, \\ \text{(b)} & C^{\alpha}_{\sigma\beta} Z^{\sigma} = 0, \end{cases}$$

which are necessary in order that $C^{\alpha}_{\beta\gamma}$ should be the components, in an affine normal coördinate system (K_{n+1}), of an affine connection for which there exists a vector satisfying (1.5). We shall obtain sufficient conditions that (1.5) are satisfied, if to (6.5) we add (1.5b) in the following form

(6.6) $$(C^{\alpha}_{\beta\gamma,\sigma} - C^{\alpha}_{\beta\sigma,\gamma} + C^{\lambda}_{\beta\gamma} C^{\alpha}_{\lambda\sigma} - C^{\lambda}_{\beta\sigma} C^{\alpha}_{\lambda\gamma}) Z^{\sigma} = 0.$$

From (6.4) we have

(6.7) $$C^{\alpha}_{\beta\sigma,\gamma} Z^{\sigma} + C^{\alpha}_{\beta\gamma} = 0.$$

From (6.4) and (6.7) the equations (6.6) reduce to

(6.8) $$C^{\alpha}_{\beta\gamma,\sigma} Z^{\sigma} + C^{\alpha}_{\beta\gamma} = 0.$$

The extra condition is, therefore, that $C^{\alpha}_{\beta\gamma}$ be homogeneous of degree minus one in the variables Z^{α}. Let Y be the normal coördinate system for the $K_{n+1}(x^{\alpha})$, and the point q^{α}, and let ξ be the components of the radius vector in (x^{α}). Then at the origin we shall have

$$\xi^{\alpha} = \left(\frac{\partial x^{\alpha}}{\partial Y^{\beta}} Z^{\beta}\right)_{Y=0} = \left(\frac{\partial x^{\alpha}}{\partial Y^{0}}\right)_{Y=0} = \delta^{\alpha}_{0}.$$

There will, therefore, be an $R_n\, y + y^0$, say, whose parametric curves are tangent at q to those of (x^{α}). Hence Y will also be the normal coördinate system for $y + y^0$ and q, and we have

THEOREM V. *The conditions* (6.5), *together with the condition that $C^{\alpha}_{\beta\gamma}$ be homogeneous of degree minus one in Z^{α}, are necessary and sufficient that* (1.5) *shall be satisfied for $C^{\alpha}_{\beta\gamma}$, and that Z^{α} shall be homogeneous projective normal coördinates for some R_n.*

Let $A^{\alpha}_{\beta\gamma\lambda_1\cdots\lambda_p}$ for $p = 1, 2, \cdots$ be the components of the normal tensors at the origin of the coördinate system Y. From (6.4) and (6.8) we have

$$(Y^{\sigma} + \delta^{\sigma}_{0}) \sum_{p=1}^{\infty} \frac{1}{p!} A^{\alpha}_{\sigma\beta\lambda_1\cdots\lambda_p} Y^{\lambda_1} \cdots Y^{\lambda_p} = 0,$$

$$\sum_{p=1}^{\infty} \frac{1}{p!} \{(Y^{\sigma} + \delta^{\sigma}_{0}) A^{\alpha}_{\beta\gamma\sigma\lambda_1\cdots\lambda_p} + A^{\alpha}_{\beta\gamma\lambda_1\cdots\lambda_p}\} Y^{\lambda_1} \cdots Y^{\lambda_p} = 0.$$

Equating the coefficients in these identities to zero we have

$$(6.9)\qquad \begin{cases} \text{(a)}\quad A^{\alpha}_{0\beta\lambda_1\cdots\lambda_p} + p\, A^{\alpha}_{\beta(\lambda_1\cdots\lambda_p)} = 0, \\ \text{(b)}\quad A^{\alpha}_{\beta\gamma 0\lambda_1\cdots\lambda_p} + (p+1)\, A^{\alpha}_{\beta\gamma\lambda_1\cdots\lambda_p} = 0. \end{cases}$$

We shall need the following relations which follow from (6.5). Multiply (6.5b) by Y^β and, summing over $0, \cdots, n$, we have, from (6.5a),

$$C^{\alpha}_{\beta 0}\, Y^{\beta} = 0,$$

and so, from (6.5b), when $\beta = 0$, we have

$$C^{\alpha}_{00} = 0,$$

and hence

$$C^{\alpha}_{bc}\, Y^{b}\, Y^{c} = 0.$$

Collecting these relations and (6.5) for reference, we have

$$(6.10)\qquad \begin{cases} \text{(a)}\quad C^{\alpha}_{\lambda\mu}\, Y^{\lambda}\, Y^{\mu} = 0, \\ \text{(b)}\quad C^{\alpha}_{\sigma\beta}\, Z^{\sigma} = 0, \\ \text{(c)}\quad C^{\alpha}_{bc}\, Y^{b}\, Y^{c} = 0, \\ \text{(d)}\quad C^{\alpha}_{\beta 0}\, Y^{\beta} = 0, \\ \text{(e)}\quad C^{\alpha}_{00} = 0. \end{cases}$$

Finally, let $P^{\alpha}_{\beta\gamma}$ be the components of connection in the normal representation $z + z^0$, related to Y by the equations

$$\begin{cases} Y^{i} = e^{z^0} z^{i}, \\ Y^{0} + 1 = Z^{0} = e^{z^0}. \end{cases}$$

From (6.10) it quickly follows that

$$(6.11)\qquad \begin{cases} \text{(a)}\quad P^{i}_{jk} = Z^{0}\,(C^{i}_{jk} - z^{i}\, C^{0}_{jk}), \\ \text{(b)}\quad P^{0}_{jk} = Z^{0}\, C^{0}_{jk}, \end{cases}$$

and

$$(6.12)\qquad \begin{cases} \text{(a)}\quad C^{i}_{jk} = \dfrac{1}{Z^0}\,(P^{i}_{jk} + z^{i}\, P^{0}_{jk}), \\ \text{(b)}\quad C^{0}_{jk} = \dfrac{1}{Z^0}\, P^{0}_{jk}, \\ \text{(c)}\quad C^{\alpha}_{\beta 0} = -\dfrac{1}{Z^0}\, C^{\alpha}_{\beta i}\, Y^{i}, \end{cases}$$

the relation (6.12c) being derived from (6.10b). Observe that the centre is a singularity for the functions $C^{\alpha}_{\beta\gamma}$.

7. **Plane hypersurfaces.** In § 2 we saw that a normal factor corresponds to a hypersurface generated by paths through some point of A_{n+1}. If this hypersurface is plane the corresponding normal factor will be the same at each point. Let z^0 be the normal factor for a plane hypersurface. Since the radius vector defines a group of affine collineations for A_{n+1}, it follows that all hypersurfaces defined by $z^0 = \text{const.}$ will be plane. Let $\Pi^\alpha_{\beta\gamma}$ be the components of connection in any representation with the factor z^0. Under changes of representation, which leave the factor unaltered, the components Π^0_{jk} transform like those of an affine tensor. Then, since in any normal representation these components vanish at the origin, and since z^0 is the normal factor for each point of the plane hypersurfaces defined by $z^0 = \text{const.}$, we shall have

$$\Pi^0_{jk} = 0. \tag{7.1}$$

If we write $\psi = e^{z^0}$, we shall have, writing x^0 for z^0,

$$\begin{aligned}\frac{\partial^2 \psi}{\partial x^\beta\, \partial x^\gamma} &= e^{x^0} \delta^0_\beta \delta^0_\gamma \\ &= e^{x^0} \Pi^0_{\beta\gamma}\end{aligned}$$

from (7.1) and (1.1). These equations may be written in invariant form as

$$\frac{\partial^2 \psi}{\partial x^\beta\, \partial x^\gamma} - \frac{\partial \psi}{\partial x^\alpha} \Pi^\alpha_{\beta\gamma} = 0. \tag{7.2}$$

Hence[28] $\psi_{,\alpha}$ are the components of a parallel vector field in A_{n+1}. Conversely, if the equations (7.2) admit a solution ψ, we shall have, for $\gamma = 0$,

$$\frac{\partial^2 \psi}{\partial x^0\, \partial x^\beta} - \frac{\partial \psi}{\partial x^\beta} = 0,$$

from which it follows that

$$\psi = e^{x^0} \varrho,$$

where ϱ depends only on $x^1, \cdots, x^n$. If we apply the change of factor defined by

$$z^0 = x^0 + \log \varrho$$

we shall have $e^{z^0} = \psi$ and, from (7.2), $\bar{\Pi}^0_{jk} = 0$, where $\bar{\Pi}^\alpha_{\beta\gamma}$ are the components of connection in $x + z^0$. The hypersurfaces $z^0 = \text{const.}$ will

[28] This result is a consequence of the special nature of A_{n+1}.

therefore be plane. Veblen, in G. P., described (7.2) as the differential equations of projective geometry, so we have:

THEOREM VI. *In order that a function* $\psi(x^0, \cdots, x^n)$ *satisfy the differential equations of projective geometry it is necessary and sufficient that* $\psi = const.$ *define plane hypersurfaces in* A_{n+1}.

8. **The equivalence problem.** Let $\Pi^\alpha_{\beta\gamma}$ and $\bar{\Pi}^\alpha_{\beta\gamma}$ be functions of $(x^1, \cdots, x^n)$ and $(\bar{x}^1, \cdots, \bar{x}^n)$ respectively, which satisfy the conditions

$$(8.1) \qquad \begin{cases} \Pi^\alpha_{0\beta} & = \bar{\Pi}^\alpha_{0\beta} & = \delta^\alpha_\beta, \\ \Pi^\alpha_{\beta\gamma,0} & = \bar{\Pi}^\alpha_{\beta\gamma,0} & = 0, \\ \Pi^\alpha_{\beta\gamma} - \Pi^\alpha_{\gamma\beta} & = \bar{\Pi}^\alpha_{\beta\gamma} - \bar{\Pi}^\alpha_{\gamma\beta} & = 0. \end{cases}$$

Then we shall take $\Pi^\alpha_{\beta\gamma}$ and $\bar{\Pi}^\alpha_{\beta\gamma}$ as the components of two projective connections in the same representation, but possibly at different points. If there exists any analytic point transformation of the $(n+1)$ variables $(x^0, \cdots, x^n)$ which carries Π into $\bar{\Pi}$ the latter will be called affinely equivalent. If there exists a transformation defined by equations of the form (1.3), which carries Π into $\bar{\Pi}$, they will be called projectively equivalent. We shall, in this section, prove the following:

THEOREM VII. *If two projective connections are affinely equivalent then they are projectively equivalent.*

Let Π and $\bar{\Pi}$ be affinely equivalent, and first suppose that there is only one radius vector for Π. Then the same will be true of $\bar{\Pi}$, and, from (8.1), this will be the vector δ^α_0. Then, if

$$(8.2) \qquad \bar{x}^\alpha = \bar{x}^\alpha(x^0, \cdots, x^n)$$

is any transformation carrying Π into $\bar{\Pi}$, we shall have $\dfrac{\partial \bar{x}^\alpha}{\partial x^0} = \delta^\alpha_0$, and so (8.2) is of the form (1.3). In this case, therefore, the theorem is established.

To prove the theorem in its general form we shall need the following lemma. *Let* Π *be an affine connection such that there are two distinct vectors satisfying* (1.5). *Then we can transform one of these into the other and, at the same time,* Π *into itself.*

Let a continuous group of transformations be defined by the equations

$$\frac{dx^\alpha}{dt} = X^\alpha.$$

If any transformation of this group carries a contravariant vector V into $\bar{V}$ we shall have

$$\bar{V}^\alpha(\bar{x}) = V^\alpha(\bar{x}) + \sum_{p=1}^{\infty} \frac{(-t)^p}{p!} V_p^\alpha, \tag{8.3}$$

where t is the value of the parameter which determines the transformation, $x \to \bar{x}$, and V_p are given by[29]

$$\begin{cases} V_1^\alpha = V^\alpha_{,\beta} X^\beta - X^\alpha_{,\beta} V^\beta, \\ V_{p+1}^\alpha = V^\alpha_{p,\beta} X^\beta - X^\alpha_{,\beta} V_p^\beta. \end{cases} \tag{8.4}$$

Since Π is symmetric, these equations will be unaltered if we use covariant differentiation with respect to Π, and this we shall do.

Now let ξ and η be two vectors each of which satisfies (1.5). Then

$$\xi^\alpha_{;\lambda} \eta^\lambda - \eta^\alpha_{;\lambda} \xi^\lambda = \eta^\alpha - \xi^\alpha,$$

and so these vectors generate a two-parameter continuous group of transformations which, as we remarked in § 1, are affine collineations for Π. Therefore the one-parameter group defined by

$$\frac{dx^\alpha}{dt} = \eta^\alpha - \xi^\alpha \tag{8.5}$$

will carry Π into itself.

Let $x + x^0$ be an R_n in which the components of the vector ξ are δ_0^α. From (1.1a), and the fact that η satisfies (1.5a), we have

$$\frac{\partial \eta^\alpha}{\partial x^0} + \eta^\alpha = \delta_0^\alpha,$$

and it follows that

$$\eta^\alpha = \delta_0^\alpha + e^{-x^0} \varphi_1^\alpha (x^1, \cdots, x^n).$$

Hence the equations (8.5) become

$$\frac{dx^\alpha}{dt} = e^{-x^0} \varphi_1^\alpha (x^1, \cdots, x^n), \tag{8.6}$$

[29] Bianchi gives the first two terms of this expansion in "Lezioni sulla teoria dei gruppi continui", p. 194. To obtain the remaining terms we observe that V^α and so, by induction, V_p^α are the components of contra-variant vectors. Let us take a coördinate system in which the components of X are δ_0^α. Then we shall have

$$\begin{aligned} \bar{V}^\alpha(\bar{x}) &= V^\alpha(\bar{x}^0 - t, x^1, \cdots, x^n) \\ &= V^\alpha + \sum_{p=1}^{\infty} \frac{(-t)^p}{p!} V_p^\alpha, \end{aligned}$$

where $V_p^\alpha = V^\alpha_{,\lambda_1 \cdots \lambda_p} \delta_0^{\lambda_1} \cdots \delta_0^{\lambda_p}$. Since $\delta^\alpha_{0,\lambda} = 0$ it follows that V_p^α are given in this, and therefore in every coordinate system, by (8.4).

and it is easily verified that

$$\frac{d^p x^\alpha}{d t^p} = e^{-p x^0} \varphi_p^\alpha(x),$$

where $\varphi_p^\alpha(x)$ are functions of $x^1, \cdots, x^n$ only. It follows that the set of solutions, $\bar{x}^\alpha(x, t)$, to (8.6), which reduce to x^α for $t = 0$, may be expanded about $t = 0$ in the power series

$$\bar{x}^\alpha = x^\alpha + \sum_{p=1}^{\infty} \frac{(e^{-x^0} t)^p}{p!} \varphi_p^\alpha(x^1, \cdots, x^n). \tag{8.7}$$

Any coördinate system, K_{n+1}, is a correspondence between a region in the affine space, A_{n+1}, and some region, X_{n+1}, in the arithmetic space of $n+1$ dimensions. An R_n has the property that if the region X_{n+1} contains the arithmetic point $(a^0, a^1, \cdots, a^n)$ it will also contain the points $(x^0, a^1, \cdots, a^n)$ for all values of x^0. Let $(a^0, \cdots, a^n)$ be in a point in X_{n+1} at which $\varphi_1^\alpha(a)$ are not all zero, and for which the series (8.7) converge for some value of t other than zero. Let t_1 be any number and δ_1 any positive number. Then it follows from the special form of the series (8.7), and from the properties of an R_n, that there is a region X_{n+1}^1 contained in X_{n+1}, which is given by inequalities of the form

$$|x^i - a^i| < \delta,$$
$$x^0 > x_1^0(t_1, \delta_1)$$

for some value of x_1^0 depending on t_1 and δ_1, such that:

(1) The series (8.7) converge for $t = t_1$ and for all values of x in X_{n+1}^1;

(2) If x^α is any point in X_{n+1}^1, and if $\bar{x}^\alpha$ is given by (8.7) with $t = t_1$, then

$$|\bar{x}^\alpha - x^\alpha| < \delta_1.$$

We can choose δ and δ_1 so small that the point $\bar{x}$, corresponding to each point x in X_{n+1}^1, will be contained in X_{n+1}. It follows that for each value of t_1 the equations (8.7) define a point transformation in A_{n+1}. Such a point transformation necessarily carries Π into itself.

In (8.3) and (8.4) let us take $\eta - \xi$ and ξ for X and V respectively. Then we shall have

$$\begin{cases} \xi_1^\alpha = X^\alpha, \\ \xi_p^\alpha = 0 \quad \text{if} \quad p > 1, \end{cases}$$

whence

$$\bar{\xi}^\alpha = \xi^\alpha - t(\eta^\alpha - \xi^\alpha),$$

and the transformation given by $t = -1$ will carry ξ into η and Π into itself.

We can now prove Theorem VII. Let $\Pi^\alpha_{\beta\gamma}$ and $\overline{\Pi}^\alpha_{\beta\gamma}$ be the components, in any R_n, of two affinely equivalent projective connections. Let T be any transformation of the form (8.2) which carries Π into $\overline{\Pi}$ and let ξ^α be the components of that vector which is carried into δ^α_0. From (8.1) it follows that δ^α_0 is a radius vector for $\overline{\Pi}$, and so, therefore, is ξ for Π. By our lemma, therefore, there exists a transformation, S, which carries δ^α_0 into ξ and Π into itself, and the transformation TS, where S is applied first, will carry Π into $\overline{\Pi}$ and δ^α_0 into δ^α_0. It is therefore defined by equations of the form (1.3) and so Π is projectively equivalent to $\overline{\Pi}$.

With the help of the lemma which we have proved in this section we can elucidate a point raised in § 1. Let Π be an affine connection for an A_{n+1}, such that there are two distinct vectors ξ_1 and ξ_2 satisfying the conditions (1.5). Then there will be two centres, O_1 and O_2, each with a bundle of rays radiating out from it. Let the rays with O_1, as centre be put into a one to one correspondence with the points of a P_n. Then, from our lemma, there will be a simple isomorphism between theorems about the rays through O_1 and theorems about the rays through O_2. The projective space may, therefore, be represented by either system. The extreme case occurs when P_n is flat,[30] and any point of A_{n+1} may be taken as centre.

9. **Non holonomic representation.** It will help us to introduce normalized projective connections if we make the generalization suggested by Veblen (G. P. footnote on p. 145). A projective connection will be defined as in G. P., except that we no longer require

$$\Pi^0_{jk} = \Pi^0_{kj}.$$

We shall consider the extended set of transformations, defined by equations of the form

$$(9.1) \qquad \begin{array}{ll} \text{(a)} & \\ \text{(b)} & \end{array} \left\{ \begin{array}{l} d\bar{x}^0 = dx^0 + v^0_i\, dx^i, \\ \bar{x}^i = \bar{x}^i(x^1, \cdots, x^n), \end{array} \right.$$

where $v^0_i\, dx^i$ is any Pfaffian form in the variables x^i. In case v^0 is a gradient the equations (9.1 a) will be said to define an integrable change of factor. We define u^0_i by the equation

[30] It is worth giving an example of a case where there is more than one radius vector without the connection being flat. The connection whose components satisfy (1.1), and at the same time the conditions

$$\Pi^\alpha_{1j} = \Pi^\alpha_{jk,1} = \Pi^0_{jk} = 0,$$

will admit the radius vectors δ^α_0 and $\delta^\alpha_0 + e^{-x^0}\delta^\alpha_1$, and the components $\Pi^i_{\varrho\sigma}$, $\varrho, \sigma = 2 \cdots n$, may be chosen so that the connection is not flat.

$$u_i^0 \, d\bar{x}^i + v_i^0 \, dx^i = 0.$$

If we apply the change of factor defined by (9.1 a), without changing coördinates we shall have

$$(9.2) \qquad \begin{cases} \text{(a)} \quad \bar{\Pi}^i_{jk} = \Pi^i_{jk} + \delta^i_j u^0_k + \delta^i_k u^0_j, \\ \text{(b)} \quad \bar{\Pi}^0_{jk} = \Pi^0_{jk} - v^0_{j,k} + v^0_i \, \Pi^i_{jk} - v^0_j \, v^0_k, \end{cases}$$

where $\bar{\Pi}^{\alpha}_{\beta\gamma}$ are the components of Π in $x + \bar{x}^0$. From these equations we see that the symmetry[31] of Π^i_{jk} is preserved, and that

$$(9.3) \qquad \bar{\pi}_{jk} = \pi_{jk} - \frac{1}{2}(v^0_{j,k} - v^0_{k,j}),$$

where $\pi_{jk} = \frac{1}{2}(\Pi^0_{jk} - \Pi^0_{kj})$. We require that there shall exist a factor for which $\pi_{jk} = 0$. From (9.3) it follows that this will be the case if, and only if, the equations

$$(9.4) \qquad v^0_{j,k} - v^0_{k,j} = 2\pi_{jk}$$

are integrable. The necessary and sufficient conditions that this should be the case are

$$(9.5) \qquad \pi_{jk,l} + \pi_{kl,j} + \pi_{lj,k} = 0,$$

and, in referring to a projective connection we shall always suppose this condition satisfied. A representation, or factor, in which $\pi_{jk} = 0$ will be called holonomic. From (9.3) we see that all holonomic factors are related by integrable transformations, and that any factor obtained from a holonomic factor by such a transformation is itself holonomic. Any projective connection will, therefore, determine in each holonomic representation, a set of components, which define a unique projective connection in the sense of G. P.

In order to define the normal representation for a given representation we shall show that we need only consider holonomic representations, and refer to sections 2, 3 and 4. At the end of § 1 we remarked that a factor determines a family of hypersurfaces in A_{n+1}. The equations to such a family in any holonomic R_n may be written

$$(9.6) \qquad \bar{x}^0 = x^0 + \varphi(x^1, \cdots, x^n),$$

from which it follows that these hypersurfaces are enveloped by a projective gradient of index[32] zero, whose null component is unity. Since any two

[31] For a definition of projective connections, based on that given by T. Y. Thomas, but in which this symmetry is discarded, see V. Hlavatý, Math. Zeit. (1928), vol. 28, 142–146.

[32] O. Veblen, A generalization of the quadratic differential form, loc. cit.

factors which differ by a constant are equivalent, we may, instead of (9.6), write

$$d\bar{x}^0 = \varphi_\alpha \, dx^\alpha \tag{9.7}$$

as the equation giving $\bar{x}^0$, where $\varphi_0 = 1$ and $\varphi_i = \varphi_{,i}$. If we remove the restriction that φ_α be a gradient we can derive any factor by an equation of the form (9.7).

If at any point q the vectors corresponding to two factors are coincident, we shall say that these factors are tangent at q.

We now define the normal factor for x^0 and the point q, as the normal factor at q, according to § 2, for any holonomic factor tangent to x^0 at this point. Any normal factor is, therefore, holonomic and we can apply the results of sections 2, 3 and 4.

We shall show that the problem of determining when two projective connections are equivalent is essentially the same whether we require the transformations defined by (9.1) to be integrable or not.

Let U and $\bar{U}$ be any two transformations of the form (9.1) such that, if[33]

$$H = U\Pi \quad \text{and} \quad \bar{H} = \bar{U}\bar{\Pi} \tag{9.8}$$

then the components H^0_{jk} and $\bar{H}^0_{jk}$ are symmetric. By § 8, and the affine theory we have a criterion for determining whether or not H and $\bar{H}$ are equivalent under integrable changes of representation. If they are equivalent, there will be an integrable transformation, T, of the form (9.1) such that

$$\bar{H} = TH.$$

From (9.8) we have, therefore,

$$\bar{\Pi} = (\bar{U}^{-1}TU)\Pi. \tag{9.9}$$

Conversely, if there is a transformation, S, of the form (9.1) such that

$$\bar{\Pi} = S\Pi$$

we shall have, from (9.8)

$$\bar{H} = (\bar{U}SU^{-1})H. \tag{9.10}$$

The equivalence problem for Π and $\bar{\Pi}$ under all changes of representation is, therefore, the same as that for H and $\bar{H}$ under holonomic transformations.

[33] If the components $\Pi^\alpha_{\beta\gamma}$ are transformed by a change of representation, U, into $H^\alpha_{\beta\gamma}$, we write

$$H = U\Pi.$$

10. **Normalized projective connections.** In this section we shall show how projective connections in the sense of § 9 are related to those first considered by T. Y. Thomas, to which we referred in the introduction. The latter give a method of handling the descriptive[34] geometry of paths. By this we mean theorems which are independent of the parameterization. We shall not appeal to the $(n+1)$ dimensional representation except when we refer to previous results, and all the terms which we use will refer to P_n.

We start with a system of paths which are given in any coördinate system, x, by the equations

$$\frac{d^2 x^i}{d\lambda^2} + \Pi^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0, \tag{10.1}$$

where Π^i_{jk} are analytic functions of position. By adding arbitrary sets of components Π^0_{jk} and $\Pi^\alpha_{\beta 0} = \Pi^\alpha_{0\beta} = \delta^\alpha_\beta$, we can define an infinite class of projective connections, each of which defines the same system of paths[35] as the equations (10.1).

From this class of projective connections we select one by requiring that, in a given representation, $x + x^0$,

$$\Pi^0_{jk} = \frac{1}{n-1} B_{jk}, \tag{10.2}$$

where

$$B^i_{jkl} = \Pi^i_{jk,l} - \Pi^i_{jl,k} + \Pi^a_{jk}\Pi^i_{al} - \Pi^a_{jl}\Pi^i_{ak},$$

and

$$B_{jk} = B^a_{jak}.$$

Let $R^\alpha_{\beta\gamma\delta}$ be the components, in $x + x^0$, of the projective curvature tensor for this connection. Then we have (P. F. § 1)

$$R^i_{jkl} = B^i_{jkl} + \delta^i_l \Pi^0_{jk} - \delta^i_k \Pi^0_{jl}, \tag{10.3}$$

and, from (10.2),

$$R^a_{jak} = 0.$$

Since $R^\alpha_{\beta 0\delta} = R^i_{0\gamma\delta} = 0$, from the symmetry of $\Pi^i_{\beta\gamma}$, this condition is equivalent to

$$R_{\beta\gamma} = 0. \tag{10.4}$$

where

$$R_{\beta\gamma} = R^\alpha_{\beta\alpha\gamma}.$$

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[34] J. Douglas, loc. cit. We shall not consider the geometry of paths in its full generality, but shall confine ourselves to these systems of paths which determine an affine connection whose components are functions of position only.

[35] This follows from (1.10a). See also G. P. (21).

factors which differ by a constant are equivalent, we may, instead of (9.6), write

$$(9.7) \qquad d\bar{x}^0 = \varphi_\alpha \, dx^\alpha$$

as the equation giving $\bar{x}^0$, where $\varphi_0 = 1$ and $\varphi_i = \varphi_{,i}$. If we remove the restriction that φ_α be a gradient we can derive any factor by an equation of the form (9.7).

If at any point q the vectors corresponding to two factors are coincident, we shall say that these factors are tangent at q.

We now define the normal factor for x^0 and the point q, as the normal factor at q, according to § 2, for any holonomic factor tangent to x^0 at this point. Any normal factor is, therefore, holonomic and we can apply the results of sections 2, 3 and 4.

We shall show that the problem of determining when two projective connections are equivalent is essentially the same whether we require the transformations defined by (9.1) to be integrable or not.

Let U and $\bar{U}$ be any two transformations of the form (9.1) such that, if[33]

$$(9.8) \qquad H = U\Pi \quad \text{and} \quad \bar{H} = \bar{U}\bar{\Pi}$$

then the components H^0_{jk} and $\bar{H}^0_{jk}$ are symmetric. By § 8, and the affine theory we have a criterion for determining whether or not H and $\bar{H}$ are equivalent under integrable changes of representation. If they are equivalent, there will be an integrable transformation, T, of the form (9.1) such that

$$\bar{H} = TH.$$

From (9.8) we have, therefore,

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Conversely, if there is a transformation, S, of the form (9.1) such that

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The equivalence problem for Π and $\bar{\Pi}$ under all changes of representation is, therefore, the same as that for H and $\bar{H}$ under holonomic transformations.

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We start with a system of paths which are given in any coördinate system, x, by the equations

$$\frac{d^2 x^i}{d\lambda^2} + \Pi^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0, \tag{10.1}$$

where Π^i_{jk} are analytic functions of position. By adding arbitrary sets of components Π^0_{jk} and $\Pi^\alpha_{\beta 0} = \Pi^\alpha_{0\beta} = \delta^\alpha_\beta$, we can define an infinite class of projective connections, each of which defines the same system of paths[35] as the equations (10.1).

From this class of projective connections we select one by requiring that, in a given representation, $x + x^0$,

$$\Pi^0_{jk} = \frac{1}{n-1} B_{jk}, \tag{10.2}$$

where

$$B^i_{jkl} = \Pi^i_{jk,l} - \Pi^i_{jl,k} + \Pi^a_{jk}\Pi^i_{al} - \Pi^a_{jl}\Pi^i_{ak},$$

and

$$B_{jk} = B^a_{jak}.$$

Let $R^\alpha_{\beta\gamma\delta}$ be the components, in $x + x^0$, of the projective curvature tensor for this connection. Then we have (P. F. § 1)

$$R^i_{jkl} = B^i_{jkl} + \delta^i_l \Pi^0_{jk} - \delta^i_k \Pi^0_{jl}, \tag{10.3}$$

and, from (10.2),

$$R^a_{jak} = 0.$$

Since $R^\alpha_{\beta 0\delta} = R^i_{0\gamma\delta} = 0$, from the symmetry of $\Pi^i_{\beta\gamma}$, this condition is equivalent to

$$R_{\beta\gamma} = 0. \tag{10.4}$$

where

$$R_{\beta\gamma} = R^\alpha_{\beta\alpha\gamma}.$$

[34] J. Douglas, loc. cit. We shall not consider the geometry of paths in its full generality, but shall confine ourselves to these systems of paths which determine an affine connection whose components are functions of position only.

[35] This follows from (1.10a). See also G. P. (21).

Conversely, given (10.4), we can deduce (10.2) from (10.3). Since $R_{\beta\gamma}$ is a projective tensor the conditions (10.2) are satisfied in all representations, and by this means a unique projective connection is defined. This will be called the *normalized projective connection* determined by a given system of paths. Since (10.2) are equivalent to (10.4) we have:

THEOREM VIII. *In order that an affine connection of* $(n+1)$ *dimensions may be taken as a normalized projective connection for a space of n dimensions, it is necessary and sufficient that* (1.5) *and* (10.4) *be satisfied.*

From (10.2) it follows that B_{jk} is symmetric in any holonomic representation, and so, corresponding to any holonomic factor there will be representations[36] in which

(10.5) $$\Pi^a_{aj} = 0.$$

A representation in which (10.5) is satisfied will be called a *selected representation.* If $\Pi^a_{aj,k} = \Pi^a_{ak,j}$ then B_{jk} will be symmetric, and so all selected representations are holonomic. Let $x + x^0$ and $\bar{x} + \bar{x}^0$ be two selected representations, related by the equations

$$\begin{cases} \bar{x}^0 = x^0 + \log \varrho, \\ \bar{x}^i = \bar{x}^i(x). \end{cases}$$

Then

(10.6) $$\bar{\Pi}^i_{jk} = (\Pi^a_{bc}\, u^b_j\, u^c_k + u^a_{j,k})\, v^i_a + \delta^i_j\, u^0_k + \delta^i_k\, u^0_j.$$

Contracting i and j, we have

$$v^i_a\, u^a_{i,k} + (n+1)\, u^0_k = 0,$$

or

$$\frac{\partial}{\partial \bar{x}^k}\left\{\log\left|\frac{\partial x}{\partial \bar{x}}\right| - (n+1)\log \varrho\right\} = 0,$$

whence

(10.7) $$\varrho = c\left|\frac{\partial x}{\partial \bar{x}}\right|^{\frac{1}{n+1}},$$

where c is an arbitrary constant other than zero. The transformation (10.6) agrees, therefore, with the transformation law for projective connections, in the sense of T. Y. Thomas. Conversely, if $x + x^0$ is a selected representation, then so is any representation $\bar{x} + \bar{x}^0$ related to $x + x^0$ by equations of the form

(10.8) $$\begin{cases} \bar{x}^0 = x^0 + \log c\left|\dfrac{\partial x}{\partial \bar{x}}\right|^{\frac{1}{n+1}}, \\ \bar{x}^i = \bar{x}^i(x). \end{cases}$$

[36] N. R. G. p. 105.

Instead of considering, as the basis of an invariant theory, all changes of representation, we shall confine our attention to transformations from one selected representation to another. From (10.7) we see that any two selected representations with the same coördinate system x^i are equivalent, and so the components of connection will be the same in both. We are left, therefore, with the theory of an invariant which determines, in each coördinate system, a unique system of components Π^i_{jk}, such that $\Pi^a_{aj}=0$, and whose transformation law is given by (10.6), subject to the condition (10.7). This is precisely the theory of a projective connection in the sense of T. Y. Thomas. Following J. Douglas we shall call this theory the descriptive geometry of paths.

The steps taken to normalize the components Π^i_{jk} have a simple interpretation in terms of volume. Consider the parallel displacement of an n-uple by means of the equations

$$d u^a_j - u^a_i \, \Pi^i_{jk} \, d x^k = 0,$$

where u^a_i are the components of n independent co-variant vectors. These equations give

$$v^i_a \, d u^a_j = \Pi^i_{jk},$$

where $v^i_a \, u^a_j = \delta^i_j$. Contracting i and j, we have

$$d \log u = \Pi^i_{ik} \, d x^k,$$

where $u = |u^a_i|$. In order that u shall be a uniform function of position it is necessary and sufficient that $\Pi^a_{aj,k} = \Pi^a_{ak,j}$. In this case the scalar density u may be taken to define a volume integral[37]

$$\int u \, d x^1 \cdots d x^n.$$

Selected representations for a normalized projective connection are those in which the volume density is constant.

11. **The equivalence problem and normal coördinates for normalized connections.** In order to apply the results obtained in the first eight sections to the descriptive geometry of paths we shall need two theorems, which we shall prove in this section.

THEOREM IX. *Any normal representation for a normalized projective connection is a selected representation.*

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[37] O. Veblen, Equiaffine geometry of paths. Proc. Nat. Ac. of Sc. (1923), vol. 9, pp. 3–4. See also N. R. G., p. 10 and p. 26.

THEOREM X. *Two affinely equivalent normalized projective connections, referred to a selected representation, are equivalent under transformations of the form* (10.8).

Once these theorems have been established we can pass on to the descriptive geometry of paths by adding to (1.5) the condition $R_{\beta\gamma}=0$.

Theorem X follows at once from Theorem VII and (10.7).

To prove Theorem IX it is necessary and sufficient to show that $P^i_{ij}=0$, where $P^\alpha_{\beta\gamma}$ are the components of connection in any normal representation $z+z^0$. Let $Y^0,\cdots,Y^n$ be affine normal coördinates (K_{n+1}) related to $z+z^0$ by the equations

$$\begin{cases} Y^i = e^{z^0}z^i, \\ Y^0+1 = e^{z^0}, \end{cases}$$

and let $C^\alpha_{\beta\gamma}$ be the components of connection Y. Then from (6.12) we have

$$\begin{aligned} C^\alpha_{\alpha j} &= \frac{1}{Y^0+1}(P^a_{aj}+z^a P^0_{aj}-Y^a C^0_{aj}) \\ &= \frac{1}{Y^0+1}P^a_{aj}, \\ C^\alpha_{\alpha 0} &= C^a_{a0}, \quad \text{since} \quad C^\alpha_{00}=0, \\ &= \frac{-z^a}{Y^0+1}P^b_{ba}. \end{aligned}$$

Collecting these results we have

$$\text{(11.1)} \qquad \begin{cases} C^\alpha_{\alpha i} = \dfrac{1}{Y^0+1}P^a_{ai}, \\ C^\alpha_{\alpha 0} = \dfrac{-z^b}{Y^0+1}P^a_{ab}. \end{cases}$$

To show that $z+z^0$ is a selected representation it is, therefore, necessary and sufficient to show that

$$\text{(11.2)} \qquad C_\beta = 0$$

in each affine normal coördinate system (K_{n+1}), where $C_\beta = C^\alpha_{\alpha\beta}$.

Let $B^\alpha_{\beta\gamma\delta}$ be the components in Y, of the curvature tensor formed out of $C^\alpha_{\beta\gamma}$. Then

$$\text{(11.3)} \qquad B_{\beta\gamma} = C_{\beta,\gamma} - C^\lambda_{\beta\gamma,\lambda} + C^\lambda_{\mu\beta}C^\mu_{\lambda\gamma} - C^\lambda_{\beta\gamma}C_\lambda = 0.$$

From these equations it follows that

$$C_{\beta,\gamma} = C_{\gamma,\beta},$$

and so there exists some uniform function g, say, such that

$$\frac{\partial g}{\partial Y^\beta} = C_\beta.$$

The value of g at any point, Y_1, is given by

$$g - g_0 = \int_0^{t_1} C_\lambda Y^\lambda \frac{dt}{t},$$

where the integral is taken along the path joining Y_1 and the origin of normal coördinates. To show that g is a constant, in which case (11.2) will be satisfied, it is, therefore, sufficient to show that

$$C_\lambda Y^\lambda = 0,$$

or, as follows from (6.10b), that

$$C_0 = 0.$$

To prove Theorem IX it is, therefore, sufficient to show that this last condition is satisfied in each affine normal coördinate system (K_{n+1}), which we shall proceed to do.

From (11.3) we have

$$\begin{aligned} B_{00} &= C_{0,0} - C^\lambda_{00,\lambda} + C^\mu_{\lambda 0} C^\lambda_{\mu 0} - C^\lambda_{00} C_\lambda \\ &= 0, \end{aligned}$$

or, since $C^\alpha_{00} = 0$,

(11.4) $$C^\lambda_{\mu 0} C^\mu_{\lambda 0} + C_{0,0} = 0.$$

From (6.10b) we have $C^\lambda_{\mu\beta} Y^\beta = -C^\lambda_{\mu 0}$, from which, and from (11.4), it follows that

(11.5) $$C^\lambda_{\mu\beta} C^\mu_{\lambda\gamma} Y^\beta Y^\gamma = -C_{0,0}.$$

Also, since $C^\alpha_{\lambda\mu} Y^\lambda Y^\mu = 0$, we have

(11.6) $$C^\alpha_{\cdot\mu,\alpha} Y^\lambda Y^\mu + C_\lambda Y^\lambda = 0.$$

So, from (11.3), (11.5), (11.6) and the condition $C^\alpha_{\lambda\mu} Y^\lambda Y^\mu = 0$, we have

(11.7) $$C_{\lambda,\mu} Y^\lambda Y^\mu + C_\lambda Y^\lambda - C_{0,0} = 0.$$

From (6.10b) we have

(11.8) $$\begin{aligned} C_{\lambda,\mu} Y_\lambda + C_\mu + C_{0,\mu} &= \frac{\partial}{\partial Y^\mu} (C_\lambda Y^\lambda + C_0) \\ &= 0, \end{aligned}$$

and so

$$C_{\lambda,\mu} Y^\lambda Y^\mu + C_\mu Y^\mu + C_{0,\mu} Y^\mu = 0,$$

or, from (11.7)

$$C_{0,\mu} Y^\mu + C_{0,0} = 0.$$

From (11.8), with $\mu = 0$, and the condition $C_{\lambda,\mu} = C_{\mu,\lambda}$, we have, therefore,

$$C_0 = 0,$$

and so Theorem IX is established[38].

We have, incidentally, the result that, in projective normal coördinates for a normalized projective connection,

$$P^a_{jak} z^j z^k = (n-1) P^0_{jk} z^j z^k = 0. \tag{11.9}$$

[38] We have two observations to make about this proof.

1. That we have used the full force of the condition $B_{\lambda\mu} = 0$. For we have assumed that $B_{\lambda\mu} Y^\lambda Y^\mu = 0$, from which it follows that $B_{\lambda\mu} = 0$ at the origin. But this is to be true in all normal coördinate systems, and so $B_{\lambda\mu}$ must vanish at all points.

2. That the centre is a singularity for the components $C^\alpha_{\beta\gamma}$. We cannot, therefore, from the condition $C_\lambda (Y^\lambda + \delta^\lambda_0) = 0$, argue that $\dfrac{\partial g}{\partial Y^\lambda} = 0$.

(Added in proof, March 1931.) Attention should be drawn to a recent publication: J. A. Schouten and St. Gołąb, Über projektive Übertragungen und Ableitungen, Math. Zeit., vol. 32 (1930), pp. 192–214. Prof. Schouten has told me about a paper by St. Gołąb called Über verallgemeinerte projektive Geometrie, which has recently appeared in a Polish journal.

DIFFERENTIAL INVARIANTS OF DIRECTION AND POINT DISPLACEMENTS.[1]

By Enea Bortolotti.

1. **Introduction.** The principal object of this note is to indicate how it is possible to reduce the theory of the differential invariants of a (linear, or non-linear) displacement of directions — that is, the invariant theory of the transformations of a vector displacement which preserve the directions of all displaced vectors — to the theory of a uniquely determined displacement of vectors yielding the same parallelism. The *linear* case has been studied in some detail (but in a somewhat different way) in my Memoir: *Sulla geometria delle varietà a connessione affine. Teoria invariantiva delle trasformazioni che conservano il parallelismo.*[2] The present paper also contains, beside some further developments of this theory, the foundations of the theory of the *non-linear* displacements of directions; further it gives an account of an analogous treatment for the theory of the (linear, or non-linear) *point displacements*. It is known that the last theory, in the linear case (projective connection) can be considered, following T. Y. Thomas and Veblen, as the theory of an affine connection (linear vector displacement);[3] which is not, however, wholly determined by the given point displacement. But we can, on the contrary, interpret in all cases a point displacement in an X_n (n-dimensional manifold) as being a *direction displacement* in an X_{n+1}; and thus, associate also to a point displacement, as will be shown in this paper, a single vector displacement.

[1] Received February 21, and June 16, 1930.

[2] This Memoir will appear in the Annali di Matematica (serie IV, tomo VIII, 1930, pp. 53–101). Some results on the same subject have been established by H. Friesecke (*Vektorübertragung, Richtungsübertragung, Metrik.* Mathem. Annalen, B. 94, 1925, pp. 101–118; esp. pp. 105–109) and by J. M. Thomas (*Asymmetric displacement of a vector.* Transactions Amer. Mathem. Society, vol. 28, 1926, pp. 658–670). See also L. P. Eisenhart: *Non-Riemannian Geometry.* New York, 1927, pp. 29–38. However, a complete and satisfactory theory of the *linear* displacements of directions has hitherto been lacking. For the *non-linear* case only a short account has been given by H. Friesecke (loc. cit.). For brevity I shall refer to my Memoir as T. C. P., and to the papers cited above as V. R. M., A. D., N. R. G., respectively.

[3] See T. Y. Thomas: *A projective theory of affinely connected manifolds*; Mathem. Zeitschrift, B. 25, 1926, pp. 723–733; Veblen, *Projective tensors and connections.* Proceedings Nation. Acad., vol. 14, 1928, pp. 154–166, and *Generalized projective Geometry,* Journal London Mathem. Society, vol. 4, 1929, pp. 140–160. I shall refer to these papers as A. M., P. T., G. P. G., respectively.

2. **Linear vector displacements (affine connections).** I shall first recall that a complete set of affine differential invariants for an affine connection Γ^i_{jk}, that is, for a linear vector displacement[4]

$$\bar{d}\,\xi^i = \nabla_k\,\xi^i\,dx^k = \left(\frac{\partial\,\xi^i}{\partial\,x^k} + \Gamma^i_{jk}\,\xi^j\right)dx^k = 0 \qquad (i, j, h, k, l, \cdots = 1, 2, \cdots, n) \tag{2.1}$$

consists of the torsion tensor[5]

$$S_{jk}^{\cdot\cdot i} = \Gamma^i_{[jk]} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) \tag{2.2}$$

and the curvature tensor[6]

$$R_{jkl}^{\cdot\cdot\cdot i} = \frac{\partial\,\Gamma^i_{lj}}{\partial\,x^k} - \frac{\partial\,\Gamma^i_{lk}}{\partial\,x^j} + \Gamma^h_{lj}\,\Gamma^i_{hk} - \Gamma^h_{lk}\,\Gamma^i_{hj} \tag{2.3}$$

together with their successive (∇)-derivatives. Another complete set of affine differential invariants for the connection Γ^i_{jk} in the *symmetrical* case ($\Gamma^i_{jk} = \Gamma^i_{kj}$) consists, as is well known, of the *normal tensors*[7]. But Veblen's definition[8], for the symmetric connections, of the *affine normal coördinates*,

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[4] We denote, following J. A. Schouten, by ∇_k the symbol of the *covariant derivative*, and (following R. Lagrange) by $\bar{d}$ the symbol of *absolute* (or *cogredient*) *differentiation*. Let us recall that $\bar{d}\,\xi^i$ is *cogredient* to ξ^i, provided we have, under an arbitrary change of the curvilinear coördinates x^i,

$$\Gamma'^{a}_{bc} = \Gamma^i_{jk}\,u^j_b\,u^k_c\,v^a_i + u^i_{bc}\,v^a_i, \tag{2.a}$$

where

$$u^i_a = \frac{\partial\,x^i}{\partial\,x'^a}, \quad u^i_{bc} = \frac{\partial^2 u^i}{\partial\,x'^b\,\partial\,x'^c}, \quad v^a_i = \frac{\partial\,x'^a}{\partial\,x^i}. \tag{2.b}$$

[5] By the *square* or the *round* brackets we denote (following J. A. Schouten) the *alternating*, or *mixing*, with respect to the indices enclosed in the brackets. Then we have

$$A_{[i_1 i_2 \cdots i_p]} = \frac{1}{p!}\,\delta^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p}\,A_{j_1 j_2 \cdots j_p}, \quad A_{(i_1 i_2 \cdots i_p)} = \frac{1}{p!}\,\sigma^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p}\,A_{j_1 j_2 \cdots j_p}, \tag{2.c}$$

where the $\delta^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p}$ are *generalized Kronecker deltas*, and $\sigma^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p}$ are also the components of a tensor, symmetric both in the subscripts and superscripts. If both the permutations $j_1 j_2 \cdots j_p$, $i_1 i_2 \cdots i_p$ contain ν_{r_1} times the element r_1, ν_{r_2} times the element $r_2, \cdots, \nu_{r_t}$ times the element r_t, $r_1 r_2 \cdots r_t$ being distinct from each other, $(\nu_{r_1} + \nu_{r_2} + \cdots + \nu_{r_t} = n)$, the value of $\sigma^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p}$ is equal to the product $\nu_{r_1}!\,\nu_{r_2}! \cdots \nu_{r_t}!$; in all other cases its value is 0. We have

$$\delta^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p} = p!\;\delta^{j_1}_{[i_1}\,\delta^{j_2}_{i_2} \cdots \delta^{j_p}_{i_p]}, \quad \sigma^{j_1 j_2 \cdots j_p}_{i_1 i_2 \cdots i_p} = p!\;\delta^{j_1}_{(i_1}\,\delta^{j_2}_{i_2} \cdots \delta^{j_p}_{i_p)}. \tag{2.d}$$

[6] Eisenhart writes L^i_{jk}, Ω^i_{jk}, L^i_{lkj} instead of our Γ^i_{jk}, $S_{jk}^{\cdot\cdot i}$, $R_{jkl}^{\cdot\cdot\cdot i}$ (N. R. G. pp. 3, 5, 6).

[7] See, e. g., T. Y. Thomas and A. D. Michal, *Differential invariants of affinely connected manifolds.* (These Annals, series 2, vol. 28, 1927, pp. 196–236.)

[8] See, e. g., *Invariants of quadratic differential forms*, Cambridge 1927, p. 85.

of the *affine normal tensors* and of the *affine extensions* of an arbitrary tensor, *preserve their significance* (as I have observed in my cited Memoir) *also in the general (asymmetric) case.* However, the series of the affine normal tensors contains, in this case, *a first term,* which vanishes in the symmetric case: the torsion tensor $S_{jk}{}^{i}$ itself, whose components $S^{*}{}_{jk}{}^{i}$ in any affine normal coördinate system x^{*i}, evaluated at the origin of this system, are equal to the quantities $(\Gamma^{*i}_{jk})_0$.

Also in the asymmetric case, *the affine normal tensors constitute a complete set of differential invariants for the affine connection.*

3. **Non-linear vector displacements.** Many results of the theory of linear vector displacements can be extended to the non-linear case, or, more precisely, to the displacements represented by the equations

$$d\xi^i + \Lambda^i_{jk}(x,\xi)\,\xi^j dx^k = 0, \tag{3.1}$$

where the functions $\Lambda^i_{jk}(x,\xi)$ of the x^i, ξ^i are homogeneous functions of zero degree in the ξ^i. These displacements have been first considered by H. Friesecke[9]. By putting

$$\Gamma^i_{jk}(x,\xi) = \Lambda^i_{jk}(x,\xi) + \frac{\partial \Lambda^i_{lk}}{\partial \xi^j}\xi^l \tag{3.2}$$

we may also write the (3.1) in the equivalent form

$$d\xi^i + \Gamma^i_{jk}(x,\xi)\,\xi^j dx^k = 0. \tag{3.3}$$

Among all systems of functions $\Lambda^i_{jk}(x,\xi)$ yielding *the same* vector displacement, the $\Gamma^i_{jk}(x,\xi)$ are characterized by the conditions

$$\frac{\partial \Gamma^i_{lk}}{\partial \xi^j}\xi^l = 0. \tag{3.4}$$

The components $\Gamma^i_{jk}(x,\xi)$, $\Gamma'^a_{bc}(x',\xi')$ in two coördinate systems x^i, x'^a are related by the law of transformation

$$\Gamma'^a_{bc}(x',\xi') = \Gamma^i_{jk}(x,\xi)\,u^j_b\,u^k_c\,v^a_i + u^i_{bc}\,v^a_i, \tag{3.5}$$

where (as in (2.b))

$$u^i_a = \frac{\partial x^i}{\partial x'^a}, \quad u^i_{bc} = \frac{\partial^2 x^i}{\partial x'^b\,\partial x'^c}; \quad v^a_i = \frac{\partial x'^a}{\partial x^i}, \text{ and } \xi'^a = v^a_i\,\xi^i. \tag{3.6}$$

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[9] Friesecke writes $\gamma^i_k(x,\xi)$ instead of our $\Lambda^i_{jk}(x,\xi)\,\xi^j$. Since his $\gamma^i_k(x,\xi)$ are supposed to be homogeneous of the first degree in the ξ^i, that assumption is no more general than the one we have chosen.

Since the (3.5) do not differ, for each value of ξ^i, from the corresponding transformation formulas (2.a) for the coefficients of an *affine* connection $\Gamma^i_{jk}(x)$, the developments of the corresponding theories must present some notable analogies. We may consider, also in the present case, the two fundamental tensors $S_{jk}{}^i(x, \xi)$ and $R_{jkl}{}^i(x, \xi)$ (the *torsion* and *curvature* tensors), obtained from $\Gamma^i_{jk}(x, \xi)$ by the formulas (2.2) and (2.3); another fundamental tensor, that I shall name *the alinearity tensor*, is the tensor $F_{ljk}{}^i$, symmetric in the subscripts l, j, given by the following formula

$$F_{ljk}{}^i = \frac{\partial \Gamma^i_{jk}(x, \xi)}{\partial \xi^l}. \tag{3.7}$$

The vanishing of $F_{ljk}{}^i$ is a necessary and sufficient condition that the displacement defined by the $\Gamma^i_{jk}(x, \xi)$ be a *linear* displacement. The vanishing of $R_{jkl}{}^i(x, \xi)$ is a condition that it be an *integrable* displacement; but the vanishing of $S_{jk}{}^i$ is a condition for the *symmetry* (or *commutability*) of the same displacement *only when we have also* $F_{ljk}{}^i = 0$. (A generalized vector displacement (3.1) cannot be symmetrical, unless it is a *linear* one.)

It is easily seen that the notions of *normal coördinates, normal tensors,* etc. can be directly applied also to the present theory, under this condition alone: that the $\Gamma^i_{jk}(x, \xi)$ and their derivatives with respect to the x^i for $\xi^i = \varrho a^i$ and $\varrho \to 0$ *have finite limits,* for each value of the a^i. The *differential invariants* of the displacement $\Gamma^i_{jk}(x, \xi)$ are in general functions of the x^i and also of the ξ^i; a *complete set* of differential invariants consists of the torsion and curvature tensors and their covariant (∇)-derivatives (obtained, as in the linear case, by means of the $\Gamma^i_{jk}(x, \xi)$)—for which we may also substitute the series of the normal tensors—and besides, of the tensors $F_{ljk}{}^i$, $F_{hljk}{}^i$, $\cdots$, obtained from $\Gamma^i_{jk}(x, \xi)$ by means of successive derivations with respect to the ξ^j.

The development of this theory will not be expounded here.

4. Linear displacements of directions. The normalized affine connection P^i_{jk}. Let us now consider a totality of vector displacements (2.1) or (3.1) yielding the same displaced directions. It defines what we may call a (linear, or non-linear) *direction displacement.*

We begin with the linear case. In this case, as H. Friesecke has shown,[10] the transformations of the vector displacement which preserve the directions of all the displaced vectors are represented by the formulas

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\delta^i_j \varphi_k, \tag{4.1}$$

[10] V. R. M., p. 106. See also A. D., p. 662; N. R. G., p. 30.

where φ_k is an arbitrary covariant vector. The invariant theory of these transformations (4.1) can be based upon the following remark:

We may obtain in a very simple way, from a given linear vector displacement (affine connection) Γ^i_{jk}, a uniquely determined covariant vector Φ_k by contracting the torsion tensor $S_{jk}^{\cdot\cdot i}$:

$$\Phi_k = S_{ki}^{\cdot\cdot i} = \Gamma^i_{[ki]} = \frac{1}{2}(\Gamma^i_{ki} - \Gamma^i_{ik}). \tag{4.2}$$

Let us call this vector Φ_k *the Einstein vector* for the connection Γ^i_{jk}.[11] From the (4.1), (4.2) we obtain

$$\bar{\Phi}_k = \Phi_k - (n-1)\varphi_k. \tag{4.3}$$

Then it is possible to reduce, by a suitable (uniquely determined) choice of φ_k, the vector $\bar{\Phi}_k$ for the transformed affine connection $\bar{\Gamma}^i_{jk}$ to the *zero* vector: for that we must assume

$$\varphi_k = \frac{1}{n-1}\Phi_k. \tag{4.4}$$

We may conclude:

Among all affine connections yielding the same displaced directions there exists a preferred one, the connection

$$P^i_{jk} = \Gamma^i_{jk} + \frac{2}{n-1}\delta^i_j\,\Phi_k, \tag{4.5}$$

characterized by the condition

$$P^i_{[ki]} = 0 \qquad (\text{or } \delta^{hl}_{ki}\,P^i_{hl} = 0), \tag{4.6}$$

that is by the vanishing of the Einstein vector $\Phi_k = S_{ki}^{\cdot\cdot i}$.

This consequence is to be observed: that *to any given linear displacement of directions a uniquely determined linear displacement of vectors* (that is, an affine connection) *can be associated, so that the arising displacement of directions be the given one.* In a like manner, as is well known, to any system of paths a uniquely determined linear displacement of points (that is, a projective connection) can be associated, so that the paths of this connection shall be the given ones (Cartan).

5. **Representation of a linear displacement of directions by means of the L^{pq}_{hjk}.** As Prof. Eisenhart has observed (compare N. R. G., p. 13, form. (7.5)) the direction displacement arising from a vector displacement Γ^i_{jk} can be represented by the equations

[11] A. Einstein makes use of this vector in a recent research (*Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität.* Sitzungsberichte Preuß. Akad. Berlin, 1928, pp. 224–227) for the representation of the elektromagnetic potential vector.

$$\delta^{pq}_{ih}\,\xi^h\,d\xi^i + L^{pq}_{hjk}\,\xi^h\,\xi^j\,dx^k = 0, \tag{5.1}$$

where we have put

$$L^{pq}_{hjk} = \delta^{pq}_{i(h}\,\Gamma^i_{j)k} = \frac{1}{2}(\delta^{pq}_{ih}\,\Gamma^i_{jk} + \delta^{pq}_{ij}\,\Gamma^i_{hk}). \tag{5.2}$$

Conversely, it will be easily shown that *an arbitrarily given system of differential equations of the form* (5.1), *only subject to the condition that it must be an algebraically consistent system for the variables* $d\xi^i$ (for all values of the x^i, dx^i, ξ^i) *always represents a linear displacement of directions*: that is, there exist sets of functions Γ^i_{jk} such that the equations (5.2) are satisfied.

In fact, if (5.1) are to be compatible, for all values of the variables x^i, dx^i and ξ^i the quantities $L^{pq}_{hjk}\,\xi^h\,\xi^j\,dx^k$ must be (as $\delta^{pq}_{ih}\,\xi^h\,d\xi^i$ are) *the components of a bivector parallel to the vector* ξ^i. From an analytical point of view the functions L^{pq}_{hjk} are first to be *skew-symmetric in the superscripts* p, q. (Besides, we may suppose, without loss of generality, that these functions are *symmetric in the subscripts* h, j.) Then, since we have identically

$$\delta^{uvw}_{pqr}\,\delta^{pq}_{ih}\,\xi^h\,d\xi^i\,\xi^r = 2\,\delta^{uvw}_{ihr}\,\xi^h\,\xi^r\,d\xi^i = 0, \tag{5.3}$$

we must have also

$$\delta^{uvw}_{pqr}\,L^{pq}_{hjk}\,\xi^h\,\xi^j\,\xi^r\,dx^k = 0 \tag{5.4}$$

for all values of the variables ξ^i and dx^i, and therefore

$$\delta^{uvw}_{pq(r}\,L^{pq}_{hj)k} = 0, \qquad (\text{or } \sigma^{st\,l}_{rhj}\,\delta^{uvw}_{pq\,s}\,L^{pq}_{t\,lk} = 0) \tag{5.5}$$

that is[12]

$$\delta^{uvw}_{pqr}\,L^{pq}_{hjk} + \delta^{uvw}_{pqh}\,L^{pq}_{jrk} + \delta^{uvw}_{pqj}\,L^{pq}_{rhk} = 0. \tag{5.6}$$

On setting $v = h$ in (5.6) and summing we obtain

$$L^{wu}_{jrk} = \delta^{wu}_{l(r}\,L^l_{j)k}, \tag{5.7}$$

where we have put

$$L^l_{jk} = \frac{2}{n}\,L^{lh}_{hjk}. \tag{5.8}$$

Then by putting $\Gamma^i_{jk} = L^i_{jk}$ the (5.2) are satisfied. However the L^i_{jk} *do not* have the law of transformation (2.a) characteristic for the coefficients

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[12] It is superfluous in our case to make use of the conditions that a skew-symmetric tensor v^{ij} be a bivector:

$$v^{ij}\,v^{hk} = \delta^{ij}_{pq}\,v^{ph}\,v^{qk}. \tag{5.a}$$

of an affine connection. But that is true, on the contrary, for the quantities[13]

$$P^i_{jk} = L^i_{jk} + \frac{1}{n-1}\,\delta^i_j\,L^h_{kh} \tag{5.9}$$

and also for

$$\Gamma^i_{jk} = P^i_{jk} - 2\,\delta^i_j\,\psi_k, \tag{5.10}$$

ψ_k being an arbitrary covariant vector. Moreover, we have also

$$L^{wu}_{jrk} = \delta^{wu}_{l(r}\,P^l_{j)k} = \delta^{wu}_{l(r}\,\Gamma^l_{j)k}. \tag{5.11}$$

We may conclude that the direction displacements represented by any consistent differential system (5.1) are the linear direction displacements, and these alone, q. e. d.

It is immediately seen, by means of (5.8), (5.9), (5.10), that the affine connection P^i_{jk} given by the (5.9) does not differ from the above obtained normalized connection (4.5). Thus we have obtained also the expressions of this connection in terms of the L^{pq}_{hjk}.

From (4.5), (5.8), (5.9) we obtain easily

$$L^i_{jk} = \Gamma^i_{jk} - \frac{1}{n}\,\delta^i_j\,\Gamma^l_{lk} = P^i_{jk} - \frac{1}{n}\,\delta^i_j\,P^l_{lk}; \quad L^i_{ik} = 0. \tag{5.12}$$

6. **The paths of a linear displacement of directions.** The *autoparallel* or *geodesic lines*, that is, *the paths* of a linear displacement of directions (5.1) are the lines represented by the differential system

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[13] We must suppose, of course, that the displacement (5.1) be invariant under an arbitrary change of coördinates. Then the components L^{pq}_{hjk}, L'^{ab}_{cde} in two coördinate systems x^i, x'^a must be related by the following law of transformation:

$$L'^{ab}_{cde} = L^{pq}_{hjk}\,u^h_c\,u^j_d\,u^k_e\,v^a_p\,v^b_q + \delta^{qp}_{kl}\,u^k_{(c}\,u^l_{d)e}\,v^a_p\,v^b_q. \tag{5.b}$$

From (5.b) by contracting for b and c we obtain:

$$L'^a_{de} = L^p_{jk}\,u^j_d\,u^k_e\,v^a_p + u^p_{de}\,v^a_p - \frac{1}{n}\,\delta^a_d\,\frac{\partial u}{\partial x'^e}, \tag{5.c}$$

where the L^p_{jk} are given by (5.8), and we have

$$u = \log\left|\frac{\partial(x^1, x^2, \cdots, x^n)}{\partial(x'^1, x'^2, \cdots, x'^n)}\right|. \tag{5.d}$$

Then by contracting again for a and e we have

$$L'^a_{da} = L^p_{jp}\,u^j_d + \frac{n-1}{n}\,\frac{\partial u}{\partial x'^d}. \tag{5.e}$$

Eliminating $\dfrac{\partial u}{\partial x'^e}$ from (5.c) by means of (5.e) we obtain at least, P^p_{jk} being given by (5.9):

$$P'^a_{de} = P^p_{jk}\,u^j_d\,u^k_e\,v^a_p + u^p_{de}\,v^a_p. \tag{5.f}$$

$$\delta^{pq}_{ih}\, d^2 x^i\, dx^h + \Pi^{pq}_{jhk}\, dx^j\, dx^h\, dx^k = 0, \tag{6.1}$$

where

$$\Pi^{pq}_{jhk} = L^{pq}_{(jhk)} = \frac{1}{3}(L^{pq}_{jhk} + L^{pq}_{hkj} + L^{pq}_{kjh}). \tag{6.2}$$

Conversely, any system of lines represented by a differential system of the form (6.1) (where the Π^{pq}_{jhk} are skew-symmetric in the superscripts, symmetric in the subscripts), provided we assume that this system, for all values of the x^i, dx^i, be a *consistent* system for the $d^2 x^i$, *can always be considered as being the system of paths of a linear displacement of directions* (and also, of an affine connection). In fact, O. Veblen and J. M. Thomas, who first considered[14] systems of the type (6.1), have pointed out that the quantities

$$Q^{uw}_{jrk} = \Pi^{uw}_{jrk} - \frac{1}{n+1}(\delta^{uw}_{rp}\, \Pi^{vp}_{vjk} + \delta^{uw}_{jp}\, \Pi^{vp}_{vkr} + \delta^{uw}_{kp}\, \Pi^{vp}_{vjr}) \tag{6.3}$$

are the components of a tensor, and that the vanishing of this tensor is a necessary and sufficient condition that the lines represented by (6.1) be the paths of an affine connection. But from (6.2), (5.7), or, more directly, by expressing the conditions that the system (6.1) be *a consistent* system[15] we obtain easily that the tensor Q^{uw}_{jrk} *is a zero tensor.* Then the geometry of paths based upon a differential system (6.1) is no more general than the *affine* geometry of paths, that is "a system of differential equations of the form (6.1) cannot be used to determine a system of paths which does not have an affine connection" as A. Church has observed (in the paper quoted in foot note [14]).

7. The differential invariants of a linear displacement of directions: a fundamental property of the normalized connection P^i_{jk}. Let us return to the normalized affine connection P^i_{jk}. We shall now demonstrate the following theorem, fundamental for the present theory[16]:

The differential invariants of a linear displacement of directions are the differential invariants of the corresponding normalized affine connection P^i_{jk}.

Such a result is easily anticipated, since the conditions (4.1) that two affine connections Γ^i_{jk}, $\bar{\Gamma}^i_{jk}$ have the same parallelism can be written also

$$\bar{P}^i_{jk} = P^i_{jk} \text{ (or } \bar{L}^i_{jk} = L^i_{jk}). \tag{7.1}$$

[14] In the paper: *Projective invariants of the affine geometry of paths.* (These Annals; 2nd series, vol. 27, 1926, pp. 279-296) p. 295. See also A. Church: *On the form of differential equations of a system of paths.* (Ibid., vol. 28, 1927, pp. 629-630.)

[15] Only under this assumption can (6.1) represent a system of paths, according to Veblen; for it is an essential property of what we call paths that we have one path from an arbitrary point, in an arbitrary direction.

[16] I have indicated another demonstration of this theorem in my Memoir: T. C. P., No. 8.

In fact, any differential invariant of the mth order[17] of the connection Γ^i_{jk} can be expressed as follows, in terms of the Γ^i_{jk} and their derivatives:

$$E^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a} = G^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a}\left(\Gamma^i_{jk}, \frac{\partial \Gamma^i_{jk}}{\partial x^l}, \ldots, \frac{\partial^{m-1}\Gamma^i_{jk}}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}}\right). \tag{7.2}$$

By means of the (4.5) we may also express the differential invariant $E^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a}$ in terms of the P^i_{jk}, Φ_k and their derivatives:

$$\begin{aligned} E^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a} = G^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a}\Bigg(&P^i_{jk} - \frac{2}{n-1}\delta^i_j\Phi_k, \frac{\partial P^i_{jk}}{\partial x^l} - \frac{2}{n-1}\delta^i_j\frac{\partial \Phi_k}{\partial x^l}, \ldots \\ &\ldots, \frac{\partial^{m-1}P^i_{jk}}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}} - \frac{2}{n-1}\delta^i_j\frac{\partial^{m-1}\Phi_k}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}}\Bigg), \end{aligned} \tag{7.3}$$

whence

$$\begin{aligned} E^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a} = H^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a}\Bigg(&P^i_{jk}, \frac{\partial P^i_{jk}}{\partial x^l}, \ldots, \frac{\partial^{m-1}P^i_{jk}}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}}; \\ &\Phi_k, \frac{\partial \Phi_k}{\partial x^l}, \ldots, \frac{\partial^{m-1}\Phi_k}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}}\Bigg). \end{aligned} \tag{7.4}$$

Let us now suppose that $E^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a}$ is also invariant under any transformation (4.1) of the connection Γ^i_{jk}. Since such a transformation leaves P^i_{jk} invariant, while it changes, under a suitable choice of φ_k, the vector Φ_k into another *arbitrary* covariant vector, or *also into the zero vector*, the functions $H^{\cdot\cdot\cdot s_1\cdot\cdot\cdot s_b}_{r_1\cdots r_a}$ must be *independent* of the arguments $\Phi_k, \frac{\partial \Phi_k}{\partial x^l}, \ldots, \frac{\partial^{m-1}\Phi_k}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}}$, and then, these functions must contain the arguments $P^i_{jk}, \frac{\partial P^i_{jk}}{\partial x^l}, \ldots, \frac{\partial^{m-1}P^i_{jk}}{\partial x^{l_1}\ldots\partial x^{l_{m-1}}}$ alone. But a differential invariant of the connection Γ^i_{jk} expressible in terms of the connection P^i_{jk} *is a differential invariant of this connection* P^i_{jk}. Conversely, anyone of these invariants is obviously also a differential invariant of the connection Γ^i_{jk} under the transformations (4.1), that is, of the direction displacement, q. e. d.

All the differential invariants of a linear displacement of directions which have been found by J. M. Thomas and Eisenhart (loc. cit.) may easily be expressed in terms of P^i_{jk}. In my cited memoir (T.C.P.) I have given the expressions and some properties of these and other tensors, all of which will be omitted here.

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[17] By some authors such an invariant would be said to be of the $(m-1)$th order. But it seems to be more suitable, according to the geometrical meaning, to consider the torsion tensor, for instance, as being a *differential invariant* of the first order and so on.

8. **Affinely connected manifolds which admit a field of vectors parallel to an arbitrary vector.** In the present paper, I shall only indicate a result concerning a single invariant, the tensor

$$(8.1)\quad L_{lkj}^{\ \ \ i} = \frac{\partial L_{jl}^{i}}{\partial x^k} - \frac{\partial L_{jk}^{i}}{\partial x^l} + L_{jl}^{h} L_{hk}^{i} - L_{jk}^{h} L_{hl}^{i} = R_{lkj}^{\ \ \ i} - \frac{1}{n}\delta_j^i R_{lkh}^{\ \ \ h}$$

already introduced by J. M. Thomas ($\mathfrak{Z}_{jlk}^{i}$: A. D., p. 668). This result is the following:

The vanishing of the tensor $L_{lkj}^{\ \ \ i}$ for an affine connection Γ_{jk}^{i} (the L_{jk}^{i} being given by (5.12)) *is a necessary and sufficient condition that the parallel displacement of directions arising from the connection Γ_{jk}^{i} be an integrable displacement*; that is, *that there exist a field of contravariant vectors parallel to an arbitrary vector.*

In fact, the condition that the parallelism for an affine connection Γ_{jk}^{i} be integrable is obviously that any vector at a given point, and that arising from it by the equipollent displacement round any infinitesimal circuit, *have proportional components.* It is easily seen that this condition can be expressed as follows:

$$(8.2)\qquad \delta_{rl}^{uv} R_{ihk}^{\ \ \ \ r} + \delta_{rk}^{uv} R_{ihl}^{\ \ \ \ r} = 0, \qquad (\text{or } \sigma_{lk}^{st}\,\delta_{rs}^{uv} R_{iht}^{\ \ \ \ r} = 0)$$

whence, setting $v = k$ and summing, we obtain

$$(8.3)\qquad R_{ihl}^{\ \ \ \ u} = \frac{1}{n}\delta_l^u R_{ihk}^{\ \ \ \ k}, \quad \text{or} \quad L_{ihl}^{\ \ \ \ u} = 0.$$

Conversely, the vanishing of $L_{ihl}^{\ \ \ \ u}$ implies that also the (8.2) are fulfilled.

Besides, the condition that $L_{ihl}^{\ \ \ \ u}$ be a zero tensor is equivalent to the vanishing of the tensor

$$(8.4)\qquad A_{lkj}^{\ \ \ i} = L_{lkj}^{\ \ \ i} + \delta_j^i L_{hlk}^{\ \ \ \ h} = R_{lkj}^{\ \ \ i} + \delta_j^i R_{hlk}^{\ \ \ \ h},$$

because we have also

$$(8.5)\qquad L_{lkj}^{\ \ \ i} = A_{lkj}^{\ \ \ i} - \frac{1}{n}\delta_j^i A_{lkh}^{\ \ \ h}.$$

This tensor $A_{lkj}^{\ \ \ i}$ has been considered (N. R. G., p. 35) by Eisenhart, who has demonstrated that its vanishing is a necessary and sufficient condition that a vector φ_k can be chosen, so that, for the corresponding connection $\overline{\Gamma}_{jk}^{i}$ given by (4.1), the curvature tensor $\overline{R}_{lkj}^{\ \ \ i}$ be zero. It suffices, indeed, to put

$$(8.5)\qquad \varphi_k = -\frac{1}{2n}\left(\gamma_k + \frac{\partial \sigma}{\partial x^k}\right),$$

where γ_k is a covariant vector such that its curl is $R_{ik\dot{l}}{}^{l}$, and $\frac{\partial \sigma}{\partial x^k}$ is an arbitrary additive gradient. Then, if $\Delta_{ikj}{}^{i} = 0$, and in this case alone, among all affine connections yielding the same parallelism as the given one Γ^i_{jk}, there exists a connection possessing an integrable *equipollence*; but that implies also that the *parallelism* of the given connection is itself an integrable one.

It the conditions (8.3) are satisfied, and in this case alone, we have then in the affinely connected manifold *an integrable direction displacement.* But the *vector* displacement is not, generally, integrable: the vector arising from a given vector by the displacement around an infinitesimal circuit is *proportional* to the given vector, but *it does not have the same length.* In general the manifold has then a *segmentary curvature* (Streckenkrümmung) according to Weyl. The segmentary curvature tensor is $\frac{1}{n} R_{ikh}{}^{h}$. The vanishing of this tensor implies, because of the (8.3), also the vanishing of the curvature tensor $R_{ikj}{}^{i}$; on the other hand, the vanishing of the segmentary curvature means that if an arbitrary *parallel* vector field ξ^i is given, there exists a scalar field φ such that the vector field $\varphi \xi^i$ is a field of *equipollent* vectors. Thus we find again a result due to Eisenhart;[18] who, restricting himself to parallel fields of vectors which can be reduced to equipollent fields by multiplying by a suitable scalar function φ, obtained, as a condition for the existence of *such* a field of vectors parallel to an arbitrary vector, the vanishing of the curvature tensor $R_{ikj}{}^{i}$.

9. **Non-linear displacements of directions.** As has been observed (No. 2) with regard to the *vector displacements,* so also for the *direction displacements* many results of the theory pertaining to the *linear* case can be extended to the *non-linear* one. H. Friesecke has also indicated the general form of the transformations of a non-linear vector displacement which preserve all displaced directions; in our representation (3.3) (where the (3.4) are assumed to be satisfied) these transformations are

$$\overline{\Gamma}^i_{jk}(x, \xi)\, \xi^j = \Gamma^i_{jk}(x, \xi)\, \xi^j + \lambda_k(x, \xi)\, \xi^i, \tag{9.1}$$

where the $\lambda_k(x, \xi)$ are homogeneous functions of zero degree in the ξ^i but otherwise arbitrary functions of their arguments.

We have also in the present case, as in the linear one, a preferred displacement of vectors, among those yielding the given direction displacement:

$$P^i_{jk}(x, \xi) = \Gamma^i_{jk}(x, \xi) + \frac{2}{n-1}\, \delta^i_j\, \Phi_k(x, \xi), \quad \Phi_k = \Gamma^{\,i}_{[ki]} \tag{9.2}$$

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[18] See: *Fields of parallel vectors in the geometry of paths.* Proceedings Nation. Acad., vol. 8, 1922, pp. 207–212; or N. R. G., pp. 18–22 (and 27–29).

which is uniquely determined by the given displacement of directions. The direction displacement belonging to a given vector displacement $\Gamma^i_{jk}(x, \xi)$ can be represented by the equations

$$(9.3) \qquad \delta^{pq}_{ih}\, \xi^h\, d\xi^i + L^{pq}_{hjk}(x, \xi)\, \xi^h\, \xi^j\, dx^k = 0,$$

where

$$(9.4) \qquad L^{pq}_{hjk}(x, \xi) = \delta^{pq}_{ih}\, \Gamma^i_{jk}(x, \xi).$$

Conversely, if a system

$$(9.5) \qquad \delta^{pq}_{ih}\, \xi^h\, d\xi^i + \Lambda^{pq}_{hjk}(x, \xi)\, \xi^h\, \xi^j\, dx^k = 0,$$

where the $\Lambda^{pq}_{hjk}(x, \xi)$ are homogeneous functions of zero degree in the ξ^i, is arbitrarily given, we may first write the (9.5) in the form (9.3), where

$$(9.6) \qquad L^{pq}_{hjk}(x, \xi) = \Lambda^{pq}_{hjk}(x, \xi) + \left(\frac{\partial \Lambda^{pq}_{ijk}}{\partial \xi^h} + \frac{\partial \Lambda^{pq}_{hik}}{\partial \xi^j}\right) \xi^i + \frac{1}{2} \frac{\partial^2 \Lambda^{pq}_{ilk}}{\partial \xi^h\, \partial \xi^j}\, \xi^i\, \xi^l;$$

then we have that a necessary and sufficient condition that the corresponding system (9.3) be *consistent* is the vanishing of the tensor

$$(9.7) \qquad L^{pq}_{hjk} - \frac{2}{n}\, \delta^{pq}_{ih}\, L^{il}_{jlk};$$

which implies that there exist systems $\Gamma^i_{jk}(x, \xi)$ satisfying the (9.4). Then the direction displacements defined by a *consistent* system (9.5) or (9.3) are not more general than the ones determined by (3.1) or (3.3), just as in the linear case.

In a set of vector displacements (3.3) yielding the same displaced directions *some linear ones* may be contained. This implies also that the corresponding displacements of directions is a *linear* one. *A necessary and sufficient condition for this is the vanishing of the tensor*[19]

$$(9.8) \qquad V_{ljk}^{\cdot\cdot\cdot i} = \frac{\partial P^i_{jk}}{\partial \xi^l}.$$

[19] This result has been given, in a somewhat different form, by H. Friesecke (V. R. M., p. 107). Instead of our P^i_{jk}, $V_{ijk}^{\cdot\cdot\cdot i}$ the author makes use of $L^i_{jk} = \Gamma^i_{jk} - \frac{1}{n}\delta^i_j \Gamma^l_{lk} = \frac{2}{n} L^{il}_{ljk}$, $W_{ijk}^{\cdot\cdot\cdot i} = \frac{\partial L^i_{jk}}{\partial \xi^l}$. (He writes F^i_{jk}, $L^i_{k,lj}$ instead of L^i_{jk}, $W_{ijk}^{\cdot\cdot\cdot i}$). The expressions of L^i_{jk}, $W_{ijk}^{\cdot\cdot\cdot i}$ in Friesecke's paper are more complicated than the above indicated ones; moreover the author has not observed that $L^i_{ip} = 0$ (so that his demonstration on p. 108 is slightly incorrect); nor that there exists (provided $W_{ijk}^{\cdot\cdot\cdot i} = 0$) a *preferred choice* of the "in der Richtungsübertragung enthaltenen linearen Übertragung", our P^i_{jk}.

The geometrical meaning is obvious; *the preferred displacement* P^i_{jk} *must then be itself a linear one.*

10. **Linear displacements of points (projective connections).** The foregoing developments may be extended to the theory of the *point displacements.* We shall first consider the *linear* point displacements (projective connections).

Let again x^i $(i, j, h, k, l, \cdots = 1, 2, \cdots, n)$ be curvilinear coördinates in an X_n, let x^0 be an independent parameter. Let us define[20] as *components of a projective tensor* in the X_n the sets of functions of the variables $x^0, x^1, x^2, \cdots, x^n$ (but containing x^0 only in a common factor e^{Nx^0}, N being the *weight* of the projective tensor) which under the transformations

$$(10.1) \qquad \begin{cases} x'^i = f^i\,(x^1, x^2, \cdots, x^n), & (i = 1, 2, \cdots, n), \\ x'^0 = x^0 + \log \varphi\,(x^1, x^2, \cdots, x^n), \end{cases}$$

transform like the components of an affine tensor in the X_{n+1} of the variables $x^0, x^1, x^2, \cdots, x^n$. Then a *projective* contravariant vector field y^α $(\alpha, \beta, \gamma, \cdots = 0, 1, 2, \cdots, n)$ in the X_n is an *affine* contravariant vector field (but only for the transformations (10.1)) in the X_{n+1}; *it can also be considered as a field of points in the projective tangential spaces associated to the points of the* X_n.

Let now $\Gamma^\alpha_{\beta\gamma}$ be the coefficients of a *projective connection* in the X_n. Geometrically the projective connection is defined by a *law of displacement for the projective contravariant vectors*:

$$(10.2) \qquad dy^\alpha + \Gamma^\alpha_{\beta\gamma}\, y^\beta\, dx^\gamma = 0,$$

that we may consider also, by interpreting the y^α as the projective coördinates of a point in the tangential space, as being *a law of point displacement.* This is, substantially, the point of view of Cartan,[21] and it will now be also our point of view.

We may suppose, as is well known,[22] without loss of generality, that

$$(10.3) \qquad \Gamma^i_{0j} = \delta^i_j$$

[20] Following Veblen, G. P. G., p. 147.

[21] Cartan, *Sur les variétés à connexion projective,* Bulletin Soc. Mathém. de France, t. 52, 1924, pp. 205–241. In fact, as H. Weyl has observed (*On the foundations of general infinitesimal geometry,* Bull. Amer. Math. Soc., vol. 35 (1929), pp. 716–725, p. 719), only through the researches of Veblen and others in Princeton have the necessary foundations been laid for a theory of projective point displacements, according to the geometrical views of Cartan.

[22] See, e. g., Cartan, loc. cit., p. 212.

and then, if the points of the tangential projective space are identified with the projective contravariant vectors *of weight* -1, we must assume that[23]

$$\Gamma^{\alpha}_{\beta 0} = \delta^{\alpha}_{\beta}. \tag{10.4}$$

The (10.2) have a signification independent of an arbitrary change (10.1) of the coördinates x^i and the parameter x^0 under the conditions

$$\Gamma'^{\varrho}_{\tau\sigma} = \Gamma^{\alpha}_{\beta\gamma}\, u^{\beta}_{\tau}\, u^{\gamma}_{\sigma}\, v^{\varrho}_{\alpha} + u^{\alpha}_{\tau\sigma}\, v^{\varrho}_{\alpha}, \tag{10.5}$$

where

$$u^{\alpha}_{\tau} = \frac{\partial x^{\alpha}}{\partial x'^{\tau}}, \qquad u^{\alpha}_{\tau\sigma} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\tau}\, \partial x'^{\sigma}}; \qquad v^{\varrho}_{\alpha} = \frac{\partial x'^{\varrho}}{\partial x^{\alpha}}. \tag{10.6}$$

Since we have, because of the (10.1),

$$\begin{aligned} u^0_0 = v^0_0 = 1, \qquad & u^i_0 = v^i_0 = 0, \\ u^0_j = \frac{\partial \log \varphi}{\partial x'^j}, \qquad & v^0_j = -\frac{\partial \log \varphi}{\partial x^j}, \end{aligned} \tag{10.7}$$

the equations (10.3), (10.4) have *an invariant signification* under the transformations (10.5).

The (10.2) define, in the X_{n+1} of the variables $x^0, x^1, x^2, \cdots, x^n$, an *affine connection*. But the given point displacement represented by the equations (10.2) *does not wholly determine this affine connection*; since the functions

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + 2\,\delta^{\alpha}_{\beta}\,\varphi_{\gamma}, \tag{10.8}$$

where φ_i is an affine covariant vector and $\varphi_0 = 0$ (whence φ_γ is a projective covariant vector) determine, if substituted for the $\Gamma^{\alpha}_{\beta\gamma}$ in (10.2) *the same point displacement*, but *another affine connection*. However, these affine connections $\Gamma^{\alpha}_{\beta\gamma}$ and $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ yield *the same displaced directions*; hence *to the given linear point displacement in X_n corresponds a uniquely determined linear direction displacement in X_{n+1}*. Then the theory of the linear point displacements for X_n do not differ by a theory of linear direction displacements in an X_{n+1}.

In particular, we can represent a given projective connection by means of the sets of functions

$$L^{\lambda\mu}_{\alpha\beta\gamma} = \delta^{\lambda}_{\nu(\alpha}{}^{\mu}\, \Gamma^{\nu}_{\beta)\gamma} \tag{10.9}$$

[23] In fact, when the coördinates x^i are unchanged and x^0 has an increment dx^0, the y^α must become multiplied by a factor e^{-dx^0}, that is (only the terms of order ≤ 1 being considered) by $1 - dx^0$. On the other hand, we have in this case $dy^\alpha = -\Gamma^{\alpha}_{\beta 0} y^\beta dx^0$, hence we must have $\Gamma^{\alpha}_{\beta 0} y^\beta = y^\alpha$, $\Gamma^{\alpha}_{\beta 0} = \delta^{\alpha}_{\beta}$.

or, more suitably, by means of the *normalized parameters*

$$P^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \frac{2}{n}\,\delta^{\alpha}_{\beta}\,\Gamma^{\delta}_{[\gamma\delta]}. \tag{10.10}$$

The $P^{\alpha}_{\beta\gamma}$ under an arbitrary change (10.1) of the x^0, x^i have *the same law of transformation* (10.5) as the $\Gamma^{\alpha}_{\beta\gamma}$; the $L^{\lambda\mu}_{\alpha\beta\gamma}$ undergo a transformation represented by the (5.b), where the *Latin* indices are replaced by the *Greek* ones, and the summations are extended from 0 to n. The $L^{\lambda\mu}_{\alpha\beta\gamma}$, as well as the $P^{\alpha}_{\beta\gamma}$, as against the $\Gamma^{\alpha}_{\beta\gamma}$, are *wholly determined by the linear point displacement* of which they give a representation. In particular, *there exists a linear vector displacement uniquely determined by a given linear point displacement*; the differential invariants of this point displacement are the differential invariants of the associated vector displacement, and these alone.

The preceding result concerning $P^{\alpha}_{\beta\gamma}$ is very simply related to the observation, due to Cartan (by whom it is expressed in a somewhat different form, loc. cit., p. 213), that a projective connection can be univocally represented also by the sets of functions

$$\mathfrak{L}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \delta^{\alpha}_{\beta}\,\Gamma^{0}_{0\gamma}. \tag{10.11}$$

For these functions we should also substitute the following ones:

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \frac{1}{n+1}\,\delta^{\alpha}_{\beta}\,\Gamma^{\delta}_{\delta\gamma}. \tag{10.12}$$

In fact, the $\mathfrak{L}^{\alpha}_{\beta\gamma}$, $L^{\alpha}_{\beta\gamma}$ are easily seen to be *invariant under any transformation* (10.8) *of the* $\Gamma^{\alpha}_{\beta\gamma}$. Then we have also

$$\mathfrak{L}^{\alpha}_{\beta\gamma} = P^{\alpha}_{\beta\gamma} - \delta^{\alpha}_{\beta}\,P^{0}_{0\gamma}, \qquad L^{\alpha}_{\beta\gamma} = P^{\alpha}_{\beta\gamma} - \frac{1}{n+1}\,\delta^{\alpha}_{\beta}\,P^{\delta}_{\delta\gamma}, \tag{10.13}$$

whence we obtain

$$P^{\alpha}_{\beta\gamma} = \mathfrak{L}^{\alpha}_{\beta\gamma} + \frac{2}{n}\,\delta^{\alpha}_{\beta}\,\mathfrak{L}^{\delta}_{[\gamma\delta]} = L^{\alpha}_{\beta\gamma} + \frac{1}{n}\,\delta^{\alpha}_{\beta}\,L^{\delta}_{\gamma\delta}. \tag{10.14}$$

Thus the $\mathfrak{L}^{\alpha}_{\beta\gamma}$ and also the $L^{\alpha}_{\beta\gamma}$ are *sufficient to determine the* $P^{\alpha}_{\beta\gamma}$ *and then, the projective connection*. But the $\mathfrak{L}^{\alpha}_{\beta\gamma}$ and $L^{\alpha}_{\beta\gamma}$, as against the $P^{\alpha}_{\beta\gamma}$, *do not have the same law of transformation* (10.5) as the $\Gamma^{\alpha}_{\beta\gamma}$, that is, that of the coefficients of an affine connection in an X_{n+1}.

Cartan has also observed (loc. cit., p. 211) that we can assume (without loss of generality), under a suitable choice of the projective coördinate systems in the projective tangential spaces, that we have

$$\Gamma^{i}_{i\gamma} - n\,\Gamma^{0}_{0\gamma} = 0, \qquad \text{(that is, } L^{\alpha}_{\beta\gamma} = \mathfrak{L}^{\alpha}_{\beta\gamma}, \text{ or } \mathfrak{L}^{\delta}_{\delta\gamma} = 0). \tag{10.15}$$

It is easily seen that *a necessary and sufficient condition that the equations* (10.15) *have a meaning independent of the choice of the coördinates* x^i *and of the parameter* x^0 *is that we have, in the* (10.1)

$$\log \varphi = -\frac{1}{n+1} u. \tag{10.16}$$

where u is given by (5.d). To that assumption also Veblen has been led, in a cited paper (P. T.) by a quite different way.

Let us now assume that the (10.15), (10.16) are true; then we obtain easily that the condition that the projective connection $\Gamma^{\alpha}_{\beta\gamma}$ be (following Cartan) a *normal* one is that we have

$$P^{\alpha}_{ij} = P^{\alpha}_{ji}, \quad P^{0}_{ij} = \frac{1}{n-1}\left(-\frac{\partial P^{h}_{ij}}{\partial x^{h}} + P^{h}_{kj} P^{k}_{ih}\right). \tag{10.17}$$

Let Γ^{i}_{jk} be the coefficients of an *affine* connection, let $B^{i}_{jk} = \Gamma^{i}_{(jk)} = \frac{1}{2}(\Gamma^{i}_{jk} + \Gamma^{i}_{kj})$ (*associated symmetric connection*). Then *the normalized coefficients* $P^{\alpha}_{\beta\gamma}$ *of the* (*uniquely determined*) *projective normal connection that has the same paths as* Γ^{i}_{jk} — the T. Y. Thomas projective connection — *are*[24]

$$\tag{10.18} \begin{cases} P^{i}_{jk} = B^{i}_{jk} - \dfrac{1}{n+1}(\delta^{i}_{j} B^{l}_{lk} + \delta^{i}_{k} B^{l}_{lj}), \quad P^{i}_{0j} = \delta^{i}_{j}, \quad P^{0}_{0j} = 0, \\ P^{\alpha}_{\beta 0} = \delta^{\alpha}_{\beta}, \quad P^{0}_{ij} = \dfrac{1}{n-1}\left(-\dfrac{\partial P^{h}_{ij}}{\partial x^{h}} + P^{h}_{kj} P^{k}_{ih}\right). \end{cases}$$

The condition that a projective connection (linear point displacement) $\Gamma^{\alpha}_{\beta\gamma}$ be *integrable* is not the vanishing of the projective tensor

$$R_{\alpha\beta\gamma}^{\cdots\delta} = \frac{\partial \Gamma^{\delta}_{\gamma\alpha}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\delta}_{\gamma\beta}}{\partial x^{\alpha}} + \Gamma^{\delta}_{\tau\beta}\Gamma^{\tau}_{\gamma\alpha} - \Gamma^{\delta}_{\tau\alpha}\Gamma^{\tau}_{\gamma\beta} \tag{10.19}$$

(curvature tensor, in the X_{n+1} of the variables $x^0, x^1, \cdots, x^n$, of the affine connection $\Gamma^{\alpha}_{\beta\gamma}$); it is, on the contrary, (compare no. 8) the vanishing of the projective tensor

$$\tag{10.20} \begin{aligned} L_{\alpha\beta\gamma}^{\cdots\delta} &= \frac{\partial L^{\delta}_{\gamma\alpha}}{\partial x^{\beta}} - \frac{\partial L^{\delta}_{\gamma\beta}}{\partial x^{\alpha}} + L^{\delta}_{\tau\beta} L^{\tau}_{\gamma\alpha} - L^{\delta}_{\tau\alpha} L^{\tau}_{\gamma\beta} \\ &= R_{\alpha\beta\gamma}^{\cdots\delta} - \frac{1}{n+1}\delta^{\delta}_{\gamma}\left(\frac{\partial \Gamma^{\sigma}_{\sigma\alpha}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\sigma}_{\sigma\beta}}{\partial x^{\alpha}}\right) \\ &= P_{\alpha\beta\gamma}^{\cdots\delta} - \frac{1}{n+1}\delta^{\delta}_{\gamma}\left(\frac{\partial P^{\sigma}_{\sigma\alpha}}{\partial x^{\beta}} - \frac{\partial P^{\sigma}_{\sigma\beta}}{\partial x^{\alpha}}\right), \end{aligned}$$

[24] Compare T. Y. Thomas, A. M., p. 726.

where $P_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}$, in the X_{n+1} of the variables $x^0, x^1, \cdots, x^n$, is the curvature tensor of the normalized affine connection $P_{\beta\gamma}^{\alpha}$. In particular, for a *projective normal* connection, *the condition that it be integrable is also the vanishing of* $P_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}$.

Some further developments of my researches on this subject will be exposed in a forthcoming Memoir.

11. **Non-linear point displacements.** It is now superfluous to detail the analogous theory for the non-linear point displacements in an X_n. It will be reduced to a theory of non-linear direction displacements in an X_{n+1}, so that the differential invariants of such a displacement, represented by a set of equations of the form

$$dy^\alpha + \Gamma_{\beta\gamma}^{\alpha}(x, y) y^\beta \, dx^\gamma = 0, \tag{11.1}$$

(where the $\Gamma_{\beta\gamma}^{\alpha}(x, y)$ are homogeneous functions of zero degree in the y^α) are also the differential invariants of a uniquely determined non-linear vector displacement in an X_{n+1}.

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A THEOREM ON GRAPHS.[1]

By Hassler Whitney.

I. Results of this paper.

1. Let a finite number of curves, or edges, whose end-points we call vertices, intersect at no other points than these vertices. Let the system be connected, that is, any two vertices are joined by a succession of edges, each two successive edges having a vertex in common. This forms a graph. A graph is planar if it can be mapped in a 1–1 continuous manner on a plane (or a sphere). If the vertices a, b are joined by an edge, we shall call the edge joining them ab, and shall say a touches b for short. A set of distinct vertices, $a, b, c, \cdots, e, f$, together with a set of distinct edges joining them in cyclic order, $ab, bc, \cdots, ef, fa$, we shall call a circuit.

A planar graph lying on the surface of a sphere divides this surface into a number of simply connected regions. The boundary of each of these regions may be a circuit. If so, we shall call these circuits elementary polygons. If all these polygons are n-gons, n fixed, we say the graph is composed of elementary n-gons.

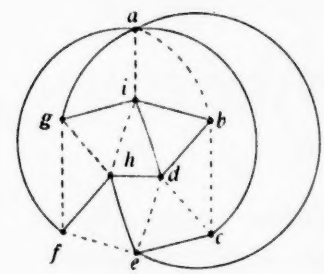

Fig. 1.

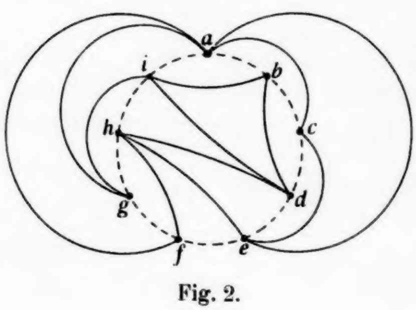

Fig. 2.

2. The fundamental theorem of this paper is the following:

Theorem I. *Given a planar graph composed of elementary triangles, in which there are no circuits of 1, 2, or 3 edges other than these elementary triangles, there exists a circuit which passes through every vertex of the graph.*

The problem of finding graphs for which this is so has been studied by several people.[2] This seems to be the first case when a large class of planar graphs has been shown to have this property.

3. This theorem gives immediately the following:

Normal Form. *Given any graph as described in Theorem I, containing*

[1] Received April 7, and July 14, 1930.—Presented to the American Mathematical Society, Febr. 22, 1930.

[2] See St. Laguë, A., Les Réseaux, Mémorial des Sciences Math., fasc. 18, Paris (1926).

n vertices, we can construct a graph homeomorphic with it as follows: Draw a regular polygon of n sides, and draw diagonals, no two of which cross, dividing the inside of the polygon into triangles. Similarly draw circular arcs, no two of which cross, dividing the outside of the polygon into circular triangles.

We have merely to find the circuit given by Theorem I, and distort it into the polygon.

4. A theorem on maps deducible immediately from Theorem I is the following, as we shall see later:

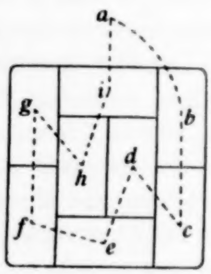

Fig. 3.

THEOREM II. *Given a map on the surface of a sphere containing at least three regions in which:*

(A_1) *The boundary of each region is a single closed curve without multiple point,*

(B) *Exactly three boundary lines meet at each vertex,*

(A_2) *No pair of regions taken together with any boundary lines separating them form a multiply connected region,*

(A_3) *No three regions taken together with any boundary lines separating them form a multiply connected region,*

we may draw a closed curve which passes through each region of the map once and only once, and touches no vertex.

5. By means of Theorem II and a lemma to be proved, we have a solution of a conundrum, which we leave to the end of the paper.

6. Finally, Theorem I gives us a new statement of the four color map problem. Given any map on the surface of a sphere, we "color" it by assigning to each region a color in such a way that no two regions with a common boundary are of the same color. Given any polygonal configuration as described in 3., we "color" it by assigning to each vertex of the polygon a color in such a way that no two vertices which are joined by a line, either a side of the polygon or a diagonal, are of the same color.

EQUIVALENT STATEMENT OF THE FOUR COLOR MAP PROBLEM. *If every polygonal configuration as described in 3. can be colored in four colors, then every map on the surface of a sphere can be colored in four colors, and conversely.*

2. Proof of Theorem I.

We consider only the graphs defined in § 1, 2. As the graph is composed of elementary triangles, there are at least three vertices present. If there are only three, the theorem is obvious. We shall assume from here on that there are at least four vertices present.

As there are no circuits of one or two edges, no vertex touches itself, and any two vertices are joined by at most a single edge.

There is no vertex touching but a single other vertex. For then the boundary of the region surrounding this other vertex would not be a circuit, and therefor not an elementary triangle.

There is no vertex touching only two others. For suppose a touched b and c alone. Then the edges ab and ac would each be sides of two triangles, whose third sides are both edges bc. But there is only one edge bc, as two vertices are joined by at most a single edge. The two triangles thus cover the whole surface of the sphere, and there are thus only three vertices in the graph, contrary to hypothesis.

Consider a vertex a touching other vertices $b, c, \cdots, f$. We read the edges emenating from a in a counter-clockwise sense, and say, a touches $b, c, \cdots, f$ in cyclic order; or, a touches b, next $c, \cdots$, next f, next b.

Remembering now that the graph is composed of elementary triangles, we have the three properties:

(α) *Each vertex touches at least three other vertices in cyclic order, distinct from each other and distinct from the first,*

(β) *If a touches b and next c, then b touches c and next a,*

(γ) *There are no triangles other than elementary triangles.*

These properties, together with the fact that the graph lies on a sphere, is all we need to prove the following lemma, from which the theorem is deduced.

Lemma. *Consider a circuit R in a graph of the type considered in Theorem* I, *together with the vertices and edges on one side, which we shall call the inside. Let A and B be two distinct vertices of R, dividing R into the two parts R_1 and R_2, in each of which we include both A and B. Suppose*

(1) *No pair of vertices of R_1 touch each other inside R (are joined by an edge which lies inside R), and*

(2) *Either no pair of vertices of R_2 touch each other inside R, or else there is a vertex C in R_2 distinct from A and B, dividing R_2 into the two parts R_3 and R_4, in each of which we include C, such that no pair of vertices of R_3 and no pair of vertices of R_4 touch each other inside R.*

Then we can draw a line from A to B, passing only along edges of and inside R, and passing through each vertex of and inside R once and only once.

In brief, if we can divide the circuit R into either two or three parts, such that in any part, including end vertices, no pair of vertices touch each other inside R, we can then draw the required curve from any one end vertex to any other end vertex of these parts.

The theorem is an immediate consequence of the lemma. For consider any elementary triangle of the graph, containing the vertices A, B, C, which we call the circuit R. The rest of the graph we call the inside of the circuit. As each pair of vertices of R touch as a part of the

circuit, and any two vertices are joined by at most one edge, it follows that no pair of them touch inside R. Thus the conditions of the lemma are fulfilled, and we can pass from A to B through every vertex of R and every vertex inside R, that is, through every vertex of the graph. We now pass from B directly to A, forming a closed curve. The edges passed over by the curve form the desired circuit.

Proof of the lemma. Assume the lemma is true for all circuits which, with the vertices inside, contain m vertices, $m = 3, 4, \cdots, n-1$. It is obviously true for the case where $m = 3$. We will prove it for all circuits which, with the vertices inside, contain n vertices. Then, by mathematical induction, it is true in general.

Take any circuit R therefore, which, with the vertices inside, contains n vertices. Let the vertices of the circuit be $A, a_1, a_2, \cdots, a_\alpha, B, b_1, b_2, \cdots, b_\beta, C, c_1, c_2, \cdots, c_\gamma, A$, (reading in a clockwise sense). We assume that no pair of the vertices $A, a_1, \cdots, a_\alpha, B$, no pair of the vertices $B, b_1, \cdots, b_\beta, C$, (or with C replaced by A, if there is no C), and no pair of the vertices $C, c_1, \cdots, c_\gamma, A$ touch inside the circuit. The vertices $C, c_1, \cdots, c_\gamma$ may be missing from the circuit, as may also the vertices $a_1, \cdots, a_\alpha$ or $b_1, \cdots, b_\beta$. We wish to draw the required curve from A to B.

We will divide the proof into four parts, according to what pairs of vertices of the circuit touch inside the circuit:

Case (1). Some vertex a_q touches a vertex b_r, C, or c_s inside R.

Case (2). There are no edges of the above form, but either B touches a vertex c_s or A touches a vertex b_r inside R.

Case (3). No pairs of vertices of the circuit touch inside the circuit.

Case (4). Some vertex b_r touches a vertex c_s inside R, but there are no edges of other forms between vertices of the circuit inside the circuit.

Case (1). Assume there is an edge of one of the forms $a_q C$, $a_q c_s$. The case where there is no edge as above, but there is an edge of the form $a_q b_r$, is reduced to this case by interchanging the rôles of A and B and of c_s and b_r. Suppose the edge nearest A is $a_i c_k$. If it is $a_i C$, we call C, c_k. The meaning of "nearest A" is obvious. Now either, Case (1a), c_k touches none of the vertices $a_{i+1}, \cdots, a_\alpha, B$, or, Case (1b), a_i touches none of the vertices $C, c_1, \cdots, c_{k-1}$ inside the circuit $c_k, a_i, \cdots, a_\alpha, B, b_1, \cdots, b_\beta, C, c_1, \cdots, c_k$. If c_k is C, the latter condition is satisfied automatically.

Consider Case (1a). We shall draw the required curve in two steps: first from A to c_k, then from c_k to B.

If first, Case (1a$_1$), a_i is not a_1, a_{i-1} exists, and does not touch c_k inside the circuit, as the edge $a_i c_k$ was the edge of this form nearest A. Therefore a_i must touch some vertex in between a_{i-1} and c_k. For if a_i touched a_{i-1} and next c_k, a_{i-1} would touch c_k and next a_i, by (β). Thus a_{i-1} would

touch c_k between a_{i-2} (or A) and a_i, and the edge $a_{i-1}\, c_k$ would therefor be inside the circuit, which it cannot be, again as the edge $a_i c_k$ was the edge of this form nearest A. As a_i touches no vertices of the set $c_{k+1}, \cdots, c_\gamma$, $A, a_1, \cdots, a_{i-1}$ inside the circuit, any vertices it touches between a_{i-1} and c_k must be vertices inside the circuit R. Call them in order $p_1, p_2, \cdots, p_\varphi$. Then, by ($\beta$), a_{i-1} touches p_1, p_1 touches $p_2, \cdots$, and p_φ touches c_k. We have thus formed a circuit $A, a_1, \cdots, a_{i-1}, p_1, \cdots, p_\varphi, c_k, \cdots, c_\gamma, A$. No pair of the vertices $A, a_1, \cdots, a_{i-1}$ touch inside this circuit, as none of the set $A, a_1, \cdots, a_\alpha, B$ touched inside the circuit R. Similarly no pair of the set $c_k, \cdots, c_\gamma, A$ touch inside the circuit. Finally, no pair of the set $a_{i-1}, p_1, \cdots, p_\varphi, c_k$ touch inside the circuit. For suppose for instance p_g touched p_h inside, $h > g$. a_i does not touch p_g and next p_h, as p_g and p_h would then touch as a part of the circuit, and therefor not inside the circuit. Therefor a touches a vertex p_s in between. But then a_i, p_g and p_h form a triangle, with p_s on one side, and other vertices, as A, on the other side, which is therefor not an elementary triangle, in contradiction to (γ). Thus all the conditions of the lemma are satisfied for this circuit, and there are fewer than n vertices in and within the circuit. We can therefor draw a line from A to c_k passing through every vertex of and inside the circuit.

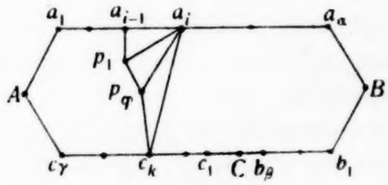

Fig. 4.

If next a_i is a_1, suppose, Case (1a$_2$), c_k is not c_γ. (If the edge nearest A is $a_1\, C$, suppose there is a vertex c_1 in R.) By hypothesis, c_k does not touch A inside the circuit. Therefor a_1 touches vertices between A and c_k. For otherwise, a_1 would touch A and next c_k, and therefor A would touch c_k and next a_1, by (β). But as c_k is not c_γ, A would touch c_k between c_γ and a_1, and the edge Ac_k would be inside the circuit. As a_1 does not touch $c_{k+1}, \cdots, c_\gamma$ inside the circuit, the vertices it touches between A and c_k must be vertices not in R. Call these vertices in order $p_1, \cdots, p_\varphi$. We get thus a circuit $A, p_1, \cdots, p_\varphi, c_k, \cdots, c_\gamma, A$. No pair of the set of vertices $c_k, \cdots, c_\gamma, A$ touch inside the circuit, nor do any of the set A, $p_1, \cdots, p_\varphi, c_k$, using exactly the same reasoning as in Case (1a$_1$). Thus the lemma applies to this circuit, and we pass from A to c_k, passing through every vertex of and inside the circuit.

In each of the two Cases (1a$_1$) and (1a$_2$) we have now passed through every vertex of and inside R which is on A's side of the edge $a_i\, c_k$. For consider the circuit $a_i, c_k, p_\varphi \cdots, p_1, a_{i-1}, a_i$, (or with a_{i-1} replaced by A, if a_i is a_1). As a_i touches every other vertex of the

circuit, there can be no vertices inside the circuit. For if there were a vertex d inside the circuit, it must then lie inside one of the triangles a_i, p_1, a_{i-1} (or A), a_i, or a_i, p_2, p_1, a_i, or $\cdots$, or a_i, c_k, p_φ, a_i. In any case, (γ) would be violated. We have thus only to pass from c_k to B on B's side of the edge $a_i c_k$, that is, through the circuit c_k, a_i, $\cdots$, a_α, B, b_1, $\cdots$, b_β, C, c_1, $\cdots$, c_k.

We have still to consider in Case (1a) the Case (1a$_3$), where the edge nearest A was the edge $a_1 c_\gamma$ (or $a_1 C$, when there is no c_γ). Draw a line directly from A to c_γ (or C). As there are no vertices inside the circuit A, a_1, c_γ, A (or A, a_1, C, A) by (γ), we have left to pass through only vertices of and inside the same circuit as in Cases (1a$_1$) and (1a$_2$).

But we can do this, by the lemma. For, no pair of the set a_i, $\cdots$, a_α, B touch inside the circuit. Also, c_k touches none of these vertices inside the circuit, by the hypothesis of Case (1a). Therefor none of the vertices c_k, a_1, $\cdots$, a_α, B touch inside the circuit. Nor do any of the set B, b_1, $\cdots$, b_β, C, or any of the set C, c_1, $\cdots$, c_k, (if these are present), by the original hypotheses. The circuit is thus divided into two or three parts, depending on whether c_k is C or not, and the lemma applies in either case. We thus pass from c_k to B, completing the required curve from A to B. This disposes of Case (1a).

Consider Case (1b), where a_i touches none of the vertices C, c_1, $\cdots$, c_{k-1}, inside the circuit (if any are present). In this case, instead of passing from A to c_k through all the vertices of and inside the circuit A, a_1, $\cdots$, a_i, c_k, $\cdots$, c_γ, A, except a_1, the same steps show we can pass from A to a_i through every vertex of and inside this circuit except c_k. We now apply the lemma to pass from a_i to B. For, no pair of vertices of the set C, c_1, $\cdots$, c_k touch inside the circuit a_i, $\cdots$, a_α, B, b_1, $\cdots$, b_β, C, c_1, $\cdots$, c_k, a_i, and a_i touches none of these vertices inside the circuit; therefor none of the set C, c_1, $\cdots$, c_k, a_i touch inside the circuit. Also, no pair of the set a_i, $\cdots$, a_α, B, and no pair of the set B, b_1, $\cdots$, b_β, C touch inside the circuit. The proof for Case (1) is now complete.

Case (2). Suppose B touches a vertex c_s inside the circuit. Of all such vertices, let the one nearest A be c_k. Exactly as we before passed from A to c_k, going through all the vertices on A's side of the edge $a_i c_k$, we now pass from A to c_k, going through all the vertices on A's side of the edge Bc_k. We have now only to pass from c_k to B, going through all the vertices on the other side of the edge Bc_k. But we can do this, by the lemma. For the vertices c_k, B do not touch inside the circuit c_k, B, b_1, $\cdots$, b_β, C, c_1, $\cdots$, c_k. Also, no vertices of the set B, b_1, $\cdots$, b_β, C, and no vertices of the set C, c_1, $\cdots$, c_k touch inside the circuit.

The proof is the same if A touches some vertex b_r inside the circuit.

Case (3). No vertices of the circuit touch inside the circuit. As any circuit contains at least three vertices, there is at least one other vertex besides A and B in the circuit. Thus if we call the vertices of the circuit $A, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_\beta, A$, either a_1 or b_β, say b_β is present. Draw a line from A to b_β. We have still to pass from b_β to B.

Suppose, Case (3a), a_1 is also present in the circuit. As a_1 and b_β do not touch inside the circuit, A does not touch b_β and next a_1, and A touches therefor other vertices in between. Calling these in order $p_1, \cdots, p_\varphi$, we have a circuit $b_\beta, p_1, \cdots, p_\varphi, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_\beta$, where at least b_β, p_1, a_1 and B are present. The lemma applies to this circuit. For, no pair of the vertices $b_\beta, p_1, \cdots, p_\varphi, a_1$, no pair of the vertices $a_1, \cdots, a_\alpha, B$, and no pair of the vertices $B, b_1, \cdots, b_\beta$ touch inside the circuit. There are no vertices inside the circuit $b_\beta, A, a_1, p_\varphi, \cdots, p_1, b_\beta$, as A touches all the other vertices of this circuit.

Suppose now, Case (3b), a_1 is not present in the circuit, but $b_1 \neq b_\beta$ is. Then, as B does not touch b_β inside the circuit, A touches vertices between b_β and B, and we obtain the circuit $b_\beta, p_1, \cdots, p_\varphi, B, b_1, \cdots, b_\beta$, to which the lemma applies. For, no pair of the vertices $b_\beta, p_1, \cdots, p_\varphi, B$, and no pair of the vertices $B, b_1, \cdots, b_\beta$ touch inside the circuit.

Consider now Case (3c), where the circuit R consists only of the vertices $A, B, b_1 = b_\beta, A$. If there are no vertices inside the circuit, we pass directly from b_β to B. If there are vertices inside the circuit, A touches vertices between b_β and B, and we obtain the circuit $b_\beta, p_1, \cdots, p_\varphi, B, b_\beta$, to which the lemma applies, as in Case (3b).

Case (4). No pair of vertices of the circuit R touch inside except for edges of the form $b_r c_s$. Of all such edges, let the one furthest from the vertex C be the edge $b_j c_k$. We will carry through the proof for this case in three steps:

(1) A chain of vertices $p_1, \cdots, p_\varphi$ with the edges joining them stretching from b_j to A and to a_1 (or to B, if there is no a_1), will be found.

(2) A subset of these vertices with the edges joining them will form another chain, $q_1, \cdots, q_\theta$.

(3) The required curve will be drawn from A to b_j on A's side of this latter chain, and from b_j to B on the other side of the chain.

(1) *The chain of p's.* As b_{j-1} (or B, if there is no b_{j-1}) does not touch c_k, the edge $b_j c_k$ being the one furthest from C, b_j touches a vertex in between, which is inside the circuit R. Call p_1 the vertex b_j touches just before c_k. Then p_1 touches c_k and forms the first vertex of the chain. If p_1 touches A, the first part of the chain is finished. If not, let c_{p_1} be the vertex of the set $c_k, \cdots, c_\gamma$ nearest A which it touches.

Suppose we have constructed the chain as far as the vertex p_i, which does not touch A, and c_{p_i} is the vertex nearest A which p_i touches. Assume the following properties hold:

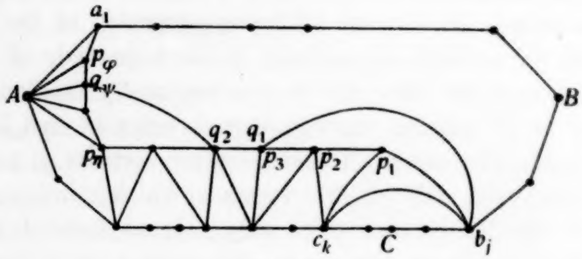

Fig. 5.

(a) All the p's are distinct.

(b) Each p_s, $s < i$, touches the vertex p_{s+1}, and each touches a vertex c_{p_s}.

(c) No p_s touches any of the vertices $c_{p_i}, \cdots, c_\gamma$ inside the circuit $A, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_j, p_1, \cdots, p_i, c_{p_i}, \cdots, c_\gamma, A$.

These properties are seen to hold when we have found the first vertex of the chain, p_1. Having found p_i, we find the next vertex, p_{i+1}, as follows. As p_i does not touch c_{p_i+1}, (or A, if c_{p_i} is c_γ), inside the circuit, c_{p_i} touches a vertex in between. Any such vertex is not a vertex of the circuit R, nor is it any of the vertices $p_1, \cdots, p_i$, by the above assumptions. Call p_{i+1} the vertex c_{p_i} touches next after p_i. If p_{i+1} touches A, the first part of the chain is finished. Otherwise, let $c_{p_{i+1}}$ be the vertex nearest A that p_{i+1} touches (which may be c_{p_i}). Now p_{i+1} is distinct from all former p's, p_i touches p_{i+1}, p_{i+1} touches $c_{p_{i+1}}$, and no vertex $p_1, \cdots, p_{i+1}$ touches $c_{p_{i+1}}$ or any vertex nearer A inside the new circuit. Thus the same properties still hold, and we continue finding vertices of the chain.

We note that, although p_{i+1} touches c_{p_i}, it touches no vertex c_s nearer C than c_{p_i}. Thus if p_i touches c_s, p_j touches c_t, and $j > i$, then $t \geq s$.

We must eventually reach A. For each time a vertex p_i does not touch A, we find a new vertex p_{i+1}, all the vertices p_s are distinct, and there are only a finite number of vertices inside the circuit.

Call the last vertex of this chain p_η. If p_η touches a_1 (or B, if there is no a_1), call it also p_φ. Otherwise, A touches vertices in between, none of which are vertices of the circuit R or of the chain $p_1, \cdots, p_\eta$. Call these in order $p_{\eta+1}, \cdots, p_\varphi$. We now have a chain of vertices $p_1, \cdots, p_\varphi$, stretching from b_j to a_1(or B), each of which touches a vertex c_s or A.

(2) *The chain of q's.* Mark in now any edges there may be joining the vertices $b_j, p_1, \cdots, p_\varphi, a_1\,(B)$ inside the circuit we now have, which includes the p's and B. Call q_1 the vertex of the set $p_1, \cdots, p_\varphi$ nearest $a_1\,(B)$ which b_j touches (which may be p_1). Thus q_1 exists. Having found q_i, if it touches $a_1\,(B)$, we cass call it q_θ. Otherwise, we take as q_{i+1} the vertex of the set $p_1, \cdots, p_\varphi$ nearest $a_1\,(B)$ which q_i touches. Continue

in this manner till we reach $a_1(B)$. Now every vertex q_i touches a vertex c_s or A. Also, no vertices of the set $b_j, q_1, \cdots, q_\theta, a_1(B)$ touch inside the circuit $b_j, q_1, \cdots, q_\theta, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_j$ (where the a's may be missing), on account of the construction of the chain. As, also, no pair of the vertices $a_1, \cdots, a_\alpha, B$, and no pair of the vertices $B, b_1, \cdots, b_j$ touch inside the circuit, we can apply the lemma and draw a line from b_j to B, passing through every vertex of and inside this circuit.

(3.) *The curve.* If there are no vertices q_s touching A, call $a_1(B)$, q_ψ. Otherwise, call the first vertex q_s which touches A, q_ψ. To finish the proof of the lemma, we have only to pass from A to b_j through every vertex on c_k's side of, but not in, the chain $b_j, q_1, \cdots, q_\psi, A$. For if q_ψ is $a_1(B)$, the chains $b_j, q_1, \cdots, q_\psi$ and $b_j, q_1, \cdots, q_\theta, a_1(B)$ are identical, and we have passed through every vertex of and on B's side of the chain in passing from b_j to B. If q_ψ is not $a_1(B)$, consider the circuit $A, a_1(B), q_\theta, \cdots, q_\psi, A$, (where q_ψ may be q_θ). As A touches each of these vertices, there can be no vertices inside the circuit, by (γ). Thus all the vertices we have not passed through on c_k's side of the chain $b_j, q_1, \cdots, q_\theta, a_1(B), A$, are also on c_k's side of the chain $b_j, q_1, \cdots, q_\psi, A$.

We will pass from A to b_j in two steps: first from A to c_k, on A's side of the edge $b_j c_k$, then from c_k to b_j, on C's side of the same edge.

Mark in all edges between the q's and the c's. Remembering that each vertex q_i, $i < \psi$, touches a vertex c_s, and that if q_i touches c_s, q_j touches c_t, and $j > i$, then $t \geqq s$, we see that these edges divide the section of the graph we must pass through into a number of sections, each of which we will pass through in turn.

Suppose q_ψ touches a vertex of the set $c_k, \cdots, c_\gamma$. Call the one nearest A that q_ψ touches c_g. If c_g is c_γ, there are no vertices inside the circuit A, q_ψ, c_γ, A, and we pass directly from A to c_γ. Otherwise, c_g does not touch A inside the circuit, and therefor q_ψ touches other vertices in between. Call these vertices in order $r_1, \cdots, r_\mu$. There are no vertices inside the circuit $A, q_\psi, c_g, r_\mu, \cdots, r_1, A$. Thus we need only pass from A to c_g through all the vertices of and inside the circuit $A, r_1, \cdots, r_\mu, c_g, \cdots, c_\gamma, A$. But we can do this, by the lemma. For, no pair of the vertices $A, r_1, \cdots, r_\mu, c_g$, and no pair of the vertices $c_g, \cdots, c_\gamma, A$ touch inside the circuit.

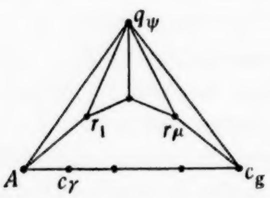

Fig. 6.

If q_ψ touches any more vertices of the set $c_k, \cdots, c_\gamma$, we pass through each of the sections thus formed in turn in exactly the same manner, till we reach the last c that q_ψ touches, c_h.

If the vertex nearest A of the c's that $q_{\psi-1}$ touches is c_i, we must now pass through the section bounded by $c_h, q_\psi, q_{\psi-1}, c_i, \cdots, c_h$.

If q_ψ did not touch any vertex c_s, we would have this section to pass through in the first place, c_h being replaced by A.

If c_i is c_h, this section is a triangle which contains no vertices inside, and we consider the next section. Suppose therefor c_i is not c_h. As then $q_{\psi-1}$ does not touch c_h, q_ψ touches vertices in between, none of which are any of the set $c_i, \cdots, c_h$. We obtain thus a chain of vertices stretching from c_h to $q_{\psi-1}$, of which the last is say d. Similarly, we obtain a chain of vertices stretching from q_ψ to c_i, of which the first is d. As there are no vertices inside the circuit $c_h, q_\psi, q_{\psi-1}, c_i, \cdots, d, \cdots, c_h$, we have only to pass from c_h to c_i through the circuit $c_h, \cdots, d, \cdots, c_i, \cdots, c_h$. We can do this, by the lemma. For, no vertices of the set $c_h, \cdots, d$, none of the set $d, \cdots, c_i$, and none of the set $c_i, \cdots, c_h$ touch inside the circuit.

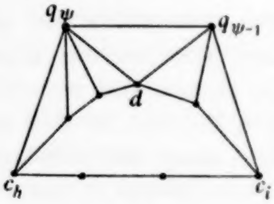

Fig. 7.

We pass in this manner through each section in turn, till we reach c_k. The last section, in particular, is bounded by the vertices $c_f, q_1, b_j, c_k, \cdots, c_f$, where c_f is either c_k or the vertex nearest c_k of the c's that q_1 touches. Thus here, b_j takes the place of what would otherwise be the next q.

We have now but to pass from c_k to b_j on C's side of the edge $b_j c_k$. We can do this, by the lemma. For, the vertices c_k, b_j, no pair of the set $b_j, \cdots, b_\beta, C$, and no pair of the set $C, c_1, \cdots, c_k$ touch inside the circuit thus described.

The proof of the lemma, and therefor of Theorem I, is now complete.

3. Proofs of the theorems on maps.

The dual representation. Given a map on the surface of a sphere, we find the dual representation in the form of a graph as follows. Mark in each region of the map a point, which will be a vertex of the graph, and which we shall call by the same name as the region of the map in which it lies. Across each boundary line of the map draw a line connecting the vertices in the two regions the boundary separates, forming an edge of the graph.

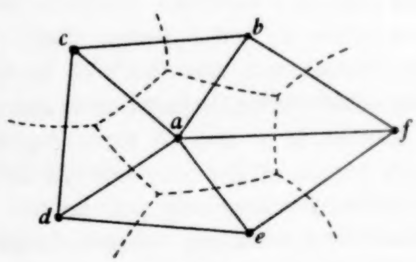

Fig. 8.

Now surrounding each vertex of the map there is a region of the graph bounded by a set of edges.

Proof of Theorem II. We will show first that in any map of the type considered in Theorem II, the dual graph holds to the properties (α), (β) and (γ) of § 2.

Each region of the map is simply connected, on account of (A_1). Each boundary is a boundary between two distinct regions. For suppose there were a boundary line QR running through a single region a. We could then, starting from a point P of QR, move into a on one side of QR, run along a path remaining always in a, and get back to P on the other side of QR. Let us now run around the boundary of a. At some time we pass along the boundary line QR. We are now inside the path we have drawn through a, and as the boundary of a is a closed curve, we must get out again. But we can only get out by passing through P, which contradicts (A_1).

Suppose we run around the boundary of a region a in a counter-clockwise sense. We are on successively sections of the boundary separating a from other regions $b, c, \cdots, f$, in cyclic order. Thus in the dual graph, a touches $b, c, \cdots, f$, in cyclic order, and these vertices are distinct from a.

Suppose a touches b and next c. Then if we pass around the boundary of the region a in a counter-clockwise sense, two successive sections of this boundary will be C, separating a and b, and B, separating a and c. C and B will meet at the vertex V. By (B), only one other boundary line abutts at V. Call it A. It must thus separate the regions b and c. Run now around the boundary of b in a counter-clockwise sense. Two successive sections of this boundary will be A and C. Thus we see that the vertex b touches c and next a, proving property (β).

Suppose now a touches in order $b, c, d, \cdots, f$. These vertices are then all distinct. For consider any two of the vertices a touches, say b and d. If a touches b and next d, or d and next b, then b touches d, and therefor b and d are distinct. Suppose now a touches a vertex c after b and before d, and a vertex f after d and before b. Here again b and d must be distinct, for otherwise the regions a and b would form a multiply connected region, separating c and f, contrary to (A_2).

Except in a map of three regions, for which Theorem II is obvious, each region of the map touches at least three others. For if there were a region touching only one or two others, that region or pair of regions would form a multiply connected region, contrary to (A_1) or (A_2). Thus each vertex of the dual graph touches at least three others. This finishes the proof of property (α).

Finally, there are no triangles in the graph other than elementary triangles. For if there were such a triangle, the regions of the map surrounding it

would form a multiply connected region, contrary to (A_3). The properties (α), (β) and (γ) are now proved.

Now, applying Theorem I to the dual graph, we find a circuit passing through every vertex of the graph. This circuit is the desired closed curve passing through every region of the map.

Proof of the equivalent statement of the four color map problem. Elementary considerations in the four color map problem show that if any map of the type considered in Theorem II can be colored in four colors, then any map on the surface of a sphere can be colored in four colors. We need therefor consider only maps of the above type.

Put the dual graph of such a map in the normal form. Suppose we can color this polygonal configuration in four colors. We then color each region of the map with the same color as the corresponding vertex of the dual graph. Any two regions with a common boundary correspond to two vertices of the graph which are joined by an edge, and are therefor of different colors.

The converse is obvious, as every polygonal configuration is the dual of a map.

Conundrum. Suppose a man, living in a certain country (state), wishes to visit all the countries about him, but does not wish to pass through any country more than once on his voyage. Can he do it? If the region he wishes to visit covers the entire globe, he can do it if the countries make up a map of the type considered in Theorem II. Suppose now the region covers but a portion of the globe. If, upon replacing the rest of the globe by a single country, we obtain a map of the type considered, he can do it also. We have but to apply the lemma to the ring of countries about the added country. By (A_3), no pair of the countries of this ring touch inside the ring. Therefor, picking out any two adjacent countries of the ring, A, B, we draw a line from one to the other, passing through every country the man wishes to visit. We now join the two ends of this line, completing the man's path.

More generally, whenever the conditions of the lemma are satisfied by the ring, calling some two adjacent countries A and B, we obtain the desired path.

4. Further remarks.

Necessity of (A_3). Theorem I would not be true if the assumption that there are no circuits of three edges other than the elementary triangles were omitted. That is, Theorem II would not be true if the assumption (A_3) were omitted. The following example shows this.[3]

[3] This example of such a map containing the least number of regions was communicated to me by C. N. Reynolds.

The number P_n. In constructing the normal form for a graph, we divide an n-sided polygon into triangles by diagonals. It is interesting to know in how many ways we can do this. The formula for this number was found by Euler. A simple proof was first given by Lamé:[4]

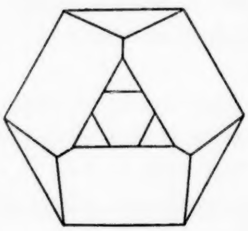

Fig. 9.

$$P_n = 2^{n-3}\,\frac{3\cdot 5\cdot 7\cdots(2n-5)}{3\cdot 4\cdot 5\cdots(n-1)}.$$

As we divide both the inside and outside of the polygon into triangles, we can construct in this manner P_n^2 different figures. Of course these are not all graphs of the type considered, and many of them give the same graph. For instance, there are 96 different circuits in the graph, Fig. 1.

[4] J. Math. Pures Appl. (1), 3 (1838), pp. 505–507.

HARVARD UNIVERSITY.

NOTE ON THE ALEXANDER DUALITY THEOREM.[1]

By Arthur B. Brown.[2]

1. **Introduction.** The theorem below was proved first by use of the Alexander duality theorem and a theorem about overlapping complexes.[3] In this paper we give a treatment which is longer, but depends only on the Alexander theorem.

Notations are as in Lefschetz's Colloquium Publication.[4] For brevity we shall describe independence with respect to homologies as simply independence. A point set D is said to have an ith topological Betti number R_i, finite or transfinite, if there exists a set of R_i independent i-cycles on D, which is maximal. Finite chains are used in testing independence.

The results of the paper hold both for absolute Betti numbers and Betti numbers modulo m, provided m is a prime integer greater than unity. We take the former case, but the same treatment is found to apply to the latter.

2. **Theorem.** *Let S be an n-manifold with the Betti numbers of an n-sphere. Let D be a proper sub-set of the points of S, possessing topological Betti numbers; and K a closed point set on a part of D which is open with respect to S. Then the following relations hold, where $R_{n-i-1}(K)$ denotes the Vietoris Betti number,[5] and the other terms denote topological Betti numbers.*

$$R_i(D-K) = R_i(D) + R_{n-i-1}(K), \qquad i = 0, 1, \cdots, n-1. \tag{1}$$

Proof. We assume that $1 \leq i \leq n-2$. The cases $i = 0$, $n-1$ may be treated by slight modifications of the following proof.

By the K-part of a given i-chain is meant the chain determined by those of its i-cells whose closures contain points of K. All chains are assumed so subdivided that their K-parts lie on the given open part of D.

Given any i-cycle, say $\mathfrak{D}_1^i$, on D, some multiple $k\mathfrak{D}_1^i$, $k \neq 0$, bounds on S. If we add to $k\mathfrak{D}_1^i$ the negative of the boundary of the K-part of a bounded $(i+1)$-chain, we obtain a new i-cycle, say D_1^i, homologous

[1] Received May 3, and October 13, 1930.

[2] National Research Fellow.

[3] A. B. Brown, "An extension of the Alexander duality theorem", Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 407–408.

[4] Solomon Lefschetz, "Topology", New York, 1930. (Lefschetz I.)

[5] See Lefschetz I for references. The "Vietoris Betti number" is as redefined by Lefschetz for absolute Betti numbers.

to $k\mathfrak{D}_1^i$ on D, and having no point on K. By this process we can replace a given maximal set of independent i-cycles on D by a new maximal set, say D^i, whose cycles are on $D-K$.

The Alexander duality theorem, as extended by Alexandroff, Lefschetz and Alexander,[6] tells us that a maximal set of independent i-cycles on $S-K$ exists, $R_{n-i-1}(k)$ in number. Given any one of these i-cycles, some multiple of it bounds on S. The boundary of the K-part of a bounded $(i+1)$-chain can be used to replace the given i-cycle; and in this way we can obtain a maximal set, say E^i, of $R_{n-i-1}(K)$ i-cycles independent on $S-K$, each of which bounds on D. If we can prove that E^i and D^i together form a maximal independent set of i-cycles on $D-K$, it will follow that (1) is correct.

Now if a combination of cycles of D^i and E^i bounded on $D-K$, since every cycle of E^i bounds on D it follows that no cycles of D^i could be involved. Since the cycles of E^i are independent on $D-K$, neither could any cycles of E^i be involved. Hence the cycles of D^i and E^i are independent on $D-K$. It remains to prove that they form a maximal set.

If Q^i is any i-cycle on $D-K$, then because of the various maximal independent sets at hand we know that an integer $r \neq 0$, combinations D_1^i and E_1^i of i-cycles of D^i and E^i respectively, and $(i+1)$-chains T_1^{i+1}, T_2^{i+1}, T_3^{i+1}, exist such that

$$T_1^{i+1} \to rQ^i + D_1^i \quad \text{on} \quad D; \tag{2}$$

$$T_2^{i+1} \to rQ^i + D_1^i + E_1^i \quad \text{on } S-K; \tag{3}$$

$$T_3^{i+1} \to E_1^i \quad \text{on} \quad D. \tag{4}$$

Therefore, if p is an integer, $p(T_1^{i+1} + T_3^{i+1} - T_2^{i+1})$ is an $(i+1)$-cycle, and hence for some $p \neq 0$ this cycle bounds an $(i+2)$-chain, say T^{i+2}, on S. Let Γ^{i+1} denote the boundary of the K-part of T^{i+2}. Then $pT_1^{i+1} + pT_3^{i+1} - \Gamma^{i+1}$ is a chain on $D-K$ bounded by $p(rQ^i + D_1^i + E_1^i)$. It follows that Q^i is dependent, on $D-K$, on cycles of D^i and E^i. Hence D^i and E^i form a maximal independent set of i-cycles on $D-K$, and the proof is complete.

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[6] See Lefschetz I, Chap. VII, for references. The theorem is usually stated with the hypothesis that S has no torsion coefficients, but this condition is unnecessary and not essential to the proof.

PRINCETON UNIVERSITY.

ON TOPOLOGICAL MANIFOLDS.[1]

By William W. Flexner.

A topological manifold M_n is a compact separable space which has a complete set of neighborhoods each of which is a combinatorial n-cell. A combinatorial n-cell is a generalization due to Alexander of the ordinary n-simplex, and has the connectivity numbers and torsion coefficients of the n-simplex. Special cases of topological manifolds have been studied before in analysis situs. The manifolds (variétés) investigated by Poincaré in his first paper[2] are topological manifolds which have a certain restricted parametric representation. Wilson[3] and Hopf[4] have investigated the singular images upon each other of manifolds whose defining neighborhoods are n-cells, but have not considered the questions here dealt with. In this and a subsequent paper it is proposed to extend, using the methods of Veblen's *Colloquium Lectures on Analysis Situs* and Lefschetz's *Colloquium Lectures on Topology*, to topological manifolds the classical duality and homology theorems. This involves defining Betti numbers and torsion coefficients, proving their topological invariance, defining Kronecker Indices and proving the duality theorems.

Vietoris[5] has introduced the homology and group invariants of a general compact metric space. To define bounding and non-bounding cycles he uses infinite sequences of chains made up of ideal cells whose diameter decreases towards zero. Section 1 of this paper defines the homology characters in another way: in terms of a complex on M_n composed of singular chains which play the rôle of cells. By means of a deformation theorem modeled on that due to Alexander[6] it is proved that any singular chain on M_n can be deformed onto the singular complex. Therefore the incidence matrices of the singular complex give topologically invariant homology characters. This method shows that the homology theory of M_n can be derived from a finite singular complex, while for an arbitrary compact metric space the complex must be infinite.

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[1] Received October 3, 1930. Presented to the American Mathematical Society, December 29, 1930.

[2] Poincaré, H., Analysis Situs. Journ. de l'Ec. Polyt. (2) 1 (1895), pp. 1–123.

[3] Wilson, W., Representations of Manifolds. Math. Ann., 100 (1928), pp. 552–578.

[4] Hopf, H., Zur Topologie der Abbildungen von Mannigfaltigkeiten. Math. Ann., 100 (1928), pp. 579–608, and 102 (1929), pp. 562–623.

[5] Vietoris, L. Math. Ann., 97 (1927), pp. 454–472, and 101 (1929), pp. 219–225.

[6] Alexander, J. W. Trans. Am. M. S., 16 (1915), pp. 148–154.

Section 2 begins by specifying when a manifold is orientable with respect to a particular set of defining neighborhoods E^i_n. Then the Kronecker Index $(\gamma_p \cdot \gamma_{n-p})$ of two singular cycles γ_p and γ_{n-p} on M_n is defined. The object of this and the next section is to serve as a basis for a proof of the Poincaré duality theorem for M_n. Since M_n cannot be cut up into cells the relations between p- and $(n-p)$-cycles cannot be obtained from a dual complex. The connectivity properties in the large are in this case brought in by considering M_n as immersed in a Euclidean space S_r of a sufficient number of dimensions,[7] r, and considering the intersection of M_n with chains C^i_{r-p} of S_r bounded by the cycles Γ^i_{n-p-1} in S_r which according to P. Alexandroff[8] link each non-bounding cycle γ^i_p on M_n. This intersection can be proved to be a cycle, γ^i_{n-p}, and to intersect γ^i_p with a Kronecker Index 1, so the duality theorem of Poincaré follows from section 2. The work just outlined has been completed by Lefschetz and Flexner[9] since this paper was first written.

Section 3 contains a proof, suggested to the writer by Professor Alexander, that M_n is homeomorphic to a subspace of Euclidean r-space.

It follows as a corollary of the duality theorem that the connectivity numbers of order higher than n are zero, a result proved in another way by Vietoris.

My thanks are due to Professor Alexander for suggesting the problem here treated to me and to both him and Professor Lefschetz for their very generous help during the course of this work.

1. Definitions; Invariance of the Betti numbers and torsion coefficients. Throughout this paper technical terms are used as in Lefschetz's *Topology*[10].

The manifold M_n here dealt with is a space satisfying two conditions.

1. M_n is a compact separable space. This condition implies that M_n is metric[11].

2. It is further required that there exist a complete set of neighborhoods $\{E^i_n\}$ for M_n each of which is a normal combinatorial n-cell. (L. T., p. 113, see also p. 106).

A normal combinatorial n-cell is an open simplicial complex whose boundary is a circuit with the Betti and torsion numbers of the $(n-1)$-sphere and which is itself the join (L. T., p. 111) of this boundary and

[7] Menger, K., Dimensionstheorie. Berlin (1929).

[8] Alexandroff, P. Annals of Math. (2) 30 (1928), pp. 101-187.

[9] Lefschetz, S., and Flexner, W. W. Proc. Nat. Acad. Sci., 16 (1930), pp. 530-533.

[10] Lefschetz, S., Topology. Am. Math. Soc., Colloquium Publications, Volume XII (1930): referred to in the sequel as "L. T.".

[11] Urysohn, P., Math. Ann., 92 (1924), pp. 275-293.

a point. A property of the cells E_n^i often used is that every cycle on E_n^i bounds. In the sequel the normal combinatorial n-cell will be called simply the n-cell and the simplicial n-cell will be referred to with the prefix simplicial. The simplicial n-cell is a special case of the n-cell so that the class of manifolds M_n includes the type originally called "topological" made up of those compact separable spaces which can be covered by a finite number of overlapping simplicial n-cells.

A finite set of n-cell neighborhoods $\{E_n^i\}$ covering M_n is called the *covering set* $\{E_n^i\}$. Because of the Borel property, finite covering sets exist. All the n-cells of the same covering set which have a point in common with a given n-cell, E_n, of the set will be called a *nest of cells* with E_n as *center*. Each n-cell of the covering of M_n is itself an open simplicial manifold. This follows from the fact that each point of E_n^i has a neighborhood of arbitrarily small diameter which is an n-cell. Thus M_n can be covered by open sets which are simplicial manifolds. This is the essential property in the proof of the duality theorems.

THEOREM 1. *Given a manifold covered by a finite set U of overlapping n-cells, the manifold can be covered by another finite set U' of n-cells each so small that every nest of cells of U' is covered by a cell of U.*

Let $\{E_n^i\}$ be the complete set of neighborhoods of M_n. Each point is covered by an E_n^i of diameter less than ε. Therefore after removal of all neighborhoods of diameter greater than ε there remains a fundamental system of neighborhoods for M_n, of which, because M_n is closed and compact, a finite subset will cover M_n. So for every ε there is a finite covering $\{E'^i_n\}$ of cells whose diameter is less than ε. Given the finite covering $U = \{E_n^i\}$, the distance from any point of M_n to the boundary of some $E_n^i \supset P$ has a lower bound, $\eta > 0$. If then $\varepsilon < \frac{1}{3}\eta$, $\{E'^i_n\}$ is a covering of the sort required in the theorem where $U' = \{E'^i_n\}$. Remove from any covering U any n-cells entirely covered by other cells of U.

Now cover M_n with $n+2$ covering sets U^{-1}, U^0, $U^1, \cdots, U^n$ such that:

i. U^{-1} is an original set covering M_n.

ii. The members of U^j are so small that theorem 1 applies and every nest of cells of U^j is completely covered by one n-cell of U^{j-1}.

Take arbitrarily a single point a_0^i in each n-cell E_n^{ni} of the covering U^n. Each such point is called an *elemental zero-cell* of M_n. The definition of an *elemental k-cell* is by induction. If $E_n^{no}, E_n^{n1}, \cdots, E_n^{nk}$ are $k+1$ n-cells of the covering U^n all contained in the same n-cell $E_n^{n-k,i}$, of the covering U^{n-k} and such that any j of them ($j = 0, 1, \cdots, k$) are covered by an n-cell of the covering U^{n-j+1} the $k+1$ points a_0^i, where

$a_0^i \subset E_n^{ni}$, are the vertices of an oriented (L. T. p. 4) singular k-complex (L. T. p. 73), a_k, on M_n, with the properties:

1. If the n-cell E_n^{n-k+1, i_p} of the covering U^{n-k+1} covers $a_{k-1}^{i_p}$, then a_k is contained in an n-cell $E_n^{n-k, j}$ of the covering U^{n-k}. Such an $E_n^{n-k, j}$ exists because each E_n^{n-k+1, i_p} covers all but one of the cells $E_n^{n0}, \cdots, E_n^{ni_k}$ and hence these cells form a nest and are covered by a cell of U^{n-k}.

2. a_k is bounded by the elemental $(k-1)$-cells $a_{k-1}^{i_0}, \cdots, a_{k-1}^{i_k}$ determined by $E_n^{n0}, \cdots, E_n^{nk}$ taken k at a time. Because the definition is inductive, properties one and two are assumed for $a_{k-1}^{i_0}, \cdots, a_{k-1}^{i_k}$. Therefore the sum of these cells properly oriented is a cycle. Since the $(k-1)$st Betti number, R_{k-1} $(E_n^{n-k, j})$ of $E_n^{n-k, j}$ is zero, this cycle bounds a singular complex not necessarily an n-cell, on $E_n^{n-k, j}$ which can be taken to be a_k. This proves the existence of an a_k satisfying conditions one and two.

The *boundary* of an elemental k-cell is the sum of the oriented elemental $(k-1)$-cells that are obtained by omitting one at a time the $k+1$ vertices defining the elemental k-cell. In the sum each oriented elemental $(k-1)$-cell is affected with a sign as in L. T. (p. 14).

The definitions of *elemental chain, elemental cycle, elemental homology* are the same as those of chain, cycle and homology in L. T. (p. 16 et seq. and p. 21) with "elemental cell" written everywhere instead of "cell".

An elemental complex, $\mathfrak{K}$, on M_n is the set of all oriented elemental cells on M_n that can be constructed using as vertices a set of elemental zero-cells, one and only one in each n-cell E_n^{ni} of the covering U^n. The incidence relations (L. T. p. 16) of these cells with their oriented boundaries give a set of incidence matrices (L. T. p. 34) for M_n just as simplicial cells do for manifolds that can be cut up into simplicial cells. From these matrices Betti numbers and torsion coefficients (L. T. p. 34) can be calculated. In the sequel these will be referred to as the *elemental Betti and torsion numbers calculated from elemental homologies.* Because $\mathfrak{K}$ is made up of a finite number of elemental cells these numbers are finite.

A complex, chain, or cycle *on* M_n is the single valued continuous image on M_n of a simplicial complex chain or cycle. If a p-cell of a chain on M_n is mapped on an s-cell, $s < p$, of $\mathfrak{K}$, the p-cell will be given a zero coefficient in the chain, as in L. T. (p. 74). This is what Lefschetz calls a singular complex, chain or cycle. (L. T. p. 72). Here the word "singular" will be omitted.

THEOREM 2. *If γ_k is any k-chain on M_n it can be cut up into subdivisions so small that each can be covered by an n-cell of the covering U^n.*

This is apparent from the definition of γ_k.

THEOREM 3. *If γ_k is a k-cycle on M_n and γ'_k is a subdivision of γ_k and γ_k is the image of G_k, then $\gamma_k \sim \gamma'_k$ on M_n.* (Notation: L. T., p. 21).

A k-cycle is a special case of a k-chain. If G'_k is the subdivision of G_k producing γ'_k then $G_k - G'_k$ bounds a degenerate (L. T., p. 74) $(k+1)$-complex on G_k. Hence its image $\gamma_k - \gamma'_k$ bounds a $(k+1)$-chain on γ_k and hence on M_n.

THEOREM 4. *Fundamental Deformation Theorem: Given an r-cycle γ_r on M_n such that each cell of γ_r is in an n-cell of U^n, and an elemental complex $\mathfrak{K}$ on M_n then:*

Part 1. *Given any k-cell p_k^i of γ_k such that $p_k^i \rightarrow \sum_j t_j^i p_{k-1}^j$ (notation: L. T., p. 15) where t_j^i is an integer, there can be associated with p_k^i an elemental k-cell a_k^i of $\mathfrak{K}$ for which the following relations hold:*

i. *$a_k^i \rightarrow \sum_j t_j^i a_{k-1}^j$ where a_{k-1}^j is associated with p_{k-1}^j.*

ii. *There exists a $(k+1)$-chain q_{k+1}^i on M_n satisfying $q_{k+1}^i \rightarrow a_k^i - p_k^i - Q_k^i$ where $Q_k^i \rightarrow \sum_j t_j^i p_{k-1}^j - \sum_j t_j^i a_{k-1}^j$.*

Part 2. *If $p_k^1, p_k^2, \cdots, p_k^s$ are k-cells of γ_r and are all in E_n^{n0} then $a_k^1, a_k^2, \cdots, a_k^s$ lie in the same n-cell $E_n^{n-k-1,1}$ containing E_n^{n0}.*

Part 3. *If γ_k is a k-cycle on M_n such that $\gamma_k = \sum_i \mu_i p_k^i$, then $\sum_i \mu_i q_{k+1}^i \rightarrow \gamma_k - \Gamma_k$ where $\Gamma_k = \sum_i \mu_i a_k^i$ is a cycle.*

The proof is made by induction.

Step 0. Part 1. If a zero-cell p_0^i is in the n-cell E_n^{ni} associate with it the elemental zero-cell a_0^i of K in E_n^{ni}. If $E_n^{n-1,1}$ covers E_n^{ni} then $q_1^i \rightarrow p_0^i - a_0^i$ where q_1^i is a segment in $E_n^{n-1,1}$ which, because the zeroth Betti number, $R_0(E_n^{n-1,1})$, of $E_n^{n-1,1}$ is one, is bounded by p_0^i and a_0^i.

Part 2. This follows immediately from the construction of a_0^i.

Part 3. Given that $\gamma_0 = \sum_i \mu_i p_0^i$ is a cycle, let $\Gamma_0 = \sum_i \mu_i a_0^i$. Since $q_1^i \rightarrow p_0^i - a_0^i$ it follows that $\sum_i \mu_i q_1^i \rightarrow \gamma_0 - \Gamma_0$. Since the boundary of γ_0 vanishes and the boundary of Γ_0 corresponds to it cell for cell, the boundary of Γ_0 vanishes.

Step k. Imagine steps 0, 1, 2, $\cdots$, $k-1$ to have been taken for each of the $(0, 1, 2, \cdots, k-1)$-cells of γ_r.

Part 1. Given $p_k^i \rightarrow \sum_j t_j^i p_{k-1}^j$. Consider $\sum_j t_j^i a_{k-1}^j$ where a_{k-1}^j is associated by step $k-1$, part 1 with p_{k-1}^j. $\sum_j t_j^i a_{k-1}^j$ is a cycle by step $k-1$, part 3 and is in an n-cell $E_n^{n-k,i}$ containing E_n^{n0} by step $k-1$, part 2. So because $R_{k-1}(E_n^{n-k,i}) = 0$, $\sum_j t_j^i a_{k-1}^j$ bounds a_k^i in $E_n^{n-k,i}$. Associate a_k^i with p_k^i. By step $k-1$, part 3

$$\sum_j t^i_j q^j_k \to \sum_j t^i_j p^j_{k-1} - \sum_j t^i_j a^j_{k-1}.$$

Therefore $p^i_k - a^i_k - \sum_j t^i_j q^j_k$ is a cycle since its boundary vanishes. q^j_k is contained in an n-cell E_n^{n-k,a_j} covering p^j_{k-1}. Hence the set of cells $\{E_n^{n-k,a_j}\}$ $(j = 0, 1, \cdots, k)$ all have points in common with $E_n^{n-k,i}$ and so form a nest of cells. Therefore $p^i_k - a^i_k - \sum_j t^i_j q^j_k$ is in an n-cell $E_n^{n-k-1,1}$ of U^{n-k-1} and, being a cycle, bounds in that n-cell a $(k+1)$-complex q^i_{k+1} because $R_k(E_n^{n-k-1}) = 0$.

$$q^i_{k+1} \to p^i_k - a^i_k - \sum_j t^i_j q^j_k.$$

Let $\sum_j t^i_j q^j_k = Q^i_k$. This completes the proof of step k part 1.

Part 2. If $p^1_k, p^2_k, \cdots, p^s_k$ are in E_n^{no} then a^i_k associated with p^i_k lies in $E_n^{n-k,i}$ which contains E_n^{no} by step k, part 1. Hence the set $\{E_n^{n-k,i}\}$ $(i = 1, 2, \cdots, s)$, since each of its members contains E_n^{no}, forms a nest of cells and is covered by $E_n^{n-k-1,1}$ of the covering U^{n-k-1} and containing E_n^{no}.

Part 3. $\gamma_k = \sum_i \mu_i p^i_k$ is a cycle. Let $\Gamma_k = \sum_i \mu_i a^i_k$. Γ_k is a cycle because its boundary cells correspond to those of γ_k which vanish. From the result of step k, part 1 follows

$$\sum_i \mu_i q^i_{k+1} \to \sum_i \mu_i p^i_k - \sum_i \mu_i a^i_k - \sum_{ij} \mu_i t^i_j q^j_k.$$

But since γ_k is a cycle $\sum_i \mu_i t^i_j = 0$ for every j. Hence $q^i_{k+1} \to \sum_i \mu_i p^i_k - \sum_i \mu_i a^i_k$ and therefore $\sum_i \mu_i q^i_{k+1} \to \gamma_k - \Gamma_k$. This completes the induction and the proof of Theorem 4.

THEOREM 5. *To every k-cycle γ_k on M_n there is an elemental k-cycle Γ_k on M_n such that $\gamma_k - \Gamma_k \sim 0$.*

By Theorems 2 and 3 $\gamma_k - \gamma'_k \sim 0$ where γ'_k is a subdivision of γ_k satisfying the conditions put on γ_k in the statement of Theorem 4. By Theorem 4 $\gamma'_k - \Gamma_k \sim 0$ and therefore $\gamma_k - \Gamma_k \sim 0$.

THEOREM 6. *If γ_k is any k-cycle on M_n and Γ_k is the elemental k-cycle associated with it and $\gamma_k \sim 0$, then $\Gamma_k \sim 0$.*

By Theorem 5 $\gamma_k - \Gamma_k \sim 0$, so $\gamma_k \sim 0$ implies $\Gamma_k \sim 0$.

THEOREM 7. *If an elemental k-cycle Γ_k bounds a $(k+1)$-chain g_{k+1} on M_n, then Γ_k bounds an elemental $(k+1)$-chain G_{k+1}* on M_n.

Cut g_{k+1} up into cells so small that each of them is contained in an n-cell of U^n. Then apply the process of Theorem 4 to the i-cells of g_{k+1} $(i = 0, 1, 2, \cdots, k+1)$ associating with each of them an elemental i-cell.

For the elemental i-cells associated with the i-elements of the boundary use the i-elements of Γ_k themselves. All the k-cells of the boundaries vanish except those making up the boundary Γ_k. Hence Γ_k bounds the chain G_{k+1} of elemental i-cells associated with the cells of g_{k+1}. Theorems 5 and 7 are analogous to theorems proved by Veblen in his *Colloquium Lectures* Chapters 3 and 4 to obtain the topological invariance of the homology characters of a simplicial complex.

Instead of using the elemental cycles and homologies to calculate the Betti and torsion numbers of M_n it is possible to use the set of all cycles on M_n and their homologies (L. T. p. 75). These cycles and homologies will be called "topological" and the numbers topological Betti and torsion numbers. It will now be proved that the topological numbers are the same as the elemental numbers which were shown on page 396 to be finite. This implies that the topological numbers are finite. By Theorem 5 every cycle of M_n is homologous to an elemental cycle, so the elemental cycles form a basis for the cycles of M_n. Every relation of bounding among elemental cycles automatically implies a topological relation of bounding. Moreover any topological bounding relation among the elemental cycles, by Theorem 7, implies an elemental bounding relation. Therefore the homology group for any cycles on M_n is isomorphic to the homology group of the elemental complex which proves Theorem 8.

THEOREM 8. *The Betti numbers and torsion coefficients calculated from elemental homologies are finite and topologically invariant.*

Corollary. The connectivity and torsion numbers modulo m (L. T. p. 18 and p. 35) as obtained from the incidence relations of the elemental cells are topologically invariant. The proof is the same as that of Theorem 8 except that in adding the boundaries of the chains the coefficients of the individual cells of the boundary are reduced modulo m. This changes none of the details of the proof.

2. Orientation; Kronecker Index. It is now necessary to draw the distinction between manifolds orientable and non-orientable with respect to a covering set $\{E_n^i\}$ of n-cells. Suppose two n-cells of the set, E_n^i and E_n^j, overlap in a number of open sets of which R is one. Orient each n-cell by means of a complex on it. Cover each region R with an infinite complex, K^i, by the method described in Lefschetz's "Topology" (L.T., p. 311) using the complex on E_n^i to give K^i. K^i is connected and only two n-cells abut on each $(n-1)$-cell of K^i so K^i is an orientable simple circuit modulo an ideal element, Δ, its boundary. (L. T., p. 47 and p. 295). Hence it has a basic oriented n-cycle Γ_n^i mod Δ (L. T., p. 46 and p. 300). Because the Betti numbers of K^i are topologically invariant, Γ_n^j, the basic oriented n-cycle mod Δ obtained using, to cover R, an infine complex K^j derived

from a complex on E_n^j, satisfies the relation $\Gamma_n^i \sim \varepsilon^j \Gamma_n^j \bmod \Delta$, where $\varepsilon^j = \pm 1$. If $p+1$ n-cells of $\{E_n^i\}$ cover R we have relations

$$\begin{array}{ccc} \Gamma_n^i \sim \varepsilon^{j_1} \Gamma_n^{j_1} \bmod \Delta, \\ \vdots \quad \vdots \quad \vdots \\ \Gamma_n^i \sim \varepsilon^{j_p} \Gamma_n^{j_p} \bmod \Delta. \end{array}$$

If the orientation of the cells E_n^i can be so chosen that for all i and j and all regions R, all the ε's corresponding to a given R are of the same sign, M_n is *orientable with respect to the covering* $\{E_n^i\}$, otherwise not.

It is now possible to define the Kronecker Index of two chains which do not intersect one another's boundaries and are on an orientable M_n. If M_n is not orientable the chains can be taken modulo 2 and the Kronecker Index computed modulo 2. Before defining the Index it is necessary to prove some theorems.

THEOREM 9. *Two cycles Γ_p and Γ_{n-p} are ε-deformable where $\varepsilon > 0$ is arbitrarily small, into Γ_p^s and Γ_{n-p}^s which intersect in a finite number of isolated points.*

Throughout the proof $F(E)$ means the boundary of E and $\bar{E}$ means $E + F(E)$, the closure of E. Suppose that $\{E_n^i\}$ is a finite set of cells covering M_n. On E_n^i take a complex K^i of mesh so small that if J^i is the sum of the closed cells of K^i with no points on $F(E_n^i)$, J^i covers M_n. Let 2τ be the maximum mesh of the complexes K^i.

Now make a subdivision of Γ_p and Γ_{n-p} into cells of mesh $\tau/4$ and call these subdivisions Γ_p and Γ_{n-p}. If the following assumptions about the reduction of the parts of Γ_p and Γ_{n-p} in $(i-1)$ of the domains $\{J^i\}$ be made, it can be proved that the reduction can be extended to another domain J and so, by induction, to all these domains. It is assumed that if

$$G^{i-1} = J^1 + J^2 + \cdots + J^{i-1},$$

then Γ_p and Γ_{n-p} are $\tau/4$-deformable into cycles Γ_p^{i-1} and Γ_{n-p}^{i-1} with mesh $\tau/4$ and such that the following three conditions are satisfied:

1. $$\Gamma_p^{i-1} = \gamma_p^{i-1} + \delta_p^{i-1}, \qquad \Gamma_{n-p}^{i-1} = \gamma_{n-p}^{i-1} + \delta_{n-p}^{i-1},$$

where γ_p^{i-1} and γ_{n-p}^{i-1} are the sets of all closed cells of Γ_p^{i-1} and Γ_{n-p}^{i-1} respectively that have points in common with G^{i-1}.

2. γ_p^{i-1} intersects γ_{n-p}^{i-1} in a finite number of isolated points.

3. There exist neighborhoods N^{i-1} and N^{*i-1} such that N^{i-1} contains the sum of closed cells of γ_p^{i-1} that meet $F(G^{i-1})$, and N^{*i-1} contains the similar sum for γ_{n-p}^{i-1}, and such that

$$\Gamma_p^{i-1} \cdot N^{*i-1} = 0, \quad \Gamma_{n-p}^{i-1} \cdot N^{i-1} = 0, \quad N^{i-1} \cdot N^{*i-1} = 0.$$

In view of (3) the intersections of the γ's are interior to G^{i-1}, that is not on its boundary, and γ_p^{i-1} does not intersect δ_{n-p}^{i-1} nor does γ_p^{i-1} intersect δ_p^{i-1}. Furthermore

$$F(\delta_p^{i-1}) \subset N^{i-1}, \qquad F(\delta_{n-p}^{i-1}) \subset N^{*i-1}.$$

Let $G^i = J^1 + J^2 + \cdots + J^i$. It will now be proved that conditions one, two and three can be satisfied for the index i.

A. Consider the following quantities:

1. The distance from $F(\delta_p^{i-1})$ (which is on N^{i-1}) to $F(N^{i-1})$ and that from $F(\delta_{n-p}^{i-1})$ to $F(N^{*i-1})$.
2. The distances from δ_p^{i-1} and δ_{n-p}^{i-1} to G^{i-1}.
3. The distance from Γ_p^{i-1} to N^{*i-1} and from Γ_{n-p}^{i-1} to N^{i-1}.
4. The distance from $F(E_n^i)$ to $F(J^i)$.

The distances just mentioned are all positive and so is their lower bound. Choose a ζ less than a quarter of that lower bound and also less than $\tau/4$.

B. Subdivide Γ_p^{i-1} and Γ_{n-p}^{i-1} into subchains whose mesh ξ is to be at all events less than $\zeta/2$. Denote by γ_p and γ_{n-p} the set of all closed cells of δ_p^{i-1} and δ_{n-p}^{i-1} on E_n^i.

C. Let K be a complex on E_n^i which has J^i as a subcomplex and K^* be its dual such that the mesh of both complexes is ξ. The mesh of the subdivisions of Γ_p^{i-1} and Γ_{n-p}^{i-1} is small enough to assure that $F(\gamma_p)$ and $F(\gamma_{n-p})$ lie entirely in $E_n^i - J^i + N^{i-1}$ and $E_n^i - J^i + N^{*i-1}$ respectively.

D. By ξ-deformations γ_p and γ_{n-p} can be reduced to subchains of K and K^* respectively. Add the deformation chains of $F(\gamma_p)$ and $F(\gamma_{n-p})$ and call the new chains γ_p and γ_{n-p} and the new cycles Γ_p^i and Γ_{n-p}^i. In addition if γ_p $(p < n)$ has p-cells on $F(J^i)$, γ_p can be modified by another ξ-deformation as in the similar case, L. T., p. 153, so as to remove them from $F(J^i)$. The process is as follows. Because of D, J^i is a subchain of K. If $p < n$ it can be arranged that γ_p has no p-cell on $F(J^i)$ for if the cell σ_p of γ_p is on $F(J^i)$, σ_p can be replaced by the p other p-faces not on $F(J^i)$ of a $(p+1)$-simplex of which σ_p is one edge. If $p = n$ no n-cell can be on $F(J^i)$ because $F(J^i)$ is $(n-1)$-dimensional. Now call γ'_p the sum of the cells of γ_p whose closure meets J^i. Also γ'_{n-p} is the similar sum of cells of γ_{n-p}.

E. Set

$$\gamma_p^i = \gamma_p^{i-1} + \gamma'_p, \qquad \delta_p^i = \Gamma_p^i - \gamma_p^i,$$

and similarly for $n-p$.

It is now necessary to construct non-overlapping neighborhoods N^i and N^{*i} such that the sum of the closed p-cells of a suitable γ_p^i that meet $F(G^i)$ is contained in N^i and such that no points of Γ_{n-p}^i meet N^i. A similar

neighborhood N^{*i} must be constructed for the $(n-p)$-chain. The first step in this construction is to create neighborhoods M^i and M^{*i} of the intersection of γ_p' and γ_{n-p}' with $F(J^i)$ such that

$$\begin{array}{lll} \text{(a)} & M^i\cdot N^{*i-1}=0, & M^{*i}\cdot N^{i-1}=0; \\ \text{(b)} & M^i\cdot M^{*i}=0; & \\ \text{(c)} & M^i\cdot \Gamma^i_{n-p}=0, & M^{*i}\cdot \Gamma^i_p=0. \end{array}$$

(a) It follows from A 1 and B that γ_p' does not enter N^{*i-1} and similarly for γ_{n-p}' and N^{i-1}, so (a) can be satisfied.

(b) γ_p' was so constructed (D) that it does not meet γ_{n-p}' on $F(J^i)$. Therefore (b) can be satisfied.

(c) To prove that (c) can be fulfilled it suffices to show that

$$[\gamma_p'\cdot F(J^i)]\cdot \Gamma^i_{n-p}=0,$$

and this will follow if

$$\begin{array}{ll} \mathrm{C^I}. & [\gamma_p'\cdot F(J^i)]\cdot\gamma^{i-1}_{n-p}=0, \\ \mathrm{C^{II}}. & [\gamma_p'\cdot F(J^i)]\cdot\gamma_{n-p}'=0, \\ \mathrm{C^{III}}. & [\gamma_p'\cdot F(J^i)]\cdot\delta^i_{n-p}=0, \end{array}$$

because

$$\Gamma^i_{n-p}=\gamma^{i-1}_{n-p}+\gamma'_{n-p}+\delta^i_{n-p}.$$

$\mathrm{C^I}$. γ_p' is outside G^{i-1} or in N^{i-1}. γ^{i-1}_{n-p} has no points in either of these sets so $\mathrm{C^I}$ holds.

$\mathrm{C^{II}}$. For the same reason as in case (b) equation $\mathrm{C^{II}}$ is true.

$\mathrm{C^{III}}$. Any closed cell of δ^i_{n-p} is either in N^{*i-1} where there are no points of γ_p' or else does not meet J^i which verifies $\mathrm{C^{III}}$.

It has been shown that a sufficiently small neighborhood M^i of $\gamma_p'\cdot F(J^i)$ will have the properties (a), (b), (c). Exactly the same proof holds for M^{*i} and the $(n-p)$-chains.

Suppose now γ_p and γ_{n-p} to be subjected to a sufficiently small subdivision and γ_p' and γ_{n-p}' to be defined as the sums of the closed cells of this new sub-division meeting J^i. Then because the old γ_p' and γ_{n-p}' did not intersect on $F(J^i)$, the sums of the closed cells of the new ones meeting $F(J^i)$ do not intersect one another. New γ^i_p, δ^i_p, γ^i_{n-p} and δ^i_{n-p} are now of course constructed from the new γ_p' and γ_{n-p}' as before from the old.

If the subdivisions have been chosen sufficiently small the M's will contain the sums of the closed cells of the respective chains γ_p' and γ_{n-p}' which meet $F(J^i)$.

F. Now let $M^i+N^{i-1}=N^i$, $M^{*i}+N^{*i-1}=N^{*i}$, and verify that condition three holds for N^i and N^{*i}.

But

$$F(G^i) = F(G^{i-1} + J^i) = \overline{F(G^{i-1}) \cdot (M_n - J^i) + F(J^i) \cdot (M_n - G^{i-1})}.$$

Cells of γ_p^{i-1} meeting $F(G^i)$ all meet $F(G^{i-1})$ and then they are in $N^{i-1} \subset N^i$. Also the closed cells of γ'_p not on N^{i-1} are exterior to G^{i-1} by A 1 and A 4, hence if they meet $F(G^i)$ they meet it in $F(J^i)$ and so lie in M^i. It follows since $\gamma_p^i = \gamma_p^{i-1} + \gamma'_p$ that the sum of the closed cells of γ_p^i meeting $F(G^i)$ is on N^i and likewise for γ_{n-p}^i and N^{*i}.

It remains to prove that $N^i \cdot N^{*i} = 0$ and that $\Gamma_{n-p}^i \cdot N^i = \Gamma_p^i \cdot N^{*i} = 0$. The first statement follows from E(a) and E(b). Due to A 3 and A 4, Γ_{n-p}^i does not meet N^{i-1} and by condition (c) it does not meet M^i. Therefore $\Gamma_{n-p}^i \cdot N^i = 0$. Similarly $\Gamma_p^i \cdot N^{*i} = 0$.

Now it has been shown that the conditions assumed for $i-1$ can all be realized for i, so the induction is complete.

The first step of the induction is possible since G^0 can be taken to be zero and $\gamma_p^0 = \gamma_{n-p}^0 = 0$, $\delta_p^0 = \Gamma_p$ and $\delta_{n-p}^0 = \Gamma_{n-p}$. Then the induction is started. It also comes to an end. Because $\{J^i\}$ covers M_n, a finite number s of steps will reduce δ_p and δ_{n-p} to zero and Γ_p^s and Γ_{n-p}^s will result which intersect in a finite number of points, each point interior to an n-cell of $\{E_n^i\}$. The deformations applied to Γ_p and Γ_{n-p} in order to produce Γ_p^s and Γ_{n-p}^s are finite in number so the total deformation is arbitrarily small.

THEOREM 10. *If Γ_p and Γ_{n-p} are two chains on M_n which do not intersect each other's boundaries, then Theorem 9 applies to them.*

The proof is the same as that of Theorem 9 provided that $F(\gamma)$ and $F(\delta)$ be everywhere replaced by the part of the boundaries of the partial chains γ and δ which are not on $F(\Gamma)$.

When chains have the following properties: their boundaries do not intersect one another, the chains intersect in a finite number of isolated points, about each intersection their cells are on a complex and its dual respectively; they are said to constitute a *regular pair*.

The Kronecker Index, $(C_p \cdot C_{n-p})$, for a regular pair of chains C_p and C_{n-p} is defined as follows. If $A^1, A^2, \cdots, A^r$ are the isolated intersections then there exist for A^i subchains C_p^i and C_{n-p}^i of C_p and C_{n-p} which include all their cells through A^i, do not intersect elsewhere and are on an n-cell E_n^i of M_n. Then $(C_p^i \cdot C_{n-p}^i)$ is defined as in L. T., p. 194. $(C_p \cdot C_{n-p})$ is defined as $\sum_i (C_p^i \cdot C_{n-p}^i)$. It follows immediately that for regular pairs the Kronecker Index is additive and obeys the same permutation laws as for simplicial manifolds.

THEOREM 11. *Let C_p and C_{n-p} be a regular pair and let there exist $C_{p+1} \to C_p$ such that C_{p+1} does not meet $F(C_{n-p})$. Then* $(C_p \cdot C_{n-p}) = 0$.

a. If C_{p+1} is on an n-cell E_n of M_n then the theorem follows from L. T., p. 170.

b. C_{p+1} can be written $C_{p+1} = \sum_{i=1}^{k} C_{p+1}^i$ where all the cells C_{p+1}^i that have a point in common are on a single n-cell, E_n^i, of the covering and no $(p-1)$-cell of C_{p+1}^i contains a point of $C_p \cdot C_{n-p}$. It will be shown that without changing $(C_p \cdot C_{n-p})$ the situation can be so modified as to replace k by $r < k$.

Let A be one of the intersections of C_p and C_{n-p} and call D the sum of the chains $F(C_{p+1}^i)$ meeting A. $D = D' + D''$ where D' is on C_p and D'' has no p-cells on C_p. $F(D'')$ does not meet $C_p \cdot C_{n-p}$. Deform D'' and C_{n-p} into a regular pair according to Theorem 10 by a deformation acting on a subchain of C_{n-p} not containing A and so small that no new intersections with C_p are brought about. Add the deformation chain of the p-cells of the old D'' to the $(p+1)$-cells to whose boundary it belongs if the $(p+1)$-cell is one of the sum whose boundary is D. If the $(p+1)$-cell is not in D subtract the deformation chain. Now replace C_p by $C_p - D$. Since $D' + D''$ bounds a sub-chain of C_{p+1} in an n-cell of M_n, and that subchain does not meet $F(C_{n-p})$, and D and C_{n-p} are a regular pair, it follows from part a. that $(C_p \cdot C_{n-p}) = ((C_p - D) \cdot C_{n-p})$. Because D'' and C_{n-p} are a regular pair, $D'' \cdot C_{n-p}$ is on a p-cell of D'' so no $(p-1)$-cell of the new $\sum_{i=1}^{r} C_{p+1}^i$ $r < k$ meets $D'' \cdot C_{n-p}$. Thus the situation is as before with k replaced by r. Repeating this process will ultimately reduce $\sum_i C_{p+1}^i$ to a sum of cells on a single n-cell of M_n which is case a.

Kronecker Index of arbitrary chains. If C_p and C_{n-p} are arbitrary chains neither of which meets the boundary of the other, then by Theorem 10 they can be deformed by a deformation T onto a regular pair, C_p' and C_{n-p}'. $(C_p \cdot C_{n-p})$ is then defined as $(C_p' \cdot C_{n-p}')$.

THEOREM 12. *The definition of* $(C_p \cdot C_{n-p})$ *just given is unique provided the deformation* T *is small enough.*

Let C_p^1, C_{n-p}^1 and C_p^2, C_{n-p}^2 be two regular pairs approximating C_p and C_{n-p}. Let A be a generic point of the intersection of C_p^1 and C_{n-p}^1 and B the same for C_p^2 and C_{n-p}^2. There are three possibilities.

a. There is no A point on C_{n-p}^2 and no B point on C_p^1.

In this case deform, using Theorem 10, a subchain of C_p^1 that is away from A and C_{n-p}^1, and deform a subchain of C_{n-p}^2 away from B and C_p^2, by a deformation T' so small that no intersections with the other pairs are changed and a regular pair, again called C_p^1 and C_{n-p}^2, results. If T and T' are small enough the deformation chain D, of dimensionality $p+1$, connecting C_p^1 and C_p^2, obtained in getting C_p^1 and C_p^2 from C_p, is very

near C_p, so $F(C^1_{n-p})$ does not meet it. Its boundary is $C^1_p - C^2_p$ plus a deformation chain which does not meet the $(n-p)$-chains. Hence by Theorem 11,

$$(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p}).$$

If $p = 0$ (or $n - p = 0$), C^1_p, C^2_{n-p} and C^2_p, C^1_{n-p} are regular pairs so $(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p})$ follows for all such cases from the argument made at the end of the last paragraph.

b. $p \neq 0$, no A point is a B point. This includes the three cases: 1) Points of type A are on C^2_{n-p} but none of type B are on C^1_p; 2) There is no A point on C^2_{n-p} but there are some B points on C^1_p; 3) A has points on C^2_{n-p} and B has points on C^1_p but no A point is a B point.

Because no points are of both type A and B, and because $n - p < n$, a small deformation applied to C^1_p away from C^1_{n-p} and to C^2_{n-p} away from C^2_p will reduce Case b to Case a by removing C^2_{n-p} from A, removing C^1_p from B, and not changing A or B.

c. A and B points coincide, $p \neq 0$.

In this case deform C^2_{n-p} in such a way that: 1) $C^2_p \cdot C^2_{n-p}$ no longer meets A points, 2) the set of points in which the new C^2_{n-p} and the old differ, can be covered by an n-cell E_n of M_n. This deformation is possible if it is taken small enough because $n - p < n$. But by Theorem 11, part a, the deformation leaves $(C^2_p \cdot C^2_{n-p})$ unchanged and reduces Case c to Case b.

So now it is proved that in all cases

$$(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p}).$$

Similarly it can be shown that $(C^2_p \cdot C^2_{n-p}) = (C^2_p \cdot C^1_{n-p})$. Therefore $(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^2_{n-p})$ which was to be proved.

THEOREM 13. *The Kronecker Index of two arbitrary chains on M_n which do not meet one another's boundaries has the permutation properties of the Index for chains on a simplicial manifold. If the chain γ_p is a cycle and $\gamma_p \sim 0$ then $(\gamma_p \cdot \gamma_{n-p}) = 0$ for every γ_{n-p}.*

The first part of the theorem follows from the definitions of the Kronecker Index. If $\gamma_p \sim 0$ and γ^0_p and γ^0_{n-p} are a regular pair constructed homologous to γ_p and γ_{n-p} according to Theorem 9, then $\gamma^0_p \sim 0$ and, by Theorem 11, $(\gamma^0_p \cdot \gamma^0_{n-p}) = 0$. Therefore $(\gamma_p \cdot \gamma_{n-p}) = 0$ which proves the second part.

3. Immersion of M_n in S_r.

THEOREM 14. *A topological manifold M_n can be homeomorphically mapped upon a subset of an Euclidean space, S_r.*

a. If C_{p+1} is on an n-cell E_n of M_n then the theorem follows from L. T., p. 170.

b. C_{p+1} can be written $C_{p+1} = \sum_{i=1}^{k} C^i_{p+1}$ where all the cells C^i_{p+1} that have a point in common are on a single n-cell, E^i_n, of the covering and no $(p-1)$-cell of C^i_{p+1} contains a point of $C_p \cdot C_{n-p}$. It will be shown that without changing $(C_p \cdot C_{n-p})$ the situation can be so modified as to replace k by $r < k$.

Let A be one of the intersections of C_p and C_{n-p} and call D the sum of the chains $F(C^i_{p+1})$ meeting A. $D = D' + D''$ where D' is on C_p and D'' has no p-cells on C_p. $F(D'')$ does not meet $C_p \cdot C_{n-p}$. Deform D'' and C_{n-p} into a regular pair according to Theorem 10 by a deformation acting on a subchain of C_{n-p} not containing A and so small that no new intersections with C_p are brought about. Add the deformation chain of the p-cells of the old D'' to the $(p+1)$-cells to whose boundary it belongs if the $(p+1)$-cell is one of the sum whose boundary is D. If the $(p+1)$-cell is not in D subtract the deformation chain. Now replace C_p by $C_p - D$. Since $D' + D''$ bounds a sub-chain of C_{p+1} in an n-cell of M_n, and that subchain does not meet $F(C_{n-p})$, and D and C_{n-p} are a regular pair, it follows from part a. that $(C_p \cdot C_{n-p}) = ((C_p - D) \cdot C_{n-p})$. Because D'' and C_{n-p} are a regular pair, $D'' \cdot C_{n-p}$ is on a p-cell of D'' so no $(p-1)$-cell of the new $\sum_{i=1}^{r} C^i_{p+1}$ $r < k$ meets $D'' \cdot C_{n-p}$. Thus the situation is as before with k replaced by r. Repeating this process will ultimately reduce $\sum_i C^i_{p+1}$ to a sum of cells on a single n-cell of M_n which is case a.

Kronecker Index of arbitrary chains. If C_p and C_{n-p} are arbitrary chains neither of which meets the boundary of the other, then by Theorem 10 they can be deformed by a deformation T onto a regular pair, C'_p and C'_{n-p}. $(C_p \cdot C_{n-p})$ is then defined as $(C'_p \cdot C'_{n-p})$.

THEOREM 12. *The definition of* $(C_p \cdot C_{n-p})$ *just given is unique provided the deformation* T *is small enough.*

Let C^1_p, C^1_{n-p} and C^2_p, C^2_{n-p} be two regular pairs approximating C_p and C_{n-p}. Let A be a generic point of the intersection of C^1_p and C^1_{n-p} and B the same for C^2_p and C^2_{n-p}. There are three possibilities.

a. There is no A point on C^2_{n-p} and no B point on C^1_p.

In this case deform, using Theorem 10, a subchain of C^1_p that is away from A and C^1_{n-p}, and deform a subchain of C^2_{n-p} away from B and C^2_p, by a deformation T' so small that no intersections with the other pairs are changed and a regular pair, again called C^1_p and C^2_{n-p}, results. If T and T' are small enough the deformation chain D, of dimensionality $p+1$, connecting C^1_p and C^2_p, obtained in getting C^1_p and C^2_p from C_p, is very

near C_p, so $F(C^1_{n-p})$ does not meet it. Its boundary is $C^1_p - C^2_p$ plus a deformation chain which does not meet the $(n-p)$-chains. Hence by Theorem 11,

$$(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p}).$$

If $p = 0$ (or $n - p = 0$), C^1_p, C^2_{n-p} and C^2_p, C^1_{n-p} are regular pairs so $(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p})$ follows for all such cases from the argument made at the end of the last paragraph.

b. $p \neq 0$, no A point is a B point. This includes the three cases: 1) Points of type A are on C^2_{n-p} but none of type B are on C^1_p; 2) There is no A point on C^2_{n-p} but there are some B points on C^1_p; 3) A has points on C^2_{n-p} and B has points on C^1_p but no A point is a B point.

Because no points are of both type A and B, and because $n - p < n$, a small deformation applied to C^1_p away from C^1_{n-p} and to C^2_{n-p} away from C^2_p will reduce Case b to Case a by removing C^2_{n-p} from A, removing C^1_p from B, and not changing A or B.

c. A and B points coincide, $p \neq 0$.

In this case deform C^2_{n-p} in such a way that: 1) $C^2_p \cdot C^2_{n-p}$ no longer meets A points, 2) the set of points in which the new C^2_{n-p} and the old differ, can be covered by an n-cell E_n of M_n. This deformation is possible if it is taken small enough because $n - p < n$. But by Theorem 11, part a, the deformation leaves $(C^2_p \cdot C^2_{n-p})$ unchanged and reduces Case c to Case b.

So now it is proved that in all cases

$$(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p}).$$

Similarly it can be shown that $(C^2_p \cdot C^2_{n-p}) = (C^2_p \cdot C^1_{n-p})$. Therefore $(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^2_{n-p})$ which was to be proved.

THEOREM 13. *The Kronecker Index of two arbitrary chains on M_n which do not meet one another's boundaries has the permutation properties of the Index for chains on a simplicial manifold. If the chain γ_p is a cycle and $\gamma_p \sim 0$ then $(\gamma_p \cdot \gamma_{n-p}) = 0$ for every γ_{n-p}.*

The first part of the theorem follows from the definitions of the Kronecker Index. If $\gamma_p \sim 0$ and γ^0_p and γ^0_{n-p} are a regular pair constructed homologous to γ_p and γ_{n-p} according to Theorem 9, then $\gamma^0_p \sim 0$ and, by Theorem 11, $(\gamma^0_p \cdot \gamma^0_{n-p}) = 0$. Therefore $(\gamma_p \cdot \gamma_{n-p}) = 0$ which proves the second part.

3. Immersion of M_n in S_r.

THEOREM 14. *A topological manifold M_n can be homeomorphically mapped upon a subset of an Euclidean space, S_r.*

The idea of the proof is as follows. A topological manifold M_n may be covered by a finite number of simplicial n-complexes K^i such that each point of M_n is interior to at least one complex K^i. Corresponding to each complex K^i can be determined a finite set of bounded, continuous functions, $x_1^i, x_2^i, \cdots, x_{r^i}^i$, defined at all points of K^i and such that:

1) The functions x_s^i all vanish on the boundary of K^i.

2) At each interior point P of K^i at least one of the functions x_s^i does not vanish.

3) If P and Q are two distinct interior points of K^i at least one of the functions x_s^i has a value at Q different from its value at P.

The domain of definition of the functions x_s^i is extended over the entire manifold M_n by putting them equal to zero at all points of M_n not on K^i. The combined functions x_s^i corresponding to all values of i will have the property that at two arbitrarily chosen distinct points P and Q of the manifold at least one of them will have a value at Q different from its value at P. They may consequently be regarded as the coordinates of a point P of M_n in a Euclidean space of $r = \sum_i r^i$ dimensions. The problem then reduces to that of defining the functions just described over the complex K^i.

Let L^i be the boundary of K^i. It may always be assumed that no two simplexes of K^i have the same vertices and that each n-simplex of K^i has at least one vertex not on the boundary of L^i. For if K^i does not have this property initially it can be replaced by its first derived which is a complex which does. Suppose K^i has α vertices and L^i has β ($\beta < \alpha$) vertices. Then K^i is homeomorphic to a sub-complex τ of an $(\alpha - 1)$-simplex σ in Euclidean $(\alpha - 1)$-space $S_{\alpha-1}$ obtained by associating each vertex A^j of σ with one of K^i and drawing between the vertices A^j the simplexes corresponding to those determined by the associated vertices of K^i. Moreover, the boundary L^i of K^i corresponds to a sub-complex of a $\beta - 1$ face δ of the simplex σ. Now, pass an $(\alpha - 2)$-dimensional hyperplane $P_{\alpha-2}$ of $S_{\alpha-1}$ through the vertices of δ in such a manner that it does not pass through any vertex of σ other than those of δ. Consider $S_{\alpha-1}$ to be a projective space and $P_{\alpha-2}$ to be the plane at infinity. Then an inversion through a point O of $S_{\alpha-1}$ not on τ will transform the interior of τ into a homeomorphic image τ' and the boundary of τ into the point O. The point O may now be taken as the origin of $\alpha - 1$ Cartesian coordinates, $x_1^i, x_2^i, \cdots, x_{\alpha-1}^i$. The values of these coordinates at a point of τ' will be by definition the values being sought of the functions x_s^i at the corresponding points of K^i. This completes the argument.

FORMAL LOGIC IN FINITE TERMS.*

By Alfred L. Foster.

I. Introduction.

The subject matter of our work lies close to the foundations of Formalism. We shall exhibit and discuss a complete finite model, of which classical formal logic is a partial realization. The significance of this for the foundations of classical logic[1] will best be appreciated at the end, where we devote a section to this matter. At this point let it merely be pointed out that a finite model, i. e., a concrete example satisfying the postulates, and sometimes called a *realization* of a discipline, may frequently be expected to shed a good bit of light on the nature of the discipline as a whole, inasmuch as it gives, in one sweep of the eye, as it were, relations which in their abstract formulation are extremely difficult to grasp. As a familiar example we mention Cayley's model of non-Euclidean geometry, which demonstrated the self-consistency of such a geometry.[2] In the present case, to take one of the results, the emptiness of pure-existence, from the purely deductive standpoint, is shown.

A model may also suggest shortcomings in the discipline which it realizes. In fact if the model suggests propositions which are foreign to the discipline, we may adopt the model as a starting point for amending the latter. As a simple example of something to which we are led which does not appear in classical logic we mention the concept which we embody in $(Dx)\,F(x)$. From its properties and from the manner of its construction it seems to have a very strong resemblance to that which the Intuitionists call "existence".

The keynote of the construction lies in a refinement of the truth table,[3]

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[1] The most exhaustive modern treatment of classical mathematical logic is Whitehead-Russell, Principia Mathematica. We shall also refer to Hilbert-Ackermann, Grundzüge der Theoretischen Logik, J. Springer, 1928, and to R. Carnap, Abriß der Logistik, Springer, 1929.

[2] A good account of this may be found, for example, in Hans Reichenbach, Philosophie der Raum-Zeit-Lehre, pp. 11, 63. W. de Gruyter & Co., 1928.

[3] The conception of the truth table as a means of representing propositional functions seems to have originated nearly simultaneously with E. L. Post (Amer. Jour. of Math., 43 pp. 163–185, 1921) and L. Wittgenstein (Ann. d. Kult. v. Nat. Phil., 14, pp. 185–262, 1921). See also R. Carnap, Abriß d. Logistik, J. Springer, 1929, pp. 6–10. Post gives a formal proof of the fact that the classical &, v, $^-$ language and the truth-table language of the logic of propositions are equivalent. This fact is assumed in the following pages.

which then permits extension into the domain of function.[4] As representatives of this domain we select functions defined over finite ranges.

The status of classical formal logic as a *language of finite sets, which has been carried over to an entirely different domain of application, that of infinite sets,* is thus suggested. It is planned to present a full discussion of the epistemological value of these and other conclusions as well as a discussion of the nature and role of logics in general in a future paper.

II. Logic of propositions.

1. **The proposition.** Let $p_1, p_2, \cdots, p_\varkappa$ be marks each of which is capable of taking on only the values "0" and "1", with the excluded middle holding, i. e., if $p \neq 0$ then $p = 1$, and if $p \neq 1$ then $p = 0$. Such marks we will call *basic variable propositions,* or, when there is no danger of confusion, simply basic propositions, or even propositions. Instead of "$p = 0$" we will sometimes say "p is true", and instead of "$p = 1$", "p is false". Similarly if anything in our work, say $\mathfrak{A} = 0$ (or 1), we shall on occasion say "$\mathfrak{A}$ is true" (or "false").

2. **The truth function.** A function of $\varkappa$ basic propositions $f(p_1, \cdots, p_\varkappa)$ is called a *truth function* of $p_1, \cdots, p_\varkappa$, if and only if (1) the range of values of the function is 0, 1 and, (2) the value which $f(p_1, \cdots, p_\varkappa)$ takes on depends only on the values taken on by $p_1, \cdots, p_\varkappa$ and on nothing else. There are $2^{(2^\varkappa)}$ different such truth functions for $\varkappa$ basic propositions. We shall introduce the following representation of these functions.

Consider the set $M^\varkappa$ of $2^\varkappa$ indexed points, each index being one of the $2^\varkappa$ coverings of $\varkappa$ linearly ordered points with 0's and 1's. Then if these points of $M^\varkappa$ are arranged in any linear order, *each of the* $2^{(2^\varkappa)}$ *coverings of these with* 0'*s and* 1'*s represents a truth function of the* $\varkappa$ *basic variables* $p_1, \cdots, p_\varkappa$. The index of each point refers (from left to right, say) to the values of $p_1, \cdots, p_\varkappa$ respectively, and the value of the particular truth function under consideration for these values of $p_1, \cdots, p_\varkappa$ is then given by the covering of this point. As examples we will write down all the truth functions for $\varkappa = 1$ and $\varkappa = 2$, with their familiar axiomatic equivalents (Hilbert notation; Cf. Grund., loc. cit.).

$M^{(1)}$ $\cdot_0$	$\cdot_1$	
0	0	eq. $p_1 \vee \bar{p}_1$
0	1	eq. p_1
1	0	eq. $\bar{p}_1$
1	1	eq. $p_1 \,\&\, \bar{p}_1$

[4] The individual, the variable, and the function are perhaps the three most fundamental concepts in mathematics. The logic of propositions is essentially the logic of the individual; the logic of functions, as its name implies, extends this to the concept of function.

$M^{(2)}$ ·00	·01	·10	·11		
0	0	0	0	eq. $p_1 \vee \bar{p}_1$	
0	0	0	1	eq. $p_1 \vee p_2$	
0	0	1	0	eq. $p_1 \vee \bar{p}_2$	
0	0	1	1	eq. p_1	
0	1	0	0	eq. $\bar{p}_1 \vee p_2$	
0	1	0	1	eq. $\bar{p}_2$	
0	1	1	0	eq. $p_1 \sim p_2$	eq. $(\bar{p}_1 \vee p_2) \,\&\, (\bar{p}_2 \vee p_1)$
0	1	1	1	eq. $p_1 \,\&\, p_2$	
1	0	0	0	eq. $\bar{p}_1 \vee \bar{p}_2$	
1	0	0	1	eq. $p_1 \nsim p_2$	eq. $(p_1 \,\&\, \bar{p}_2) \vee (\bar{p}_1 \,\&\, p_2)$
1	0	1	0	eq. p_2	
1	0	1	1	eq. $p_1 \,\&\, \bar{p}_2$	
1	1	0	0	eq. $\bar{p}_1$	
1	1	0	1	eq. $\bar{p}_1 \,\&\, p_2$	
1	1	1	0	eq. $\bar{p}_1 \,\&\, \bar{p}_2$	
1	1	1	1	eq. $p_1 \,\&\, \bar{p}_1$	

3. **Implication.** If $f(p_1, \cdots, p_x)$ and $g(p_1, \cdots, p_x)$ are any two truth functions such that g is true whenever f is true, then we write

$$f(p_1, \cdots, p_x) \rightarrow g(p_1, \cdots, p_x)$$

and read it "f implies g". *Intuitively the process of implication is the deduction of weaker truths from stronger ones.*

From our covering or *matrix* representation we can at once determine when two functions are such that $f \rightarrow g$. For this to be the case it is necessary and sufficient that whenever there is a 0 in the covering representing f there be also one in the covering representing g. We give a few examples.

a) $1_{00}\ 1_{01}\ 0_{10}\ 0_{11} \rightarrow 0_{00}\ 1_{01}\ 0_{10}\ 0_{11}$ eq. $p_2 \rightarrow (\bar{p}_1 \vee p_2)$,

b) $0\ 1\ 1\ 1 \rightarrow 0\ 0\ 1\ 1$ eq. $p_1 \,\&\, p_2 \rightarrow p_1$,

c)[5] $0\ 0\ 1\ 1 \rightarrow 0\ 0\ 0\ 1$ eq. $p_1 \rightarrow p_1 \vee p_2$.

All possible implications can be read off at once from the set of representing coverings.

4. **Consistency.** A *vacuous implication* is one of the form

$$1\ \ 1\ \ 1\ \ 1\ \ \cdots\ \ 1 \rightarrow g.$$

[5] Compare with Hilbert's axioms for the logic of propositions, loc. cit., p. 22. Note that axioms a) and c) there have no significance for us. $p_1 \vee p_1 \rightarrow p_1$ and $p_1 \vee p_2 \rightarrow p_2 \vee p_1$, are really *equivalences*, each side of the implication being merely different language for the other. See also II, 1.

If, barring vacuous implications, it is impossible that

$$f \rightarrow g, \qquad f \rightarrow \bar{g}$$

obtain simultaneously in our system, we call it *consistent*. That it is consistent is at once evident, for let

$$(\alpha_1 \alpha_2 \cdots \alpha_{2\varkappa}) \rightarrow (\alpha'_1 \alpha'_2 \cdots \alpha'_{2\varkappa}), \quad \alpha_i, \quad \alpha'_i = 0, 1$$

be any implication. Then $\alpha'_i = 0$ if $\alpha_i = 0$, and since $\bar{f}$ is obtained from f by taking the complementary covering (i. e., the 0's and 1's interchanged), it is clear that $f \rightarrow g$ and $f \rightarrow \bar{g}$ together is impossible.

We observe in passing that this representation enables one to determine at once, in any application (cf. for example those in Hilbert Ackermann, *Grundzüge*, loc. cit., pp. 20, 21), 1) what conclusions can be drawn from a set of axioms[6] 2) whether the axioms are consistent 3) whether they are redundant. It is easily seen that they are consistent if and only if the product-0-covering of all of the axioms is not empty. Two axioms are redundant if a 0-covering of one is also a 0-covering of the other. (By "0-covering" is meant the set of 0's in the covering. By "product-0-covering" of two coverings is meant the covering which has 0's only where each of the coverings has a 0.)

We now pass on to the logic of functions.

III. Logic of functions.

1. A mathematical function $\boldsymbol{f}(x)$ consists of a set of things $\{x_i\}$ and a device for answering "$\boldsymbol{f}(x_i)$" when x_i is inserted. In logic we shall be concerned only with functions $\boldsymbol{f}(x)$ for which $\boldsymbol{f}(x_i)$ is confined to "0" and "1", ("true" and "false"). Such functions are called propositional functions.

In conformity with our program we consider only functions of a finite number of variables, the range of each of which is finite.

2. **On universal language.** We commence our discussion of the logic of functions with certain considerations on "language" in general, in the light of which we may later best appreciate the role which classical logic plays in the complete logic.

Regard the general problem of setting up a universal language over a given body of discourse. Suppose we have a set of "objects" $\varepsilon_1, \cdots, \varepsilon_\mu$. With each of these objects ε_i let there be associated a set of $\varkappa$ "characteristics", $\varepsilon_{i1}, \varepsilon_{i2}, \cdots, \varepsilon_{i\varkappa}$, such that two objects are identical if and only if their corresponding sets of characteristics are identical, in some order. An object is given when a list of its characteristics is given.

[6] An axiom will correspond to one of the f's. This of course only applies to the case where the axioms can be put in propositional form.

Consider now a "language" which consists of a set of "expressions"

$$\begin{array}{llll} S_1(\varepsilon_1), & S_1(\varepsilon_2), & \cdots, & S_1(\varepsilon_\mu), \\ S_2(\varepsilon_1), & & \cdots, & S_2(\varepsilon_\mu), \\ \vdots & & & \vdots \\ S_\varrho(\varepsilon_1), & & \cdots, & S_\varrho(\varepsilon_\mu). \end{array}$$

If now we have a second language $\mathfrak{S}(\varepsilon, \alpha_1, \alpha_2, \cdots, \alpha_\varkappa)$ such that when ε is replaced by ε_i, and the variables $\alpha_1, \cdots, \alpha_k$ in a suitable way by the characteristics $\varepsilon_{i1}, \varepsilon_{i2}, \cdots, \varepsilon_{i\varkappa}$ of ε_i, any of the expressions $S_1(\varepsilon_i), S_2(\varepsilon_i), \cdots, S_\varrho(\varepsilon_i)$ may be obtained, then we call $\mathfrak{S}$ a *universal* and *complete* language over the S language. If however there is some expression in the S which cannot be obtained from the $\mathfrak{S}$, we call the latter *incomplete*.

In general a universal language will break up into a number of "statements", $\mathfrak{S}_1$, $(\varepsilon, \alpha_1, \cdots, \alpha_\varkappa), \cdots, \mathfrak{S}_\sigma(\varepsilon, \alpha_1, \cdots, \alpha_\varkappa)$, which we call *universal language statements,* such that each $\mathfrak{S}_\alpha$ is a universal language over a specific set of S's. For example, a particular language $\mathfrak{S}_1$ might be the universal language over S_1; $\mathfrak{S}_2$ over S_2, S_5, S_6; etc. As a special case, σ may of course be 1 or ϱ. Again as an important special case, $\varkappa$ might be 1.

As a simple example we have the familiar idea of "variable" over a given range. If the range is $x_1, x_2, \cdots$, then the "expressions" of our discourse are "x_1", "x_2", $\cdots$, and "x" is the universal and complete language over this.

Consider next a definite proposition "a". (A proposition is by definition either "0" or "1".) Then we can build all the propositional functions of a, $2^{(2^\varkappa)}$ in number, which we write a, $\bar{a}$, $a \vee \bar{a}$, $a \,\&\, \bar{a}$. Take another definite proposition "b", and again build the propositional functions of b. They are b, $\bar{b}$, $b \vee \bar{b}$, $b \,\&\, \bar{b}$. And similarly for any other given proposition. If now we regard each such set of propositional functions, written down for a definite proposition, as one of the "expressions" in our language, and we seek a universal and complete language over this, we at once find it in the familiar p, $\bar{p}$, $p \vee \bar{p}$, $p \,\&\, \bar{p}$ (or, as we have written it, $\underset{0\,1}{0\,1}$, $\underset{0\,1}{1\,0}$, $\underset{0\,1}{0\,0}$, $\underset{0\,1}{1\,1}$). After going through the same argument for several variables, we see that the *classical logic of propositions is a universal and complete language over the propositional functions of propositions.*

We consider next the logic of functions. Here, however, in contrast to the result obtained above for the logic of propositions, we shall find that *the classical logic of functions is an incomplete language over the functions of functions.*

3. Let us take a definite range (of cardinal number 3, say), Δ, δ, d. (This is the analogue of the proposition "a" in the above argument.)

A variable over this range will be denoted by $\mathfrak{d}$, or, in case several are needed, by $\mathfrak{d}_{(1)}$, $\mathfrak{d}_{(2)}$ etc. There are precisely 2^3 propositional functions over this given range, which we denote by $\boldsymbol{D}_1(\mathfrak{d}), \cdots, \boldsymbol{D}_8(\mathfrak{d})$. Thus for example:

$$\begin{array}{llll}
\boldsymbol{D}_1(\mathfrak{d}): & \boldsymbol{D}_1(\Delta) = 0 & \boldsymbol{D}_1(\delta) = 0 & \boldsymbol{D}_1(d) = 0 \\
\boldsymbol{D}_2(\mathfrak{d}): & \qquad\;\; 0 & \qquad\;\; 0 & \qquad\;\; 1 \\
\vdots & \qquad\;\; \vdots & \qquad\;\; \vdots & \qquad\;\; \vdots \\
\boldsymbol{D}_8(\mathfrak{d}): & \qquad\;\; 1 & \qquad\;\; 1 & \qquad\;\; 1
\end{array}$$

Let $\boldsymbol{D}$ (or $\boldsymbol{D}_{(1)}$, $\boldsymbol{D}_{(2)}$, etc. if several were needed) be a variable over the set $\boldsymbol{D}_1, \cdots, \boldsymbol{D}_8$. Consider now the $2^{(2^3)}$ propositional functions of the propositional functions $\boldsymbol{D}_1(\mathfrak{d}), \cdots, \boldsymbol{D}_8(\mathfrak{d})$, which we write $\boldsymbol{D}_1^*(\boldsymbol{D}), \cdots, \boldsymbol{D}_{2^8}^*(\boldsymbol{D})$. Thus, for example:

$$\begin{array}{lllll}
\boldsymbol{D}_1^*(\boldsymbol{D}): & \boldsymbol{D}_1^*(\boldsymbol{D}_1) = 0 & \boldsymbol{D}_1^*(\boldsymbol{D}_2) = 0 & \cdots & \boldsymbol{D}_1^*(\boldsymbol{D}_8) = 0 \\
\boldsymbol{D}_2^*(\boldsymbol{D}): & \qquad\quad 0 & \qquad\quad 0 & \cdots\ 0 & \qquad\quad 1 \\
\vdots & \qquad\quad \vdots & \qquad\quad \vdots & & \qquad\quad \vdots \\
\boldsymbol{D}_{2^8}^*(\boldsymbol{D}): & \qquad\quad 1 & \qquad\quad 1 & \cdots & \qquad\quad 1
\end{array}$$

By establishing the correspondence

$$(*) \qquad \boldsymbol{D}(\Delta) = p_1, \qquad \boldsymbol{D}(\delta) = p_2, \qquad \boldsymbol{D}(d) = p_3,$$

these may also be represented by the 2^8 coverings of $M^{(3)}$ (Cf. II, 3; these same coverings in the logic of propositions, were the representation of the $f(p_1, p_2, p_3)$). We list several of these coverings, together with their equivalents in formalistic language.[7]

	000	001	010	011	100	101	110	111		
a) $\boldsymbol{D}_1^*(\boldsymbol{D})$:	0	0	0	0	0	0	0	0	:	$(\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d}) \vee \overline{(\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d})}$
b) $\boldsymbol{D}_2^*(\boldsymbol{D})$:	0	0	0	0	0	0	0	1	:	$(E\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d})$[8]
c) $\boldsymbol{D}_5^*(\boldsymbol{D})$:	0	1	1	1	1	1	1	1	:	$(\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d})$
d) $\boldsymbol{D}_8^*(\boldsymbol{D})$:	0	0	0	0	1	1	1	1	:	$\boldsymbol{D}(\Delta)$
e) $\boldsymbol{D}_{11}^*(\boldsymbol{D})$:	0	0	1	1	0	0	1	1	:	$\boldsymbol{D}(\delta)$

[7] I should like to emphasize the non-unique character of the ordinary (including formalistic) language equivalents; i. e., a given covering may be translated into ordinary language in <u>any</u> number of different ways. For example, in the logic of propositions, $p \vee q$ eq. $q \vee p$; $p \& q$ eq. $\bar{p} \vee \bar{q}$; $p \sim q$ eq. $\bar{p} \vee q \;\&\; \bar{q} \vee p$. This flexibility is both a source of strength and of weakness, depending on the use to which language is to be put. It is the essence of poetry and a continual source of danger to the mathematician.

The Hilbert notation is here employed, as it shall be throughout whenever formalistic equivalents are given. Cf. Grundzüge, loc. cit.

[8] To illustrate the reading in ordinary language, consider $\boldsymbol{D}_2^*(\boldsymbol{D})$. Here $\boldsymbol{D}_2^*(\boldsymbol{D}_i) = 0$, $i = 1, \cdots, 7$; $\boldsymbol{D}_2^*(\boldsymbol{D}_8) = 1$. That is, $\boldsymbol{D}_2^*(\boldsymbol{D}_i)$ is true for such and only such $\boldsymbol{D}_i(\mathfrak{d})$ as are not false for all $\mathfrak{d}$; which is also true for $(E\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d})$.

	·000	·001	·010	·011	·100	·101	·110	·111	
f)	1	0	1	0	1	1	1	1 :	$\boldsymbol{D}(\Delta) \,\&\, \overline{\boldsymbol{D}(d)}$
g)	0	1	1	0	1	1	1	1 :	$(\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d}) \vee [\boldsymbol{D}(\Delta) \,\&\, \overline{\boldsymbol{D}(\mathfrak{d})}$ for $\mathfrak{d} \neq \Delta]$
h)	1	0	0	1	1	0	1	1 :	$[\boldsymbol{D}(\Delta) \,\&\, (E\mathfrak{d} \neq \Delta)\,\boldsymbol{D}(\mathfrak{d}) \,\&\, \overline{(\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d})}] \vee [\overline{\boldsymbol{D}(\Delta)} \,\&\, \boldsymbol{D}(\delta)]$
i)	1	0	0	0	1	0	1	1 :	$[\boldsymbol{D}(\Delta) \,\&\, \overline{(\mathfrak{d})\,\boldsymbol{D}(\mathfrak{d})}] \vee [\overline{\boldsymbol{D}(\Delta)} \,\&\, \boldsymbol{D}(\delta) \,\&\, (\mathfrak{d} \neq \delta)\,\overline{\boldsymbol{D}(\mathfrak{d})}]$
j)	1	0	1	0	1	0	0	1 :	$[\boldsymbol{D}(\Delta) \,\&\, \boldsymbol{D}(\delta) \,\&\, (\mathfrak{d} \neq \Delta, \delta)\,\overline{\boldsymbol{D}(\mathfrak{d})}] \vee$ [$\boldsymbol{D}(\mathfrak{d})$ is true for precisely one value of $\mathfrak{d}$].

Similarly for any other definite range (of the same cardinal number), for example γ, g, g we can write down the $\boldsymbol{G}(\mathfrak{g})$, $\boldsymbol{G}^*(\boldsymbol{G})$ corresponding to the $\boldsymbol{D}(\mathfrak{d})$ and the $\boldsymbol{D}^*(\boldsymbol{D})$. Now let the "expressions" of our discourse be the propositional functions of propositional functions over any given range (of cardinal number 3), e. g., the $\boldsymbol{D}^*$ or the $\boldsymbol{G}^*$ etc., and let us seek a universal and complete language over these.

Consider first, for example, $\boldsymbol{D}_2^*(\boldsymbol{D})$. The universal language statement corresponding to this is clearly $(Ex)\,F^{(3)}(x)$. Here x is a symbol which, when any definite range (of cardinal number 3) is given, may be replaced by any variable on that range; and F may be replaced by a variable over the set of (2^3) propositional functions over that range. $F^{(3)}$ we call a variable function of kind (3). Similarly, for example, the universal language statement corresponding to $\boldsymbol{D}_5^*(\boldsymbol{D})$ is $(x)\,F(x)$. The two cases $\boldsymbol{D}_2^*(\boldsymbol{D})$, $\boldsymbol{D}_5^*(\boldsymbol{D})$ are examples of a special subclass of the $\boldsymbol{D}^*$'s which is characterized by the fact that each element of this class has a different universal language statement corresponding to it. The general case however, is that in which to each of a certain class of $\boldsymbol{D}_i^*(\boldsymbol{D})$ (presently defined as an *equivalent* class) corresponds the same universal language statement. We shall illustrate this by a simple example, which may serve as a prototype.

Let us consider the universal language statement corresponding to $\boldsymbol{D}_8^*(\boldsymbol{D})$ i. e., $\boldsymbol{D}(\Delta)$. $\boldsymbol{D}(\Delta)$ is that $\boldsymbol{D}_i^*(\boldsymbol{D})$ which is true for those and only those propositional functions over the range Δ, δ, d that are true for the value Δ. But in a general range there is nothing to distinguish that definite value which corresponds to Δ from that which corresponds to δ (or d). Hence the universal language statement must correspond to $\boldsymbol{D}_{11}^*(\boldsymbol{D})$ and to $\boldsymbol{D}_{14}^*(\boldsymbol{D})$ as well as to $\boldsymbol{D}_8^*(\boldsymbol{D})$. We shall write it $F(\alpha)$, or sometimes $(Dx)\,F(x)$. Here α is a symbol which, when any definite range is given, may be replaced by any *specific* element of that range; F is as above.

In general, since our language is to be universal over the propositional functions of propositional functions over *any* given range, (which in general

has no structure), any set of $\boldsymbol{D}_i^*$ which can be distinguished only by means of a structure in the set Δ, δ, d will be represented by the same universal language statement.

4. **Metafunction $\Phi(F)$.** To formulate the above mathematically we introduce the notion of equivalence of two $\boldsymbol{D}_i^*(\boldsymbol{D})$. Two propositional functions of propositional functions $\boldsymbol{D}_i^*(\boldsymbol{D})$ are called *equivalent* if and only if one can be made identical with the other by some permutation of the elements of the independent variable range Δ, δ, d. In terms of the coverings which represent the $\boldsymbol{D}_i^*(\boldsymbol{D})$, two coverings are equivalent if and only if one goes into the other by some permutation of places in the index matrix. Thus the following three coverings (respectively representing $\boldsymbol{D}(\Delta)$, $\boldsymbol{D}(\delta)$, $\boldsymbol{D}(d)$) are equivalent.

$\dot{0}00$	$\dot{0}01$	$\dot{0}10$	$\dot{0}11$	$\dot{1}00$	$\dot{1}01$	$\dot{1}10$	$\dot{1}11$
0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1.

The $\boldsymbol{D}_i^*(\boldsymbol{D})$ (or the coverings of $M^{(3)}$) thus separate into equivalent classes of coverings. Each such equivalent class of coverings will be called a *metafunction*. A variable metafunction will be denoted by $\Phi(F)$. Each covering in an equivalent class is called a *component* of the corresponding $\Phi_i(F)$. Two $\Phi_i(F)$ will be called non-equivalent if their corresponding classes of coverings are not identical.

The *metafunctions* $\Phi(F)$ form then the universal and complete language over the propositional functions of propositional functions.[9] It may be verified that those $\Phi_i(F)$ which have only one component, i. e., which are *invariant* under the symmetric permutation group on the independent variable range, correspond to the propositional functions of propositional functions of *classical logic*. It will thus be seen to what extent classical logic is incomplete (cf. 2) though universal.

5. **Reading of the $\Phi(F)$.** The translation of the $\Phi_i(F)$ into ordinary language requires a word of explanation. It follows quite easily from the definition of a metafunction that a $\Phi_i(F)$ may be read by first reading any component of the $\Phi_i F)$ and then abstracting from all structure in the independent variable range. The test of a correct reading of a $\Phi_i(F)$ is furnished by constructing the coverings by means of the reading alone; if this is identical with the coverings of the corresponding $\Phi_i(F)$ the reading is correct.

[9] The objects ε_i are "ranges"; the "characteristics" are elements of the range; the "expressions" S_i, are defined above, e. g., $\boldsymbol{D}_i^*$ or $\boldsymbol{G}_i^*$, etc.; the $\mathfrak{S}_\alpha$ are the Φ_α.

To illustrate this reading several examples will be given. For ease of comparison the numbering of the examples runs parallel to those of the $\mathfrak{D}_i^*$ previously considered.

$$\begin{array}{lcccccccc}
 & \dot{}000 & \dot{}001 & \dot{}010 & \dot{}011 & \dot{}100 & \dot{}101 & \dot{}110 & \dot{}111 \\
\hline
\text{b}') & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\text{c}') & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

b') eq. $(Ex)\,F(x)$.

c') eq. $(x)\,F(x)$.

d') $$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$ eq. $(Dx)\,F(x)$ eq. $F(a)$ eq. $F(x)$ is true for a specific x.

f') $$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$ eq. $F(x)$ is true for one specific value of x and false for one (other) specific value of x.

g') $$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$ eq. $F(x)$ is either true for all values of x, or else true for precisely one specific value of x.

j') $$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$ eq. $F(x)$ is either true for precisely two, specific x, or else $F(x)$ is true for precisely one value of x.

6. **Metafunctions of several arguments.** In the foregoing we have, for concreteness, confined ourselves to $\varkappa = 3$. The metafunctions $\Phi(F^{(3)})$ are the non-equivalent classes of coverings of $M^{(3)}$. In the same way the non-equivalent classes of coverings of $M^{(\varkappa)}$ give the metafunctions $\Phi(F^{(\varkappa)})$.

The Φ's of several functions, and of functions of several variables, e. g., $\Phi(F_{(1)}^{(\varkappa)}, F_{(2)}^{(\varkappa)}, F^{(\varkappa)(\lambda)})$, etc., are obtained by a simple extension of the method discussed for $\Phi(F^{(\varkappa)})$. We have only to start with a more general set of indexed points than $M^{(\varkappa)}$, and a corresponding generalization of the definition of the equivalence of two coverings of this general M. This will be shown in IV., 5. In view of the fact that all the characteristic results manifest themselves already with functions of a single variable, we shall however largely confine ourselves to these.

7. **Negation.** The negation of the Φ's is so important in logic that we must say a few words about it. Every proposition has its negative.

The $2^{(2^z)} f(p_1, \cdots, p_z)$ can be paired off, each f with its negative $\bar{f}$, whose covering is the complementary of that of f. (By "complementary" we mean, 0's and 1's interchanged). Similarly all the $\Phi(F)$ can be paired off, each with the Φ determined by the complementary class of equivalent coverings, which we denote by $\bar{\Phi}$ and which is obviously "not Φ". By its definition

$$\overline{(\bar{\Phi})} = \Phi$$

which is the excluded middle for metafunctions. A few formal properties of negation which may easily be verified are:

$$\overline{(Ex)F(x)} \text{ eq. } (x)\overline{F(x)}, \quad \overline{(x)F(x)} \text{ eq. } (Ex)\overline{F(x)}, \quad \overline{(Dx)F(x)} \text{ eq. } (Dx)\overline{F(x)}.$$

The first two are equivalences which are familiar from classical logic. The last is not. If we recall the meaning of (Dx) it says, in words: "It is false that $F(\alpha)$ is true" (where α is a specific x), is equivalent to "$F(\alpha)$ is false".

8. **& and v.** From all the metafunctions (for a given number and kind of variable functions) select any two X, Ψ. In classical logic $X \,\&\, \Psi$, and $X \vee \Psi$ have a unique meaning. This cannot be true in general for us, as X and Ψ may each involve "specific x", in which case X and Ψ may be added, or multiplied (logically) in different ways, depending on how the definite x's in X and Ψ are related.

Now for Φ's with only one component, which is the case of strictly classical logic, if X, Ψ each have only one component, $X \vee \Psi$ is given by the product-0-covering (see II, 5) and $X \,\&\, \Psi$ by the sum-0-covering of X and Ψ respectively.

In the general case, if f is any component of X, and g any component of Ψ, then $f \,\&\, g$ and $f \vee g$ determine Φ's which we may write $X \,\&_{(i)}\, \Psi$, and $X \vee_{(i)} \Psi$ respectively. A subscript is necessary since if f', g' are other components, $f \,\&\, g$ and $f \vee g$ will not in general determine the same Φ as above. These different $X \,\&_{(i)}\, \Psi$, $X \vee_{(i)} \Psi$ when expressed in words, correspond, as already indicated, to the different relations which may be established between the "specific x's" of X and Ψ respectively. All this is very clearly seen from the geometrical representation of the metafunctions developed in IV.

If for two given metafunctions X, Ψ we write down all non-equivalent $X \,\&_{(i)}\, \Psi$, $X \vee_{(i)} \Psi$, $\bar{X} \,\&_{(i)}\, \bar{\Psi}$, $\bar{X} \vee_{(i)} \bar{\Psi}$ it may readily be shown that

$$\overline{X \,\&_{(i)}\, \Psi} \text{ eq. to one of the } \bar{X} \vee_{(i)} \bar{\Psi},$$

$$\overline{X \vee_{(i)} \Psi} \text{ eq. to one of the } \bar{X} \,\&_{(i)}\, \bar{\Psi}.$$

IV. Classification of the $\Phi(F)$'s.

1. In the covering representation of the $f(p_1, \cdots, p_\varkappa)$, each of the $2^{(2^\varkappa)}$ different coverings of $M^{(\varkappa)}$ represented a different f_i. In the case of the $\Phi(F(x))$, each covering of $M^{(\varkappa)}$ *determines* a $\Phi_i(F^{(\varkappa)})$, but different coverings may or may not determine the same $\Phi_i(F^{(\varkappa)})$ depending on whether or not they are equivalent coverings. The problem of completely characterizing the $\Phi(F)$ appears to be quite difficult. It is, on the other hand, very easy to give *necessary* conditions that two coverings of $M^{(\varkappa)}$ be equivalent. In the following this problem is only touched. In the first part of this section the problem is formulated in what appears to be its most natural form, namely as a certain representation (in the group-theoretical sense) of the symmetric permutation group $P_\varkappa$ on the $\varkappa$ elements of the independent variable range. The explicit form of the representation of $P_\varkappa$ is found in the case of a certain fundamental sub-class of the $\Phi(F)$. In the latter part of the section the covering-representation of the $\Phi(F)$ is transformed into geometric garb, which has the advantage of greater intuitive clarity. Here each $\Phi(F)$ is determined by a generalized $(\varkappa - 1)$-dimensional complex,[10] and conversely, each generalized $(\varkappa - 1)$-dimensional complex determines a $\Phi(F^{(\varkappa)})$.

2. We recall a few necessary facts from the theory of groups and their representation. Let $\mathfrak{G}$ be any abstract group, with elements $s, s', s'', \cdots$. If we have a space (in the widest sense, i. e., merely a set of objects with or without structure) and a set of transformations $T(s)$ which carry the space into itself and which obey the same multiplication law as the group, i. e.,

$$s \to T(s), \qquad s' \to T(s'), \qquad ss' \to T(s)\,T(s') = T(ss'),$$

we speak of a *realization* of the abstract group $\mathfrak{G}$. The space which undergoes transformation we will call a *realization space*. A realization is *faithful* if to different s correspond different $T(s)$. A *representation* is a linear realization, and we may write for such

$$s \to U_{ij}(s)$$

or else

$$\xi'_i = \sum_{j=1}^{n} U_{ij}(s)\,\xi_j$$

where $U_{ij}(s)$ is a square matrix. The number n is called the *degree* of the representation. A realization is *irreducible* if the realization space does not break up into subspaces, any of which is again a realization space.

[10] "Generalized complex" is used in the sense of O. Veblen, Cambridge Colloquium Lectures, Analysis Situs, 1916, namely, as an arbitrary subset of an $(\varkappa - 1)$-dimensional complex.

Let $\xi_i' = \sum U_{ij}(s)\,\xi_j$ be a representation of $\mathfrak{G}$, call it U. Then from this we may derive other representations, in particular the "skew-symmetrical" one whose coördinates are given by the r-rowed determinants of the matrix

$$\begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \eta_1 & \eta_2 & \cdots & \eta_n \\ \vdots & & & \\ \zeta_1 & \zeta_2 & \cdots & \zeta_n \end{pmatrix}$$

where $\xi_i, \eta_i, \cdots, \zeta_i$ are "vectors" in the representation space U. This representation we shall call $\{U^r\}$. In the particular case when $U_{ij}(s) \geqq 0$ for all s; $i, j, = 1, \cdots, n$, we may choose as coördinates of the representation space of $\{U^r\}$ the absolute values of the r-rowed determinants of the above matrix.[11]

3. Set

$$M^{(\varkappa)} = \sum_{\tau=0}^{\varkappa} M_\tau^{(\varkappa)}$$

where $M_\tau^{(\varkappa)}$ contains those points of $M^{(\varkappa)}$ which have precisely τ 0's in their indices. (Here "$\sum$" means simple a grouping together, or logical addition of the points.) It is clear that a necessary (but by no means sufficient) condition that two coverings determine the same Φ is that the number of 0's in each $M_\tau^{(\varkappa)}$ be the same for each. This is so because the points of each $M_\tau^{(\varkappa)}$ constitute a realization space of $P_\varkappa$, and moreover an irreducible one. Thus, for example, $M_0^{(\varkappa)}$ consists of only one point, $(\cdot_{111\cdots1})$ and is the realization space of the trivial realization $P_\varkappa \to \mathbf{1}$. $M_1^{(\varkappa)}$ consists of $\varkappa$ points and is the realization space of a faithful realization of $P_\varkappa$.

Consider the sub-class of the $\Phi(F)$'s whose determining coverings lie within a single $M_\tau^{(\varkappa)}$. Such a Φ we will denote by $\Phi^{(\tau)}$. Now, as already observed, $M_\tau^{(\varkappa)}$ is an irreducible (and hence transitive) realization space of $P_\varkappa$; but since it is not multiply transitive we have that even here the number of 0's in the $M_\tau^{(\varkappa)}$ of the covering, though a necessary condition that two $\Phi^{(\tau)}$'s be equivalent, is not sufficient.

Let $\boldsymbol{P}_\varkappa$ be the simplest realization of the permutation group $P_\varkappa$, namely that whose realization space consists of the $\varkappa$ points being permuted. Denote these points by $\xi_1, \xi_2, \cdots, \xi_\varkappa$, which, for purposes of visualization, we may regard as $\varkappa$ mutually orthogonal unit vectors in a Euclidean $\varkappa$-space, the point ξ_i being the vector $(0, 0, \cdots, \xi_i, 0, \cdots, 0)$. We have here a representation of $P_\varkappa$, where the ξ_i are coördinates of the representation space. For example to

[11] For example for $r = 2$, the coördinates of the representation $\{U^2\}$ may be taken as $(\xi_i\eta_j - \xi_j\eta_i)$. In case $U_{ij}(s) \geqq 0$ for all s; $i, j, 1, \cdots, n$, we may choose as coördinates of the representation space of $\{U^2\}$ the quantities $|(\xi_i\eta_j - \xi_j\eta_i)|$.

$$
(132) \text{ of } P_\varkappa \rightarrow \begin{array}{l} \xi_1 \rightarrow \xi_3, \\ \xi_3 \rightarrow \xi_2, \\ \xi_2 \rightarrow \xi_1, \\ \xi_\mu \rightarrow \xi_\mu, \quad \mu \neq 1, 2, 3, \end{array}
$$

with the matrix

$$
U_{ij}((132)) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & 0 & \\
 & & & & & & 1 & & & & \\
 & & & & & & & 1 & & & \\
 & & & & 0 & & & & \ddots & & \\
 & & & & & & & & & & 1
\end{pmatrix}.
$$

Consider now $\{\boldsymbol{P}_\varkappa^\tau\}$, that is, the τth skew symmetrical representation of $P_\varkappa$; it corresponds to the transformations of the linear r-spaces spanned by the r vectors $\xi_i, \eta_i, \cdots, \zeta_i$ of $\boldsymbol{P}_\varkappa$. The degree of this representation is ${}_\varkappa C_\tau$ (number of combinations of $\varkappa$ things τ at a time). We now assert:

The representation of $P_\varkappa$ determined by the space $M_\tau^{(\varkappa)}$ is equivalent to the representation of $\{\boldsymbol{P}_\varkappa^\tau\}$.

To prove this we observe that the coördinates of the space $\{\boldsymbol{P}_\varkappa^\tau\}$ reduce, by our choice of coördinates in $\boldsymbol{P}_\varkappa$, to the form $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\tau}$ where $i_1, i_2, \cdots, i_\tau$ are all different, and independently take on the values $1, 2, \cdots, \varkappa$.[12] These ${}_\varkappa C_\tau$ coördinates obviously can be put in one-to-one correspondence with the ${}_\varkappa C_\tau$ points of $M_\tau^\varkappa$ in such a way that corresponding points always transform together under $P_\varkappa$. To $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\tau}$ we need only associate that point of $M_\tau^{(\varkappa)}$ whose index is 0 at the places $i_1, i_2, \cdots, i_\tau$.

Let r be the number of points in a sub-space of $M_\tau^{(\varkappa)}$. Any such sub-space determines a new representation of $P_\varkappa$. Let $M_{\tau,r}^{(\varkappa)}$ be the space of ${}_{\varkappa c_\tau}C_r$ points each of which consists of an r-space of $M_\tau^{(\varkappa)}$. $M_{\tau,r}^{(\varkappa)}$ is then a representation space of $P_\varkappa$, and because of its fundamental role in the classification of the Φ's we will call it the *fundamental representation* of $P_\varkappa$. This representation however bears exactly the same relation to $M_\tau^{(\varkappa)}$ as $M_\tau^{(\varkappa)}$ does to $\boldsymbol{P}_\varkappa$, and hence we have the theorem:

The fundamental representation of $P_\varkappa$ is given by $\{\{\boldsymbol{P}_\varkappa^\tau\}^r\}$, where each $\{\ \}$ refers to the corresponding derived skew-symmetrical representation. The

---

[12] Since $U_{ij}(s) \geqq 0$ for all s; $i, j, 1, \cdots, n$, we choose as coördinates of the representation $\{U^\tau\}$ the *absolute values* of the r-rowed determinants discussed in the footnote to § 2.

composition of the fundamental representation is quite complicated. It is not, for example, irreducible. The number of irreducible representations into which it can be decomposed, whether they are equivalent from the group-theoretical standpoint or not, gives the number of non-equivalent $\Phi^{(\tau)}$'s. We shall not attempt to give a decomposition of $\{\{\boldsymbol{P}_{\varkappa}^{\tau}\}^{r}\}$, but what we shall do is to transform this representation into geometric garb, in which form it is easily grasped intuitively, and show in this way how the characterization *fundamental representation* is justified. We shall illustrate with $\varkappa = 4$.

4.

$$M^{(4)} \left| \begin{array}{llllll} M_4^{(4)} \cdot_{0000} & & & & & \\ M_3^{(4)} \cdot_{0001} & \cdot_{0010} & \cdot_{0100} & \cdot_{1000} & & \\ M_2^{(4)} \cdot_{0011} & \cdot_{0101} & \cdot_{0110} & \cdot_{1001} & \cdot_{1010} & \cdot_{1100} \\ M_1^{(4)} \cdot_{0111} & \cdot_{1011} & \cdot_{1101} & \cdot_{1110} & & \\ M_0^{(4)} \cdot_{1111}. & & & & & \end{array} \right.$$

$M_0^{(4)}, \cdots, M_4^{(4)}$ are the irreducible sub-spaces of $M^{(4)}$.

$M_0^{(4)}$ is the representation space to which corresponds the representation $P_4 \to \mathbf{1} \equiv \{\boldsymbol{P}_4^0\}$,

$M_1^{(4)}$ " $P_4 \to \boldsymbol{P}_4 \equiv \{\boldsymbol{P}_4^1\}$,

$M_2^{(4)}$ " $P_4 \to \{\boldsymbol{P}_4^2\}$,

$M_3^{(4)}$ " $P_4 \to \{\boldsymbol{P}_4^3\} \equiv \{\boldsymbol{P}_4^1\}$,[13]

$M_4^{(4)}$ " $P_4 \to \{\boldsymbol{P}_4^4\} \equiv \{\boldsymbol{P}_4^0\} = \mathbf{1}$.

Consider the 4 points

$$\begin{array}{cc} .1 & .2 \\ .4 & .3 \end{array}$$

Representation $\boldsymbol{P}_4$

where

$$\cdot_{0111} = 1,$$
$$\cdot_{1011} = 2,$$
$$\cdot_{1101} = 3,$$
$$\cdot_{1110} = 4.$$

This is the representation $\boldsymbol{P}_4$. From what has been said it follows that we can regard the points of the representation $\{\boldsymbol{P}_4^2\}$ as the ${}_4C_1$, 1-cells formed from the points of $\boldsymbol{P}_4$, namely

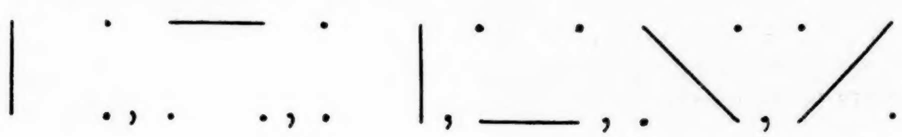


Fig. 1. Representation $\{\boldsymbol{P}_4^2\}$

[13] This duality $\{\boldsymbol{P}_{\varkappa}^{\sigma}\} = \boldsymbol{P}_{\varkappa}^{\varkappa-\sigma}$ is obviously true quite generally.

Similarly the points of the representation $\{\boldsymbol{P}_4^2\}$ may be taken as the 2-cells formed from the points of $\boldsymbol{P}_4$, namely

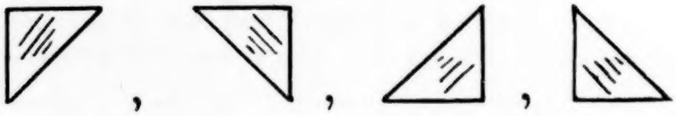
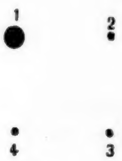


Fig. 2. Representation $\{\boldsymbol{P}_4^2\}$.

And so on for the other $\{\boldsymbol{P}_4^i\}$. Consider a $\Phi^{(1)}$. Since $\boldsymbol{P}_1$ is a faithful representation of P_4, the number of 0's in $\Phi^{(1)}$ completely characterize it, and there are accordingly precisely 4 $\Phi^{(1)}$'s. They are *determined* by the following 4 figures. (When we speak of a Φ being *determined* by a given covering or figure we mean the Φ which is given by all the coverings or figures equivalent to it under P_χ. Any *component* of a Φ thus *determines* the Φ.)



determines "$F^{(4)}(x)$ is true for precisely one, definite x".

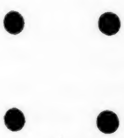
Fig. 3.

determines "$F^{(4)}(x)$ is true for precisely one of two, definite x".

Fig. 4

determines "$F^{(4)}(x)$ is true for precisely one of three, definite x".

Fig. 5.



determines "$F(x)$ is true for precisely one x".

Fig. 6.

Similarly there are 10 different $\Phi^{(2)}$'s, determined by the 10 figures:

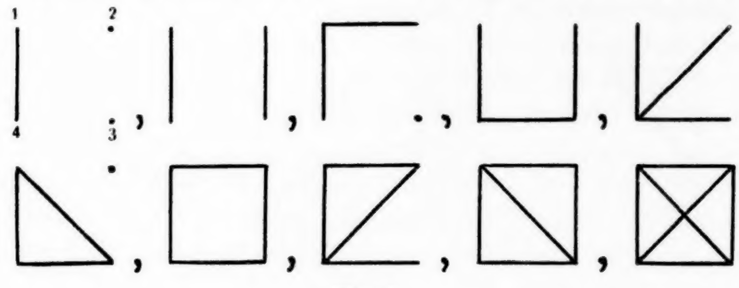


Fig. 7.

And so on for the other $\Phi^{(\tau)}$'s. In the case of a general Φ we have more complicated determining figures,—complexes built up of cells of different dimensionality.[14] We give two examples, in which $\varkappa$ has been taken as 3:

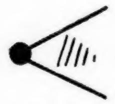
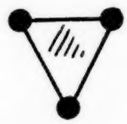
 determines $(Dx)\,F(x)$,  determines $(Ex)\,F(x)$.

Fig. 8. Fig. 9.

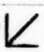
From this we can at once formulate the geometrical representation in the general case. *With $\varkappa$ points as possible vertices construct complexes of* $0, 1, 2, \cdots, (\varkappa-1)$ *dimensional simplexes in any manner whatever. Each such complex determines a $\Phi(F)$; two complexes determine the same Φ if and only if they are equivalent under $P_\varkappa$.*[15]

5. Metafunctions of several functions, and of several variables. Consider first the representation of $\Phi(F^{(\varkappa)(\lambda)}(x, y))$. In this case we have $F(x_i, y_j) = p_{ij}$. The set $M^{(\varkappa)(\lambda)}$ now consists of points indexed with two dimensional matrices. For example for $\varkappa = 2$, $\lambda = 3$ we have

$$M^{(\varkappa)(\lambda)}\cdot\ \overset{\cdot}{\substack{000\\000}}\quad \overset{\cdot}{\substack{001\\000}}\quad \overset{\cdot}{\substack{010\\000}}\quad \cdots\quad \overset{\cdot}{\substack{111\\111}}\cdot$$

The matrix is ordered to correspond with

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{pmatrix}.$$

[14]  for example determines the Φ one of whose components is

$$\underset{\substack{\cdot\\0110}}{0}\quad \underset{\substack{\cdot\\1010}}{0}\quad \underset{\substack{\cdot\\1100}}{0}\quad \underset{\cdot}{1}\quad \underset{\cdot}{1}\quad \cdots\quad \underset{\cdot}{1}$$

[15] Prof. Veblen has pointed out to me that the more natural geometrical representation along these lines is not in terms of complexes but in terms of the equivalences of various linear subspaces of a projective space, i. e., those determined by the above complexes. It is clear that one mode of representation may be translated into the other, but it is probable that the latter method may be more elegantly treated.

The group under which two coverings of M are to be regarded as equivalent must be such that it, a) recognizes no structure in either the x's or the y's, and b) does distinguish between the x's and the y's.[16] This means interchange of rows and columns in the index matrix.

Next consider $\Phi(F^{(\varkappa)}(x), G^{(\lambda)}(y))$. Here we have $F(x_i) = p_i$, $G(y_i) = q_i$. For $\varkappa = 2$, $\lambda = 3$, for example,

$$M = \cdot_{00,000} \quad \cdot_{00,001} \quad \cdot_{00,010} \quad \cdots \quad \cdot_{11,111}.$$

The equivalence group is plainly $P_\varkappa$ and P_λ applied respectively to the left and right of the "," in the index matrix. In case we are dealing with $\Phi(F^{(\varkappa)}(x), G^{(\varkappa)}(x))$, the equivalence group is $P_\varkappa$ applied simultaneously to the right and left of ",".

Since there is nothing fundamentally new introduced by more than one variable function we will not pursue this in detail. A few examples follow. Take $\varkappa = 2$, $\lambda = 2$

$$M = \begin{array}{cccccccc} \cdot_{00,00} & \cdot_{00,01} & \cdot_{00,10} & \cdot_{00,11} & \cdot_{01,00} & \cdot_{01,01} & \cdot_{01,10} & \cdot_{01,11} \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot_{10,00} & \cdot_{10,01} & \cdot_{10,10} & \cdot_{10,11} & \cdot_{11,00} & \cdot_{11,01} & \cdot_{11,10} & \cdot_{11,11} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

This is, $(x)(F(x) \mathbin{\&} G(x))$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

This is, $(x) F(x) \vee (Dx) G(x)$.

$$\begin{array}{cccccccccccccccc} \cdot\,00 & \cdot\,00 & \cdot\,00 & \cdot\,00 & \cdot\,01 & \cdot\,01 & \cdot\,01 & \cdot\,01 & \cdot\,10 & \cdot\,10 & \cdot\,10 & \cdot\,10 & \cdot\,11 & \cdot\,11 & \cdot\,11 & \cdot\,11 \\ 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}.$$

This is $(x)(Ey) F(x, y)$.

$$0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1.$$

This is $(Ey)(x) F(x, y)$.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This is $(Dy)(x) F(x, y)$.

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[16] This is of course evident if $\varkappa \neq \lambda$. If $\varkappa = \lambda$ we can conceive of a still more general case in which not even this rudimentary structure is recognized. We practically never meet this case in practice as in almost all cases we can distinguish between the "first" and the "second" variables of a function $f(x, y)$. The only gain, if we admitted this kind of variable function, would be of the kind we are already familiar with; e. g., we should have such classifications in the metafunctions as "there exists a variable ...", and "there exists a definite variable ...". The group corresponding is the above group enlarged by the operation *transposition of rows and columns* in the index matrix.

V. Implication.

1. In the logic of propositions, given any two functions $f(p_1, \cdots, p_k)$ and $g(p_1, \cdots, p_k)$ we could immediately tell whether $f \to g$ was a valid implication or not. We wish now to investigate implication between metafunctions. Implication, we recall, is the deduction of weaker from stronger truths.

Let X and Ψ be any two metafunctions. By means of the relation corresponding to (*), III, 3, any component of X (or Ψ) may be read as an $f(p_1, \cdots, p_k)$. Take now f, g, any components of X and Ψ respectively, and consider $f \to g$. This may or may not be a valid implication. If it is, however, this property remains invariant when $P_\varkappa$ is applied to $f \to g$; in other words $f' \to g'$, $f'' \to g''$, $\cdots$, (where $f', g'; f'', g''; \cdots$ are corresponding transforms of f and g), are all valid implications. Conversely, if $f \to g$ is not a valid implication, this property also remains invariant under P_k. This is at once clear if we recall that the validity of $f \to g$ means that a 0 appears in the covering which represents g at those places where f has a 0; and since by all permutations of the places the 0's go with them, this property is preserved.

From this it is but a step to the general formulation of implication between metafunctions:

If X, Ψ are any two metafunctions, and if f, g are any components of X, Ψ respectively such that $f \to g$ is a valid implication, then $f \to g$ determines an implication between X and Ψ. $f \to g$ is then a *component* of the implication, the implication proper being represented by the class equivalent to $f \to g$. A little reflection shows that, just as the metafunctions themselves were certain representations of the group $P_\varkappa$, so also are the implications between metafunctions. By means of the geometrical representation developed in IV, a very clear intuitive picture of implication is gained. This will be illustrated in the several examples to follow.

It is to be noted that there will in general be more than one non-equivalent implication between two metafunctions which fact we may conveniently indicate by the notation $X \underset{(1)}{\to} \Psi$, $X \underset{(2)}{\to} \Psi$, $\cdots$.

In the examples, the geometrical equivalent, where supplied, is set up for small values of $\varkappa$.

$(x)\ F(x) \to (Ex)\ F(x)$ determined by

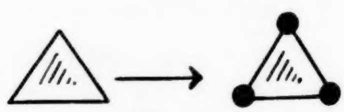

Fig. 10.

$(Dx)\ F(x) \to (Ex)\ F(x)$ determined by
$(Ey)\ (x)\ F(x, y) \to (x)\ (Ey)\ F(x, y)$.

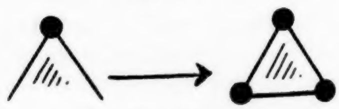

Fig. 11.

As an example of two non-equivalent implications between the same two metafunctions consider the following:

$$X \underset{(1)}{\rightarrow} \psi$$

$$\begin{pmatrix} 1_{000} & 0_{001} & 0_{010} & 1_{011} & 1_{100} & 1_{101} & 1_{110} & 1_{111} & \rightarrow & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$X \underset{(2)}{\rightarrow} \psi$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & \rightarrow & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

In words, 1) is: "$F(x)$ is true for precisely two values of x, one of these being a definite value" implies "$F(x)$ is either true for precisely two values of x, or else true for the same definite value of x as in X, and for no other value of x".

And 2) is: "$F(x)$ is true for precisely two values of x, one of these being a definite value" implies "$F(x)$ is either true for precisely two values of x, or else true for one definite value of x, *not* the same as in X, and for no other x".

VI. Logic of functions of higher types.

1. Our development so far has carried us through the logic of functions in the restricted sense[17], or, in Russell's terminology, the logic of *type 1*[18], though we have not explicitly mentioned types in the preceding. Strictly speaking, to what we heretofore have called propositional functions, truth functions, and metafunctions, should be added *of type 1*. It is now our purpose to show how the development may be naturally extended to the higher types.

We shall confine ourselves to the case of functions of kind $F^{(\varkappa)}(x)$, and shall carry the logic of such through the second type. This, together with the theory of type 1, with which we are already familiar, will indicate the characteristic features of the development, both with respect to all other types, and with respect to several functions F and of several variables.

2. **Propositional functions of type 2.** There are $2^{\varkappa}$ propositional functions of type 1, of a single variable, x, of range $\varkappa$. Denote these

[17] Compare with Hilbert-Ackermann "Grundzüge", loc. cit., Chap. III, IV.

[18] Cf., Whitehead Russell, Principia Mathematica, Vol. I, pp. 36–55.

by $F_1, F_2, \cdots, F_{2^\varkappa}$, and, as before, let $x_1, \cdots, x_\varkappa$ be the x range. For example, for $\varkappa = 2$, $F_1(x)$ might be:

$$F_1(x_1) = 0, \quad F_1(x_2) = 0.$$

Again $F_2(x)$ might be:

$$F_2(x_1) = 0, \quad F_2(x_2) = 1; \quad \text{etc.}$$

$F^{(x)}$, $G^{(x)}$ etc., or else $F^{(x)}_{(1)}$, $F^{(x)}_{(2)}$, etc.[19] denote variables over this range $F_1, \cdots, F_{2^\varkappa}$.

Consider the set $\{F_\alpha(x)\}$, $\alpha = 1, \cdots, 2^\varkappa$. A propositional function in which the set $\{F_\alpha(x)\}$ is the independent variable range is called a *propositional function of 2nd Russell type.* We will consider only such propositional functions of one argument as the theory here is typical. There are $2^{(2^\varkappa)}$ of them, which we denote by $\mathfrak{U}_1(F(x))$, $\mathfrak{U}_2(F(x))$, $\cdots$, $\mathfrak{U}_{2^{(2^\varkappa)}}(F(x))$. Let $\mathfrak{U}_{(1)}$, $\mathfrak{U}_{(2)}$, etc. be variables over this set.[20])

It is of course clear that the $\Phi_i(F(x))$ and the $\mathfrak{U}_\mu(F(x))$ are quite distinct functions. The Φ's are invariant under $P_\varkappa$ on the x_i; the $\mathfrak{U}$'s are not.

3. Metafunctions. The metafunctions of type 2 are represented similarly to type 1, with the necessary change in the equivalence group. Corresponding to $(*)$ we have

$$(**) \qquad \mathfrak{U}(F_\alpha(x_i)) = p_{\alpha i}, \qquad \alpha = 1, \cdots, 2^\varkappa,\ i = 1, \cdots, \varkappa$$

where the $p_{\alpha i}$ are variable propositions.

The $\Phi_i(F(x))$ recognized no structure in the $\{x_i\}$. Similarly, *a metafunction of type* 2 *must recognize no structure* a) *in the* $\{x_i\}$, b) *in the* $\{F_\alpha\}$. Let us denote variable metafunctions of type 2 by R, S, or else $R_{(1)}$, $R_{(2)}$, etc.

From this and our knowledge of the theory of type 1, we can formulate the representation of the $R_\sigma\big(\mathfrak{U}(F(x))\big)$ as follows. (Compare with III, IV).

[19] Note "$F^{(x)}_{(1)}$" (with parenthesis), is a variable, while "$F^{(x)}_1$", is a definite one of this set of all propositional functions.

[20] In mathematics we frequently have to do with functions $\mathfrak{U}(f(x))$, etc., where $f(x)$ is not a variable propositional function, but a more general kind. In analysis, for example, we are concerned with *individual* functions, i.e., functions whose values are individuals (that is, x_i). There are $\varkappa^\varkappa$ individual functions $f_1(x), \cdots, f_{\varkappa^\varkappa}(x)$ where the cardinal number of the individual range $\{x_i\}$ is $\varkappa$. It is not essential to treat these separately as it will easily be seen that the truth functions and implications are of the same kind as in the case of the propositional functions. (f) will, for example, refer to the totality of individual functions (1st type), whereas (F) referred to the totality of propositional functions (1st type).

Construct all the $f(p_{11}, \cdots, p_{2^\varkappa\varkappa})$; *then every set of equivalent* f*'s represents an* $R_\sigma\big(\mathfrak{U}(F(x))\big)$, *and non equivalent* f*'s represent non equivalent* R*'s. The possible* $R(\mathfrak{U})$ *are exhausted in this way.*

A few examples follow, in which for convenience $\varkappa$ is taken $= 2$. Here $M = \cdot_{0000;0000}\ \cdot_{0000;0001}\ \cdot_{0000;0010} \cdots \cdot_{0000;1111}\ \cdot_{0001;1111} \cdots \cdot_{1111;1111}$ where the index matrix corresponds to the value of $(p_{11}, p_{21}, p_{31}, p_{41}; p_{12}, p_{22}, p_{32}, p_{42})$. Two coverings of M are *equivalent*, (see second paragraph above) if they go into each other by their equivalence group which consists of a) the symmetric permutation group P_2 applied to the two parts (separated by ";") of the index matrix, and b) the symmetric permutation group P_4 applied simultaneously to each side of the ";" in the index matrix.

$$\text{a)}\quad 0\ 1\ 1\ \cdots\ 1 = (F)(x)\,\mathfrak{U}(F(x)),$$

$$\text{b)}\quad 0\ 0\ 0\ \cdots\ 0\ 1 = (EF)(Ex)\,\mathfrak{U}(F),$$

$$\text{c)}\quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 & 0 & \cdots & 1 \end{pmatrix} = (Dx)(F)\,\mathfrak{U}(F).$$

The *definite* F naturally makes its appearance alongside of the definite x.

From this and our complete treatment of type 1 it will be clear *mutatis mutandis* how the general case of $R\big(\mathfrak{U}(F(x, y, \cdots, z))\big)$, $R\big(\mathfrak{U}(F_{(1)}(x), F_{(2)}(x))\big)$, $R\big(\mathfrak{U}(F_{(1)}(x), F_{(2)}(y))\big)$, etc., $R(\mathfrak{U}_{(1)}, \mathfrak{U}_{(2)})$, etc., is represented. In each case we have a relation corresponding to $(**)$ and an equivalence group depending on the argument of R. For example, in $R\big(\mathfrak{U}(F(x), F(y))\big)$, we have $\mathfrak{U}(F_\alpha(x_i), F_\alpha(y_j)) = p_{\alpha ij}$, $\alpha = 1, \cdots, 2^\varkappa$; $i, j = 1, \cdots, \varkappa$. The equivalence group recognizes no structure a) in the $\{F_\alpha\}$, b) in the $\{x_i\}$, c) in the $\{y_i\}$, which is easily translated into the corresponding permutation group when applied to the coverings of M.

It will also be clear how we may proceed to all higher types.

4. Implication. This may also be treated very briefly. It is again carried back to the logic of propositions.

Let $f(p_{11}, \cdots, p_{2^\varkappa\varkappa}) \rightarrow g(p_{11}, \cdots, p_{2^\varkappa\varkappa})$ *be any valid implication; then this determines an implication between the metafunctions* R *and* S *determined by* f *and* g *respectively, and we write it*

$$R\big(\mathfrak{U}(F(x))\big) \underset{(i)}{\rightarrow} S\big(\mathfrak{U}(F(x))\big).$$

All possible implications between the different R*'s are obtained in this way.*

The need for the subscript (i) is as in the case of the $X \underset{(i)}{\rightarrow} \Psi$ (see V). We give two examples which may easily be verified.

$$\text{a)}\quad (DF)(x)\,\mathfrak{U}(F(x)) \rightarrow (EF)(x)\,\mathfrak{U}(F) \rightarrow (EF)(Dx)\,\mathfrak{U}(F),$$

$$\text{b)}\quad (Ex)(F)\,\mathfrak{U}(F) \rightarrow (F)(Ex)\,\mathfrak{U}(F).$$

5. **Negation.** Every R has its negative $\bar{R}$, which satisfies

$$\overline{(\bar{R})} = R.$$

If f is any component of the representation of R, then $\bar{R}$ is determined by $\bar{f}$.

6. **Mixed logic.** If we are interested in the logic of propositions and functions simultaneously, we have only to incorporate functions of the kind $F^{(1)}(x)$ in the logic of functions, for

$$F^{(1)}(x) = p.$$

In this way it would be possible to develop the logic of propositions as a special case of that of functions of type 1. Thus, for example,

$$(x)\,(F^{(1)}(x) \mathbin{\&} (Ey)\, G^{(x)}(y))$$

may be read

$$p \mathbin{\&} (Ey)\, G^{(x)}(y).$$

7. **Consistency.** It is clear that, barring vacuous implications, the system is consistent in that

$$R \underset{(i)}{\rightarrow} S, \qquad R \underset{(i)}{\rightarrow} \bar{S}$$

are simultaneously impossible. The proof is easily supplied, and is similar to II, 5. The consistency is assured for the logic up to any type.

VII. Evaluation.

In classical mathematical logic we have to do with a tool which will guide us, (consistently, we hope) in all mathematics,—for example this was the chief aim of Principia Mathematica. Since the mathematician immediately plunges into infinite sets, the logician was forced to do so also. Moreover, since the former continues to employ the principle of excluded middle (far beyond the limits of constructive thought) the latter did likewise. As far as infinite sets are concerned, realizing that the jump from the finite to the infinite was not to be followed by constructive thought, the logician simply introduced a new language to deal with infinite sets. This new language is built in the image of finite sets, and has as its hub the pure existence proposition (Ex) or (EF), etc. This has from the beginning been very close to the nucleus of the Intuitionism vs. Formalism controversy, and has been the cause of much misunderstanding.

Now formal logic need not have introduced any such discontinuity into the language of the infinite *so long as it built it in the image of the finite*, in particular, obeying the principle of excluded middle. Indeed this is what has been shown in the body of our work. *We have built up the*

complete language of mathematical logic as an uninterrupted, self-consistent, deductive system without ever passing from the finite.

In so doing the special position of classical formal logic in the *complete* logic is clearly brought out. Again the two kinds of existence statements, (Ex) and (Dx) (and of course the corresponding expressions in the higher types) make their appearance side by side. Let us examine (Ex) more closely.

When the body of possible implications is examined it is at once seen that

$$(Ex)\,F(x) \rightarrow (Ex)\,F(x)$$
$$(Ex)\,F(x) \rightarrow 0\;0 \cdots 0$$

are the only implications possible from $(Ex)\,F(x)$, These are both trivial, the latter being the metafunction which is always true. $(Ex)\,F(x)$ thus appears as the second weakest metafunction. This result also holds for the higher types; for example $(EF)\,\mathfrak{U}(F)$ implies nothing except the trivial metafunction corresponding to the above. With the fact in mind that our whole theory of implication is simply the theory of *deductive reasoning* via the excluded middle, we have (excepting the trivial cases mentioned above):

In a mathematics based on the excluded middle and proceeding purely deductively, the pure-existence statement $(Ex)\,F(x)$ (*or* (EF) *etc.*) *is empty in the sense that nothing can be deduced from it.*

The emptiness of pure-existence is one of the basic intuitionistic contentions which has often been denied by Formalists.[21] The precise meaning of "empty" is, I believe, clearly given by the above result. It is furthermore seen that it is not the infinite which is responsible for the emptiness of pure-existence, but rather the insistence upon a purely deductive structure.

We observe that

$$(Dx)\,F(x) \rightarrow (Ex)\,F(x)$$

but not the reverse. This reverse is however often practiced in mathematics in the form of the non-deductive step: "let a be a value of x for which $F(x)$ is true".

We have only demonstrated the emptiness of *pure*-existence. Such existence theorems as $(Ex)\,(y)\,F(x, y)$ are certainly not empty; for example,

$$(Ex)\,(y)\,F(x, y) \rightarrow (Ex)\,(Dy)\,F(x, y).$$

The right hand side is however again an existence proposition, weaker than $(Ex)\,(y)\,F(x, y)$. The question arises, when are we to define a given Φ (or R, etc.) as an *existence* proposition,—a question which offered no trouble

[21] Cf. for example Hilbert Grundlagen, loc. cit., pp. 13, 14.

in the case of *pure*-existence. The solution of this problem, which seems to be fairly difficult and which we shall not further deal with here, is to be looked for in the same way as the general definition of "$^{-}$", "$\&_{(i)}$", "$\mathrm{v}_{(i)}$", in terms of the respective coverings of Φ (or R, etc.).[22] When the proper definition of an existence Φ, $\Phi_{(\text{exist.})}$ is set up quite generally,—and it must of course include as special cases all the simple classical $\Phi_{(\text{exist.})}$'s such as $(Ex)\,(y)\,F(x,y)$, $(Ex)\,(Dy)\,F(x,y)$, etc., which are obviously existence propositions—I believe it will be found that the set of all Φ's (or R's etc., and of course with a given set of arguments) will split up into two classes

$$\{\Phi\} = \{\Phi_{(\text{exist.})}\} + \{\Phi_{(\text{non exist.})}\}$$

with the fundamental property

$$(\dagger) \qquad \Phi_{(\text{exist.})} \underset{(i)}{\rightarrow} \Phi_{(\text{non exist.})}$$

is impossible. Indeed this together with the theorem on the emptiness of pure-existence might be the clue to the proper general definition of a $\Phi_{(\text{exist.})}$.

It must not be concluded that existence theorems are of no interest, even though pure-existence propositions are empty, and the proposed theorem (†) be true. An existence theorem and a non-existence theorem (logical sum) may yield a *stronger* non-existence theorem. For example, consider

1) $(Dx)\,(Ey)\,F(x,y)$, or $(Ey)\,F(\alpha,y)$,

and

2) $(Dx)\,[F(x,y)$ is independent of $y]$, which may be written

$$(y)\,F(\alpha,y) \ \mathrm{v}\ (y)\,\overline{F(\alpha,y)}.$$

Here 1) is clearly an existence, and 2) a non-existence theorem. We have, however, as may be directly verified,

$$1)\ \&\ 2) = (Ey)\,F(\alpha,y)\ \&\ [(y)\,F(\alpha,y)\ \mathrm{v}\ (y)\,\overline{F(\alpha,y)}]\ \text{eq.}\ (y)\,F(\alpha,y),$$

which is no longer an existence theorem, but a theorem *stronger* than 2), as

$$(y)\,F(\alpha,y) \rightarrow 2).$$

In conclusion I wish to express my thanks and indebtedness to Professors O. Veblen and A. Church for their valuable and generous counsel.

[22] See III A, 8.

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THE INVARIANT THEORY OF FUNCTIONAL FORMS UNDER THE GROUP OF LINEAR FUNCTIONAL TRANSFORMATIONS OF THE THIRD KIND.[1]

By A. D. Michal and T. S. Peterson.

Introduction. The subject of "algebraic" functional forms and their functional invariants has been studied to a certain extent in previous papers.[2] In these papers the underlying group of transformations was the Fredholm group

$$y^i = \bar{y}^i + K^i_\alpha \bar{y}^\alpha \tag{0.1}$$

for which unity is not a characteristic value of the kernel K^i_α. The quadratic forms considered were of the specialized "Fredholm type"

$$g_{\alpha\beta}\, y^\alpha y^\beta + \int_a^b (y^\alpha)^2\, d\alpha. \tag{0.2}$$

The above formulae stand, respectively, for

$$y(i) = \bar{y}(i) + \int_a^b K(i, \alpha)\bar{y}(\alpha)\, d\alpha \tag{0.1'}$$

and

$$\int_a^b \int_a^b g(\alpha, \beta)\, y(\alpha)\, y(\beta)\, d\alpha\, d\beta + \int_a^b [y(\alpha)]^2\, d\alpha. \tag{0.2'}$$

These conventions

1) *of representing the arguments of functions as continuous indices and*
2) *of letting the repetition of a continuous index, once as a subscript and once as a superscript, signify a Riemannian integration with respect to that index throughout the fundamental interval* (a, b)

will be used throughout the paper. A parenthesis around any continuous index will be used to denote that the integration convention 2) does *not* apply to that index.

The object of this paper is to effect a generalization both in the type of forms and in the functional transformations. To this end, we have in the first part of the paper considered the quadratic functional form

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[1] Received March 27, 1930.—Presented at the New York meeting of the American Mathematical Society March 29, 1929.

[2] A. D. Michal, "*Affinely connected function space manifolds,*" American Journal of Mathematics, vol. 50 (1928), pp. 473–517. See also, T. S. Peterson, "*A class of invariant functionals of quadratic functional forms,*" American Journal of Mathematics, vol. 51 (1929), pp. 417–430.

$$g_{\alpha\beta}\, y^{\alpha}\, y^{\beta} + g_{\alpha}\, (y^{\alpha})^2$$

and its invariantive properties. In the second part, we consider this form under the more general linear functional transformations of the third kind

$$y^i = K^i\, \bar{y}^i + K^i_{\alpha}\, \bar{y}^{\alpha}.$$

In the last part of this paper, we confine our attentions to a study of simultaneous functional invariants of various types of functional forms and to the related subject of Fréchet differentials of functionals.

The paper is confined entirely to functions which are continuous in all their arguments—each argument ranging over the fundamental interval (a, b), and all integrations in the sense of Riemann.

The methods and results of this paper, besides being of interest in themselves, are most important in their applications to various function-space geometries.

1. **Generalized quadratic functional form.** Let us consider the laws of transformation induced in the coefficients $g_{\alpha\beta}$ and g_{α} of the quadratic functional form

$$(1.1)\qquad g_{\alpha\beta}\, y^{\alpha}\, y^{\beta} + g_{\alpha}(y^{\alpha})^2, \qquad (g_{\alpha\beta} = g_{\beta\alpha};\ g_{\alpha} \neq 0)$$

when this form is an absolute form under the Fredholm group of transformations

$$(1.2)\qquad y^i = \bar{y}^i + K^i_{\alpha}\, \bar{y}^{\alpha}$$

with non-vanishing Fredholm determinants. We shall deal only with forms in which the coefficients are continuous functions of all their arguments in the fundamental interval (a, b). Substituting (1.2) in the expression

$$\bar{g}_{\alpha\beta}\, \bar{y}^{\alpha}\, \bar{y}^{\beta} + \bar{g}_{\alpha}\, (\bar{y}^{\alpha})^2 = g_{\alpha\beta}\, y^{\alpha}\, y^{\beta} + g_{\alpha}(y^{\alpha})^2$$

we are led to the functional identity

$$(1.3)\qquad \varphi_{\alpha\beta}\, \bar{y}^{\alpha}\, \bar{y}^{\beta} + \varphi_{\alpha}(\bar{y}^{\alpha})^2 = 0, \qquad (\varphi_{\alpha\beta} = \varphi_{\beta\alpha})$$

where

$$(1.4)\qquad \begin{cases} \varphi_{\alpha} = \bar{g}_{\alpha} - g_{\alpha}, \\ \varphi_{\alpha\beta} = \bar{g}_{\alpha\beta} - g_{\alpha\beta} - g_{\lambda\beta}\, K^{\lambda}_{\alpha} - g_{\alpha\mu}\, K^{\mu}_{\beta} - g_{\lambda\mu}\, K^{\lambda}_{\alpha}\, K^{\mu}_{\beta} \\ \qquad\qquad - g_{(\alpha)}\, K^{\alpha}_{\beta} - g_{(\beta)}\, K^{\beta}_{\alpha} - g_{\sigma}\, K^{\sigma}_{\alpha}\, K^{\sigma}_{\beta}. \end{cases}$$

Before proceeding, we shall find it of value to prove two fundamental lemmas.

LEMMA 1. *A necessary and sufficient condition that*

$$(1.5)\qquad \psi_{\alpha\sigma}\, y^{\sigma} + \psi_{\alpha}\, y^{(\alpha)} = 0$$

for all continuous functions y^i, *such that* $y^a = y^b = 0$, *is that*

$$\psi_{\alpha\sigma} \equiv 0, \qquad \psi_{\alpha} \equiv 0$$

in the field of continuous functions.

Let us first consider the subclass S of continuous functions which are zero at the point a and throughout the closed interval (α, b) but are otherwise arbitrary. By hypothesis, we have

$$\int_a^{\alpha} \psi_{\alpha\sigma} y^{\sigma} \, d\sigma = 0$$

for all functions y^i in S subject to the above restrictions. Therefore by the fundamental lemma of the calculus of variations, we have for an arbitrary value of α in the interval (a, b)

$$\psi_{\alpha\sigma} \equiv 0, \qquad (a \leqq \sigma \leqq \alpha).$$

By a similar reasoning, we obtain

$$\psi_{\alpha\sigma} \equiv 0, \qquad (\alpha \leqq \sigma \leqq b),$$

and hence

$$\psi_{\alpha\sigma} \equiv 0, \qquad (a \leqq \alpha, \sigma \leqq b).$$

This then implies that

$$\psi_{\alpha} y^{(\alpha)} = 0$$

for all continuous functions y^i such that $y^a = y^b = 0$. Clearly ψ_{α} must vanish for all interior points of (a, b) and hence

$$\psi_{\alpha} \equiv 0$$

throughout (a, b) in the class of continuous functions. Q. E. D.

LEMMA 2. *A necessary and sufficient condition that*

$$(1.6) \qquad \psi_{\alpha\beta} y^{\alpha} y^{\beta} + \psi_{\alpha} (y^{\alpha})^2 = 0, \quad (\psi_{\alpha\beta} = \psi_{\beta\alpha})$$

be true for all continuous functions y^i, *for which* $y^a = y^b = 0$, *is that*

$$\psi_{\alpha\beta} \equiv 0, \quad \psi_{\alpha} \equiv 0.$$

Taking the differential of (1.6), we obtain

$$(\psi_{\alpha\beta} y^{\beta} + \psi_{\alpha} y^{\alpha}) \, \delta y^{\alpha} = 0.$$

Again, by the fundamental lemma of the calculus of variations, this condition reduces to

$$\psi_{\alpha\beta}\, y^{\beta} + \psi_{\alpha}\, y^{(\alpha)} = 0.$$

The truth of this lemma then follows from Lemma 1. Q. E. D.

In virtue of the above lemma, we see that (1.3) yields the transformations

$$(1.7)\qquad \begin{cases} \bar{g}_{\alpha} = g_{\alpha} \\ \bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{\lambda\beta}\, K^{\lambda}_{\alpha} + g_{\alpha\mu}\, K^{\mu}_{\beta} + g_{\lambda\mu}\, K^{\lambda}_{\alpha}\, K^{\mu}_{\beta} + g_{(\alpha)} K^{\alpha}_{\beta} \\ \qquad\quad + g_{(\beta)}\, K^{\beta}_{\alpha} + g_{\sigma}\, K^{\sigma}_{\alpha}\, K^{\sigma}_{\beta}. \end{cases}$$

THEOREM 1—I. *A necessary and sufficient condition that* (1.1) *be an absolute form under the Fredholm group of transformations* (1.2) *is that the coefficients* g_{α} *and* $g_{\alpha\beta}$ *have the respective laws of transformation as given in* (1.7).

It readily follows with but little difficulty that the transformations (1.7) form a group.[3] Let us define

$$(1.8)\qquad \begin{cases} \mathfrak{K}^{\lambda}_{\mu} = K^{(\lambda)}_{\mu} \sqrt{g_{\lambda}/g_{\mu}}\,, \\ f_{\alpha\beta} = g_{\alpha\beta}/\sqrt{g_{\alpha} g_{\beta}}\,. \end{cases}$$

With this notation the second transformation of (1.7) takes on the form

$$(1.9)\qquad \begin{aligned} \bar{f}_{\alpha\beta} = {} & f_{\alpha\beta} + f_{\lambda\beta}\, \mathfrak{K}^{\lambda}_{\alpha} + f_{\alpha\mu}\, \mathfrak{K}^{\mu}_{\beta} + f_{\lambda\mu}\, \mathfrak{K}^{\lambda}_{\alpha}\, \mathfrak{K}^{\mu}_{\beta} \\ & + \mathfrak{K}^{\alpha}_{\beta} + \mathfrak{K}^{\beta}_{\alpha} + \int_a^b \mathfrak{K}^{\sigma}_{\alpha}\, \mathfrak{K}^{\sigma}_{\beta}\, d\sigma. \end{aligned}$$

Hence by a theorem due to one of us,[4] it follows that

$$D[\bar{f}_{\alpha\beta}] = (D[\mathfrak{K}^{\lambda}_{\mu}])^2\, D[f_{\alpha\beta}]$$

or

$$D\left[\bar{g}_{\alpha\beta}/\sqrt{\bar{g}_{\alpha}\bar{g}_{\beta}}\,\right] = \left(D\left[K^{(\lambda)}_{\mu} \sqrt{g_{\lambda}/g_{\mu}}\,\right]\right)^2 D\left[g_{\alpha\beta}/\sqrt{g_{\alpha} g_{\beta}}\,\right].$$

Let us consider the Fredholm determinant

$$(1.10)\qquad D[\mathfrak{K}^{\lambda}_{\mu}] = 1 + \mathfrak{K}^{\sigma_1}_{\sigma_1} + \frac{1}{2!}\mathfrak{K}^{\sigma_1\,\sigma_2}_{\sigma_1\,\sigma_2} + \cdots + \frac{1}{r!}\mathfrak{K}^{\sigma_1\cdots\sigma_r}_{\sigma_1\cdots\sigma_r} + \cdots$$

where

$$(1.11)\qquad \mathfrak{K}^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n} = \begin{vmatrix} \mathfrak{K}^{\alpha_1}_{\beta_1}, & \mathfrak{K}^{\alpha_1}_{\beta_2}, & \cdots, & \mathfrak{K}^{\alpha_1}_{\beta_n} \\ \mathfrak{K}^{\alpha_2}_{\beta_1}, & \mathfrak{K}^{\alpha_2}_{\beta_2}, & \cdots, & \mathfrak{K}^{\alpha_2}_{\beta_n} \\ \cdot & \cdot & \cdot & \cdot \\ \mathfrak{K}^{\alpha_n}_{\beta_1}, & \mathfrak{K}^{\alpha_n}_{\beta_2}, & \cdots, & \mathfrak{K}^{\alpha_n}_{\beta_n} \end{vmatrix}.$$

[3] Cf. T. S. Peterson, *loc. cit.*, Theorem I, p. 417

[4] Cf. A. D. Michal, *loc. cit.*, Theorem 2, II, p. 479.

Since, however,

$$\mathfrak{K}^{\alpha}_{\beta} = K^{(\alpha)}_{\beta} \sqrt{g_\alpha / g_\beta}$$

we readily see that

$$(1.12) \qquad \mathfrak{K}^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n} = \frac{\sqrt{g_{\alpha_1} g_{\alpha_2} \cdots g_{\alpha_n}}}{\sqrt{g_{\beta_1} g_{\beta_2} \cdots g_{\beta_n}}} K^{(\alpha_1)(\alpha_2)\cdots(\alpha_n)}_{\beta_1\ \beta_2\cdots\ \beta_n}.$$

From which it follows that

$$(1.13) \qquad \mathfrak{K}^{\sigma_1\sigma_2\cdots\sigma_n}_{\sigma_1\sigma_2\cdots\sigma_n} = K^{\sigma_1\sigma_2\cdots\sigma_n}_{\sigma_1\sigma_2\cdots\sigma_n},$$

and therefore

$$(1.14) \qquad D[\mathfrak{K}^{\lambda}_{\mu}] = D[K^{\lambda}_{\mu}].$$

Thus we have

THEOREM 1—II. *The functional $D[g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}]$ is a relative scalar functional invariant of weight two of the absolute functional form* (1.1), *i. e.*,

$$(1.15) \qquad D[\bar{g}_{\alpha\beta}/\sqrt{\bar{g}_\alpha \bar{g}_\beta}] = (D[K^{\lambda}_{\mu}])^2\, D[g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}]$$

under the Fredholm group of transformations (1.2).

We shall now prove the following

THEOREM 1—III. *The totality of functional transformations*

$$(1.16) \qquad y^i = \bar{y}^i + \mathfrak{K}^i_\alpha \bar{y}^\alpha; \qquad \mathfrak{K}^i_\alpha = K^{(i)}_\alpha \sqrt{g_i/g_\alpha}$$

form a group with inverses, when g_α is an arbitrarily given non-vanishing continuous function and K^i_α is an arbitrary continuous function for which the Fredholm determinant is not zero.

We saw above that

$$D[\mathfrak{K}^{\lambda}_{\mu}] = D[K^{\lambda}_{\mu}]$$

and so it follows that

$$D[\mathfrak{K}^{\lambda}_{u}] \neq 0.$$

Let k^i_α be the resolvent kernel of K^i_α and λ^i_α that of $\mathfrak{K}^i_\alpha$. We shall show that

$$(1.17) \qquad \lambda^i_\alpha = k^{(i)}_\alpha \sqrt{g_i/g_\alpha}.$$

The expression for the first Fredholm minor $D^i_j[\mathfrak{K}^{\lambda}_{\mu}]$ is given by

$$(1.18) \qquad D^i_j[\mathfrak{K}^{\lambda}_{\mu}] = \mathfrak{K}^i_j + \mathfrak{K}^{i\sigma_1}_{j\sigma_1} + \cdots + \frac{1}{r!} \mathfrak{K}^{i\sigma_1\cdots\sigma_r}_{j\sigma_1\cdots\sigma_r} + \cdots$$

where $\mathfrak{K}^{i\alpha_1\cdots\alpha_n}_{j\beta_1\cdots\beta_n}$ has the same significance as in (1.11). Clearly, we have

$$\mathfrak{K}^{i\alpha_1\cdots\alpha_n}_{j\beta_1\cdots\beta_n} = \frac{\sqrt{g_i\, g_{\alpha_1}\cdots g_{\alpha_n}}}{\sqrt{g_j\, g_{\beta_1}\cdots g_{\beta_n}}} K^{(i)(\alpha_1)\cdots(\alpha_n)}_{j\ \beta_1\cdots\beta_n} \tag{1.19}$$

and consequently

$$D^i_j[\mathfrak{K}^\lambda_\mu] = \sqrt{g_i/g_j}\, D^{(i)}_j[K^\lambda_\mu]. \tag{1.20}$$

Dividing (1.20) by (1.14), we obtain

$$\lambda^i_j[\mathfrak{K}^\lambda_\mu] = \sqrt{g_i/g_j}\, k^{(i)}_j[K^\lambda_\mu]$$

and the theorem follows readily.

From a theorem proved elsewhere[5] and (1.9), we see that the resolvent kernel $f^{\alpha\beta}$ of $f_{\alpha\beta}$,

$$f^{\alpha\beta} = -D^\alpha_\beta[f_{\lambda\mu}]/D[f_{\lambda\mu}] \tag{1.21}$$

where[6] we assume $D[f_{\lambda\mu}] \neq 0$, transforms in the following manner

$$\bar{f}^{\alpha\beta} = f^{\alpha\beta} + f^{\sigma\beta}\lambda^\alpha_\sigma + f^{\alpha\tau}\lambda^\beta_\tau + f^{\sigma\tau}\lambda^\alpha_\sigma\lambda^\beta_\tau + \lambda^\alpha_\beta + \lambda^\beta_\alpha + \int_a^b \lambda^\alpha_\sigma\lambda^\beta_\sigma\, d\sigma. \tag{1.22}$$

If we make use of the notation

$$h^{\alpha\beta} = f^{(\alpha)(\beta)}/\sqrt{g_\alpha g_\beta}, \qquad h^\alpha = 1/g_\alpha, \tag{1.23}$$

we find that (1.22) assumes the form

$$\bar{h}^{\alpha\beta} = h^{\alpha\beta} + h^{\lambda\beta}k^\alpha_\lambda + h^{\alpha\mu}k^\beta_\mu + h^{\lambda\mu}k^\alpha_\lambda k^\beta_\mu + h^{(\beta)}k^\alpha_\beta + h^{(\alpha)}k^\beta_\alpha + h^\sigma k^\alpha_\sigma k^\beta_\sigma. \tag{1.24}$$

From the character of the transformation (1.24), we find that

$$h^{\alpha\beta}\xi_\alpha\xi_\beta + h^\alpha(\xi_\alpha)^2, \qquad (h^{\alpha\beta} = h^{\beta\alpha};\ h^\alpha \neq 0) \tag{1.25}$$

is an absolute quadratic functional form under the transformations

$$\xi_i = \bar{\xi}_i + k^\sigma_i\bar{\xi}_\sigma \tag{1.26}$$

contragredient to (1.2).

Let us take

$$Q[\xi,\eta] = h^{\alpha\beta}\xi_\alpha\eta_\beta + h^\alpha\xi_\alpha\eta_\alpha\ ^{7} \tag{1.27}$$

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[5] Cf. A. D. Michal, *loc. cit.*, Theorem 2—III, p. 480.

[6] An application of Theorem 1—II shows that this property persists under transformations of the group (1.2).

[7] η^α transforms cogrediently to ξ_α.

and consider the quartic functional form

$$(1.28)\qquad \Delta = \begin{vmatrix} Q[\xi,\xi], & Q[\xi,\eta] \\ Q[\eta,\xi], & Q[\eta,\eta] \end{vmatrix}.$$

Clearly Δ is an absolute quartic functional form under the contragredient transformations (1.26). On expanding we obtain

$$(1.29)\qquad \begin{aligned} \Delta = h^{\alpha_1\beta_1,\alpha_2\beta_2}\,\xi_{\alpha_1}\,\xi_{\beta_1}\,\eta_{\alpha_2}\,\eta_{\beta_2} + h^{\alpha\beta}\,h^{\gamma}\begin{vmatrix} \xi_\alpha, & \xi_\gamma \\ \eta_\alpha, & \eta_\gamma \end{vmatrix}\cdot\begin{vmatrix} \xi_\beta, & \xi_\gamma \\ \eta_\beta, & \eta_\gamma \end{vmatrix} \\ + h^{\gamma_1}\,h^{\gamma_2}\,\xi_{\gamma_1}\,\eta_{\gamma_2}\begin{vmatrix} \xi_{\gamma_1}, & \xi_{\gamma_2} \\ \eta_{\gamma_1}, & \eta_{\gamma_2} \end{vmatrix} \end{aligned}$$

where

$$(1.30)\qquad \begin{cases} h^\alpha = 1/g_\alpha, \\ h^{\alpha\beta} = -\dfrac{D_\beta^{(\alpha)}[g_{\lambda\mu}/\sqrt{g_\lambda\, g_\mu}\,]}{\sqrt{g_\alpha\, g_\beta}\,D[g_{\lambda\mu}/\sqrt{g_\lambda, g_\mu}]}, \\ h^{\alpha_1\beta_1,\alpha_2\beta_2} = \begin{vmatrix} h^{\alpha_1\beta_1}, & h^{\alpha_1\beta_2} \\ h^{\alpha_2\beta_1}, & h^{\alpha_2\beta_2} \end{vmatrix} = \dfrac{D_{\beta_1\beta_2}^{(\alpha_1)(\alpha_2)}[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]}{\sqrt{g_{\alpha_1} g_{\alpha_2} g_{\beta_1} g_{\beta_2}}\,D[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]}. \end{cases}$$

In (1.30), $D_{\beta_1\beta_2}^{\alpha_1\alpha_2}[\omega_{\lambda\mu}]$ stands for the second Fredholm minor of the function $\omega_{\lambda\mu}$.

Before investigating the intrinsic character of the coefficients of such quartic functional forms as Δ, it will be useful to establish the following lemma.

LEMMA 3. *In order that*

$$(1.31)\qquad \begin{aligned} A^{\alpha\beta\gamma\delta}\,\xi_\alpha\,\xi_\beta\,\eta_\gamma\,\eta_\delta + B^{\sigma\gamma\delta}(\xi_\sigma)^2\,\eta_\gamma\,\eta_\delta + C^{\sigma\gamma\delta}\,\xi_\sigma\,\xi_\gamma\,\eta_\gamma\,\eta_\delta + D^{\gamma\delta}\,\xi_\gamma\,\xi_\delta\,\eta_\gamma\,\eta_\delta \\ + E^{\alpha\beta\sigma}\,\xi_\alpha\,\xi_\beta\,(\eta_\sigma)^2 + F^{\sigma\tau}(\xi_\tau)^2(\eta_\sigma)^2 = 0 \end{aligned}$$

where $A^{\alpha\beta\gamma\delta}$ and $E^{\alpha\beta\sigma}$ are symmetric in α and β, and $A^{\alpha\beta\gamma\delta}$, $B^{\sigma\gamma\delta}$, and $D^{\gamma\delta}$ are symmetric in γ and δ, be true for all continuous functions ξ_i and η_i for which $\xi_a = \xi_b = \eta_a = \eta_b = 0$, it is necessary and sufficient that

$$A^{\alpha\beta\gamma\delta} = B^{\sigma\gamma\delta} = C^{\sigma\gamma\delta} = D^{\gamma\delta} = E^{\alpha\beta\sigma} = F^{\sigma\tau} \equiv 0.$$

Considering ξ_i for the moment a fixed function, we may write (1.31) in the form

$$\begin{aligned} \{A^{\alpha\beta\gamma\delta}\,\xi_\alpha\,\xi_\beta + B^{\sigma\gamma\delta}(\xi_\sigma)^2 + C^{\sigma\gamma\delta}\,\xi_\sigma\,\xi_\gamma + D^{\gamma\delta}\,\xi_\gamma\,\xi_\delta\}\,\eta_\gamma\,\eta_\delta \\ + \{E^{\alpha\beta\sigma}\,\xi_\alpha\,\xi_\beta + F^{\sigma\tau}(\xi_\tau)^2\}(\eta_\sigma)^2 = 0; \end{aligned}$$

from which it follows by Lemma 2 that

$$(1.32)\qquad \begin{cases} 2A^{\alpha\beta\gamma\delta}\xi_\alpha\xi_\beta + 2B^{\sigma\gamma\delta}(\xi_\sigma)^2 + C^{\sigma(\gamma)\delta}\xi_\sigma\xi_\gamma \\ \qquad + C^{\sigma(\delta)\gamma}\xi_\sigma\xi_\delta + 2D^{(\gamma)(\delta)}\xi_\gamma\xi_\delta = 0, \\ E^{\alpha\beta\sigma}\xi_\alpha\xi_\beta + F^{\sigma\tau}(\xi_\tau)^2 = 0. \end{cases}$$

From the second equation of (1.32), it again follows by Lemma 2 that

$$E^{\alpha\beta\sigma} = F^{\sigma\tau} \equiv 0$$

for each value of σ and hence for all values of σ. Next let us take the variation of the first equation of (1.32) with respect to ξ_i and obtain

$$(1.33)\qquad \begin{aligned} &\{4A^{\sigma\beta\gamma\delta}\xi_\beta + 4B^{\sigma\gamma\delta}\xi_\sigma + C^{\sigma(\gamma)\delta}\xi_\gamma + C^{\sigma(\delta)\gamma}\xi_\delta\}\,\delta\xi_\sigma \\ &+\{C^{\sigma(\gamma)\delta}\xi_\sigma + 2D^{(\gamma)(\delta)}\xi_\delta\}\,\delta\xi_\gamma + \{C^{\sigma(\delta)\gamma}\xi_\sigma + 2D^{(\gamma)(\delta)}\xi_\gamma\}\,\delta\xi_\delta = 0 \end{aligned}$$

where $\delta\xi_\alpha$ is an arbitrary continuous function of α that vanishes at the endpoints of the interval (a, b). This particular equation is of a little more general character than that of Lemma 1. However, it is easily seen that the coefficients must vanish, for consider

$$(1.34)\qquad f^{\sigma\gamma\delta}\,\delta\xi_\sigma + \varphi^{(\gamma)\delta}\,\delta\xi_\gamma + \psi^{\gamma(\delta)}\,\delta\xi_\delta = 0.$$

Assuming first that $\gamma \leqq \delta$, let us consider the subclass L of continuous functions $\delta\xi_i$ which are zero at the point a and vanish throughout the closed interval (δ, b), but are otherwise arbitrary. Equation (1.34) then reduces to

$$\int_a^\delta f^{\sigma\gamma\delta}\,\delta\xi_\sigma\,d\sigma + \varphi^{(\gamma)\delta}\,\delta\xi_\gamma = 0$$

and by Lemma 1 we have

$$\varphi^{\gamma\delta} \equiv 0, \qquad (\gamma \leqq \delta).$$

Returning to equation (1.34), we then have

$$f^{\sigma\gamma\delta}\,\delta\xi_\sigma + \psi^{\gamma(\delta)}\,\delta\xi_\delta = 0.$$

Since γ is arbitrary, we have again by Lemma 1

$$f^{\sigma\gamma\delta} \equiv 0, \qquad \psi^{\gamma\delta} \equiv 0, \qquad (\gamma \leqq \delta).$$

A similar procedure shows that the above results hold also for $\gamma \geqq \delta$ and hence

$$f^{\sigma\gamma\delta} \equiv 0, \qquad \varphi^{\gamma\delta} \equiv 0, \qquad \psi^{\gamma\delta} \equiv 0.$$

In virtue of (1.33), we obtain the conditions

$$4\,A^{\sigma\beta\gamma\delta}\,\xi_\beta+4\,B^{(\sigma)\gamma\delta}\,\xi_\sigma+C^{\sigma(\gamma)\delta}\,\xi_\gamma+C^{\sigma(\delta)\gamma}\,\xi_\delta=0,$$
$$C^{\sigma\gamma\delta}\,\xi_\sigma+2\,D^{\gamma(\delta)}\,\xi_\delta=0.$$

By Lemma 1, we readily see that the above conditions reduce to

$$A^{\sigma\beta\gamma\delta}=B^{\sigma\gamma\delta}=C^{\sigma\gamma\delta}=D^{\gamma\delta}\equiv 0. \quad \text{Q.E.D.}$$

Let us now investigate the mode of transformation of the coefficients of an absolute quartic functional form

$$U=H^{\alpha_1\beta_1,\alpha_2\beta_2}\,\xi_{\alpha_1}\,\xi_{\beta_1}\,\eta_{\alpha_2}\,\eta_{\beta_2}+H^{\alpha\beta}\,H^{\gamma}\begin{vmatrix}\xi_\alpha, & \xi_\gamma\\ \eta_\alpha, & \eta_\gamma\end{vmatrix}\cdot\begin{vmatrix}\xi_\beta, & \xi_\gamma\\ \eta_\beta, & \eta_\gamma\end{vmatrix}$$
$$(1.35)\qquad +H^{\gamma_1}\,H^{\gamma_2}\,\xi_{\gamma_1}\,\eta_{\gamma_2}\begin{vmatrix}\xi_{\gamma_1}, & \xi_{\gamma_2}\\ \eta_{\gamma_1}, & \eta_{\gamma_2}\end{vmatrix}$$

under the transformations (1.26). We may facilitate the work by adopting the notations

$$(1.36)\qquad \begin{cases}\overline{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}}=(\theta^\alpha_\beta+k^\alpha_\beta),\\ \quad l^\alpha_\beta=H^{(\alpha)}\,k^\beta_\alpha+H^{(\beta)}\,k^\alpha_\beta+H^\sigma\,k^\alpha_\sigma\,k^\beta_\sigma,\end{cases}$$

where θ^α_β is a symbolic operator such that[8]

$$\varphi^{i\beta}\,\theta^\alpha_\beta=\varphi^{i\alpha},\qquad \varphi_{i\alpha}\,\theta^\alpha_\beta=\varphi_{i\beta},\quad \text{etc.}$$

It is clear from these definitions that the transformations (1.26) take the form

$$(1.37)\qquad \xi_i=\overline{\begin{pmatrix}\sigma\\ i\end{pmatrix}}\,\bar\xi_\sigma,\qquad \eta_i=\overline{\begin{pmatrix}\sigma\\ i\end{pmatrix}}\,\bar\eta_\sigma.$$

Substituting (1.37) into the equation $\bar U=U$, we obtain

$$\bar U=H^{\lambda_1\mu_1,\lambda_2\mu_2}\overline{\begin{pmatrix}\alpha_1\\ \lambda_1\end{pmatrix}}\,\overline{\begin{pmatrix}\beta_1\\ \mu_1\end{pmatrix}}\,\overline{\begin{pmatrix}\alpha_2\\ \lambda_2\end{pmatrix}}\,\overline{\begin{pmatrix}\beta_2\\ \mu_2\end{pmatrix}}\,\bar\xi_{\alpha_1}\,\bar\xi_{\beta_1}\,\bar\eta_{\alpha_2}\,\bar\eta_{\beta_2}$$
$$(1.38)\qquad +H^{\lambda\mu}\,H^{\nu}\overline{\begin{pmatrix}\alpha_1\\ \lambda\end{pmatrix}}\,\overline{\begin{pmatrix}\beta_1\\ \mu\end{pmatrix}}\,\overline{\begin{pmatrix}\gamma_1\\ \nu\end{pmatrix}}\,\overline{\begin{pmatrix}\gamma_2\\ \nu\end{pmatrix}}\begin{vmatrix}\bar\xi_{\alpha_1}, & \bar\xi_{\gamma_1}\\ \bar\eta_{\alpha_1}, & \bar\eta_{\gamma_1}\end{vmatrix}\cdot\begin{vmatrix}\bar\xi_{\beta_1}, & \bar\xi_{\gamma_2}\\ \bar\eta_{\beta_1}, & \bar\eta_{\gamma_2}\end{vmatrix}$$
$$+H^{\nu_1}\,H^{\nu_2}\overline{\begin{pmatrix}\gamma_1\\ \nu_1\end{pmatrix}}\,\overline{\begin{pmatrix}\gamma_2\\ \nu_2\end{pmatrix}}\,\overline{\begin{pmatrix}\gamma_3\\ \nu_1\end{pmatrix}}\,\overline{\begin{pmatrix}\gamma_4\\ \nu_2\end{pmatrix}}\,\bar\xi_{\gamma_1}\,\bar\eta_{\gamma_2}\begin{vmatrix}\bar\xi_{\gamma_3}, & \bar\xi_{\gamma_4}\\ \bar\eta_{\gamma_3}, & \bar\eta_{\gamma_4}\end{vmatrix}.$$

[8] The part played here by the symbol θ^α_β is analogous to the part played by the Kronecker delta δ^i_j in n-dimensional geometry.

Since however

$$H^{\sigma} \overline{\binom{\alpha}{\sigma}} \overline{\binom{\beta}{\sigma}} \varphi_{\alpha\beta} = H^{\sigma} \varphi_{\sigma\sigma} + \int_a^b l^{\alpha}_{\beta} \varphi_{\alpha\beta} \, d_{\beta},$$

we may collect terms in (1.38) and group them in a form similar to U. Doing this, we obtain

$$\overline{U} = \left\{ H^{\lambda_1\mu_1, \lambda_2\mu_2} \overline{\binom{\alpha_1}{\lambda_1}} \overline{\binom{\beta_1}{\mu_1}} \overline{\binom{\alpha_2}{\lambda_2}} \overline{\binom{\beta_2}{\mu_2}} - \begin{vmatrix} 0, & \overline{\binom{\beta_1}{\mu}}, & \overline{\binom{\beta_2}{\mu}} \\ \overline{\binom{\alpha_1}{\lambda}}, & l^{\alpha_1}_{\beta_1}, & l^{\alpha_1}_{\beta_2} \\ \overline{\binom{\alpha_2}{\lambda}}, & l^{\alpha_2}_{\beta_1}, & l^{\alpha_2}_{\beta_2} \end{vmatrix} H^{\lambda\mu} \right.$$

$$\left. + \begin{vmatrix} l^{\alpha_1}_{\beta_1}, & l^{\alpha_1}_{\beta_2} \\ l^{\alpha_2}_{\beta_1}, & l^{\alpha_2}_{\beta_2} \end{vmatrix} \right\} \overline{\xi}_{\alpha_1} \overline{\xi}_{\beta_1} \overline{\eta}_{\alpha_2} \overline{\eta}_{\beta_2}$$

$$+ \left\{ H^{\lambda\mu} \overline{\binom{\alpha}{\lambda}} \overline{\binom{\beta}{\mu}} + l^{\alpha}_{\beta} \right\} H^{\nu} \begin{vmatrix} \overline{\xi}_{\alpha}, & \overline{\xi}_{\nu} \\ \overline{\eta}_{\alpha}, & \overline{\eta}_{\nu} \end{vmatrix} \cdot \begin{vmatrix} \overline{\xi}_{\beta}, & \overline{\xi}_{\nu} \\ \overline{\eta}_{\beta}, & \overline{\eta}_{\nu} \end{vmatrix}$$

$$+ H^{\nu_1} H^{\nu_2} \overline{\xi}_{\nu_1} \overline{\eta}_{\nu_2} \begin{vmatrix} \overline{\xi}_{\nu_1}, & \overline{\xi}_{\nu_2} \\ \overline{\eta}_{\nu_1}, & \overline{\eta}_{\nu_2} \end{vmatrix}.$$

Hence, in virtue of Lemma 3, we have

THEOREM 1—IV. *A necessary and sufficient condition that the quartic functional form U be an absolute form under the transformations* (1.37), *when $\overline{H}^{\alpha} = H^{\alpha}$, is that the coefficients $H^{\alpha_1\beta_1, \alpha_2\beta_2}$ and $H^{\alpha\beta}$ of this form have the following laws of transformation*

$$\overline{H}^{\alpha_1\beta_1, \alpha_2\beta_2} = H^{\lambda_1\mu_1, \lambda_2\mu_2} \overline{\binom{\alpha_1}{\lambda_1}} \overline{\binom{\beta_1}{\mu_1}} \overline{\binom{\alpha_2}{\lambda_2}} \overline{\binom{\beta_2}{\mu_2}}$$

$$- \begin{vmatrix} 0, & \overline{\binom{\beta_1}{\mu}}, & \overline{\binom{\beta_2}{\mu}} \\ \overline{\binom{\alpha_1}{\lambda}}, & l^{\alpha_1}_{\beta_1}, & l^{\alpha_1}_{\beta_2} \\ \overline{\binom{\alpha_2}{\lambda}}, & l^{\alpha_2}_{\beta_1}, & l^{\alpha_2}_{\beta_2} \end{vmatrix} H^{\lambda\mu} + \begin{vmatrix} l^{\alpha_1}_{\beta_1}, & l^{\alpha_1}_{\beta_2} \\ l^{\alpha_2}_{\beta_1}, & l^{\alpha_2}_{\beta_2} \end{vmatrix},$$

$$\overline{H}^{\alpha\beta} = H^{\lambda\mu} \overline{\binom{\alpha}{\lambda}} \overline{\binom{\beta}{\mu}} + l^{\alpha}_{\beta}.$$

As an immediate consequence of the above theorem, we have the following[9]

THEOREM 1—V. *The system of functionals*

$$h^{\alpha}[g_\lambda] = 1/g_\alpha,$$

$$h^{\alpha\beta}[g_{\lambda\mu}, g_\nu] = -\frac{D_\beta^{(\alpha)}[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]}{\sqrt{g_\alpha g_\beta}\, D[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]},$$

$$h^{\alpha\beta,\gamma\delta}[g_{\lambda\mu}, g_\nu] = \frac{D_{\beta\delta}^{(\alpha)(\gamma)}[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]}{\sqrt{g_\alpha g_\beta g_\gamma g_\delta}\, D[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]},$$

constitute an invariant class of functionals of the quadratic functional form (1.1), *that is, they transform in accordance with Theorem* 1—IV *with the understanding that* $\bar{h}^{\alpha} \equiv h^{\alpha}[\bar{g}_\lambda]$, $\bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}[\bar{g}_{\lambda\mu}, \bar{g}_\nu]$, *and* $\bar{h}^{\alpha\beta,\gamma\delta} \equiv h^{\alpha\beta,\gamma\delta}[\bar{g}_{\lambda\mu}, \bar{g}_\nu]$.

2. **Transformations of the third kind.** Let us now consider the problem of finding those invariantive functionals of the quadratic functional form

$$g_{\alpha\beta}\, y^{\alpha} y^{\beta} + g_\alpha (y^{\alpha})^2, \qquad (g_{\alpha\beta} = g_{\beta\alpha};\ g_\alpha \neq 0) \tag{2.1}$$

under the more general transformations of the third kind

$$y^i = K^i \bar{y}^i + K^i_\alpha \bar{y}^{\alpha}, \qquad (K^i \neq 0 \text{ in } (a, b)). \tag{2.2}$$

By Picard's observation,[10] we may write (2.2) as a Fredholm integral equation

$$y^i = z^i + \frac{K^i_\alpha}{K^\alpha} z^\alpha; \qquad z^\alpha = K^\alpha \bar{y}^\alpha. \tag{2.3}$$

THEOREM 2—I. *The totality of transformations* (2.2), *for which the Fredholm determinant* $D[K^\sigma_\tau/K^{(\tau)}]$ *is not zero, form a group.*

To prove this theorem, let us consider two transformations T_1 and T_2 of (2.2)

$$T_1:\ \ y^i = L^i \bar{y}^i + L^i_\alpha \bar{y}^\alpha,$$
$$T_2:\ \ \bar{y}^i = M^i \bar{\bar{y}}^i + M^i_\alpha \bar{\bar{y}}^\alpha.$$

By taking the symbolic product $T_1 T_2$ of the transformations T_1 and T_2, we obtain

$$y^i = N^i \bar{\bar{y}}^i + N^i_\alpha \bar{\bar{y}}^\alpha$$

[9] See last sentence of § 2.

[10] Cf. Picard, "*Sur les équations intégrales de troisième espèce,*" Annales de l'École Norm. Super., (3° Serie), tome 28 (1911), pp. 459–472.

where

$$N^i = L^i M^i,$$
$$N^i_\alpha = L^i M^i_\alpha + L^i_\alpha M^{(\alpha)} + L^i_\sigma M^\sigma_\alpha.$$

Since $L^i \neq 0$ and $M^i \neq 0$, it is clear that $N^i \neq 0$ for $a \leqq i \leqq b$. To prove that, $D[N^\sigma_\tau/N^{(\tau)}] \neq 0$, we observe that

$$\frac{N^i_\alpha}{N^{(\alpha)}} = \frac{L^i}{L^{(\alpha)}}\frac{M^i_\alpha}{M^\alpha} + \frac{L^i_\alpha}{L^{(\alpha)}} + \frac{L^\sigma}{L^{(\alpha)}}\frac{L^i_\sigma}{L^\sigma}\frac{M^\sigma_\alpha}{M^\alpha}. \tag{2.4}$$

If we put $A^i_\alpha = L^i_\alpha/L^{(\alpha)}$ and $B^i_\alpha = L^i M^i_\alpha/L^\alpha M^{(\alpha)}$, we see that (2.4) reduces to

$$N^i_\alpha/N^{(\alpha)} = A^i_\alpha + B^i_\alpha + A^i_\sigma B^\sigma_\alpha \tag{2.5}$$

from which it follows at once that

$$D[N^i_\alpha/N^{(\alpha)}] = D[A^i_\alpha]\, D[B^i_\alpha]. \tag{2.6}$$

By an argument similar to that of (1.14), it is clear that

$$D[B^i_\alpha] = D[L^i M^i_\alpha/L^\alpha M^{(\alpha)}] = D[M^i_\alpha/M^{(\alpha)}]$$

and so (2.6) reduces to

$$D[N^i_\alpha/N^{(\alpha)}] = D[L^i_\alpha/L^{(\alpha)}]\, D[M^i_\alpha/M^{(\alpha)}]. \tag{2.7}$$

Since the hypothesis $D[L^i_\alpha/L^{(\alpha)}] \neq 0$ and $D[M^i_\alpha/M^{(\alpha)}] \neq 0$, it is evident that $D[N^i_\alpha/N^{(\alpha)}] \neq 0$. From (2.3) we have the unique inverse

$$z^i = y^i + \Gamma^i_\alpha y^\alpha \tag{2.8}$$

where Γ^i_α is the resolvent kernel of $K^i_\alpha/K^{(\alpha)}$. z^i, however, represents $K^i \bar{y}^i$ and so we have the unique inverse transformation to (2.2)

$$\bar{y}^i = k^i y^i + k^i_\alpha y^\alpha \tag{2.9}$$

where

$$k^i = 1/K^i; \qquad k^i_\alpha = \Gamma^i_\alpha/K^i.$$

By a procedure essentially the same as that for the functional form (1.1), we arrive at the following theorem.

THEOREM 2—II. *A necessary and sufficient condition that* (2.1) *be an absolute form under the group of transformations* (2.2) *is that the coefficients* g_α *and* $g_{\alpha\beta}$ *have the respective laws of transformation*

$$\bar{g}_\alpha = (K^\alpha)^2 g_{(\alpha)} \tag{2.10}$$

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} K^{(\alpha)} K^{(\beta)} + g_{\sigma\beta} K^{(\beta)} K^\sigma_\alpha + g_{\alpha\tau} K^{(\alpha)} K^\tau_\beta + g_{\sigma\tau} K^\sigma_\alpha K^\tau_\beta + g_{(\alpha)} K^\alpha K^\alpha_\beta + g_{(\beta)} K^\beta K^\beta_\alpha + g_\sigma K^\sigma_\alpha K^\sigma_\beta. \tag{2.11}$$

Using the notation[11]

$$f_{\alpha\beta} = g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}; \qquad \mathfrak{K}^\alpha_\beta = \{K^{(\alpha)}_\beta/K^{(\beta)}\}\sqrt{g_\alpha/g_\beta} \tag{2.12}$$

and with the help of (2.10), we may write (2.11) as follows

$$\bar{f}_{\alpha\beta} = f_{\alpha\beta} + f_{\sigma\beta}\mathfrak{K}^\sigma_\alpha + f_{\alpha\tau}\mathfrak{K}^\tau_\beta + f_{\sigma\tau}\mathfrak{K}^\sigma_\alpha\mathfrak{K}^\tau_\beta + \mathfrak{K}^\alpha_\beta + \mathfrak{K}^\beta_\alpha + \int_a^b \mathfrak{K}^\sigma_\alpha\mathfrak{K}^\sigma_\beta\, d\sigma. \tag{2.13}$$

It is evident from (1.9) and (1.13) that

$$D[\bar{f}_{\alpha\beta}] = (D[K^\sigma_\tau/K^{(\tau)}])^2\, D[f_{\alpha\beta}] \tag{2.14}$$

and so we have the theorem

THEOREM 2—III. *The Fredholm determinant* $D[g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}]$ *is a relative scalar functional invariant of weight two of the absolute functional form* (2.1) *under the group of functional transformations of the third kind* (2.2), *i. e.,*

$$D[\bar{g}_{\alpha\beta}/\sqrt{\bar{g}_\alpha\bar{g}_\beta}] = (D[K^\sigma_\tau/K^\tau])^2\, D[g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}]. \tag{2.15}$$

As for (1.22), we may write the law of transformation of the resolvent kernel $f^{\alpha\beta}$ of $f_{\alpha\beta}$ as follows,

$$\bar{f}^{\alpha\beta} = f^{\alpha\beta} + f^{\sigma\beta}\Lambda^\alpha_\sigma + f^{\alpha\tau}\Lambda^\beta_\tau + f^{\sigma\tau}\Lambda^\alpha_\sigma\Lambda^\beta_\tau + \Lambda^\alpha_\beta + \Lambda^\beta_\alpha + \int_a^b \Lambda^\alpha_\sigma\Lambda^\beta_\sigma\, d\sigma \tag{2.16}$$

where

$$\Lambda^\alpha_\beta = \Gamma^{(\alpha)}_\beta\sqrt{g_\alpha/g_\beta}$$

and Γ^α_β is the resolvent kernel of $K^\alpha_\beta/K^{(\beta)}$ as before.

If we place $h^{\alpha\beta} = f^{(\alpha)(\beta)}/\sqrt{g_\alpha g_\beta}$, then by (2.16) and (2.10), we have

$$\begin{aligned} \bar{h}^{\alpha\beta} = {} & h^{\alpha\beta}k^\alpha k^\beta + h^{\sigma\beta}k^\beta k^\alpha_\sigma + h^{\alpha\tau}k^\alpha k^\beta_\tau + h^{\sigma\tau}k^\alpha_\sigma k^\beta_\tau + h^{(\alpha)}k^\alpha k^\beta_\alpha \\ & + h^{(\beta)}k^\beta k^\alpha_\beta + h^\sigma k^\alpha_\sigma k^\beta_\sigma \end{aligned} \tag{2.17}$$

where

$$h^\alpha = 1/g_\alpha.$$

THEOREM 2—IV. *The quadratic functional form*

$$h^{\alpha\beta}\xi_\alpha\xi_\beta + h^\alpha(\xi_\alpha)^2 \tag{2.18}$$

is an absolute functional form under the group of functional transformations

$$\xi_i = k_i\bar{\xi}_i + k^\alpha_i\bar{\xi}_\alpha \tag{2.19}$$

contragredient to (2.2).

[11] Hereafter we shall use this generalized definition of $\mathfrak{K}^\alpha_\beta$ rather than that of (1.8).

By the same reasoning as for (1.29), we may show that the quartic functional form

$$
(2.20)\qquad \begin{aligned} & h^{\alpha_1\beta_1,\alpha_2\beta_2}\,\xi_{\alpha_1}\,\xi_{\beta_1}\,\eta_{\alpha_2}\,\eta_{\beta_2}+h^{\alpha\beta}\,h^{\gamma}\begin{vmatrix}\xi_\alpha, & \xi_\gamma\\ \eta_\alpha, & \eta_\gamma\end{vmatrix}\cdot\begin{vmatrix}\xi_\beta, & \xi_\gamma\\ \eta_\beta, & \eta_\gamma\end{vmatrix}\\ & \qquad +h^{\gamma_1}\,h^{\gamma_2}\,\xi_{\gamma_1}\,\eta_{\gamma_2}\begin{vmatrix}\xi_{\gamma_1}, & \xi_{\gamma_2}\\ \eta_{\gamma_1}, & \eta_{\gamma_2}\end{vmatrix}\end{aligned}
$$

is an absolute form under the contragredient transformations (2.19). Theorem 1—V also holds for the coefficients of (2.20) with the understanding that the expressions $\overline{\binom{\alpha}{\beta}}$ and l^{α}_{β} are to be interpreted in accordance with the definitions (3.4) and (3.6) respectively.

3. **Simultaneous invariants.** Up to this point we have been concerned wholly with one quadratic functional form and its functional invariants. In this concluding paragraph we shall build up a class of simultaneous functional invariants of two quadratic functional forms.

THEOREM 3—I. *The expression*

$$
(3.1)\qquad g_{\alpha\beta}\,h^{\alpha\beta}+g_{\sigma\sigma}\,h^{\sigma}+g_{\sigma}\,h^{\sigma\sigma}+g_{\sigma}\,h^{\sigma}
$$

is an absolute invariant of the coefficients $g_{\alpha\beta}$ and g_α of a form

$$
(3.2)\qquad g_{\alpha\beta}\,y^{\alpha}\,y^{\beta}+g_{\alpha}(y^{\alpha})^2,\quad (g_{\alpha\beta}=g_{\beta\alpha};\ g_\alpha\neq 0)
$$

and $h^{\alpha\beta}$ and h^{α} of a form

$$
(3.3)\qquad h^{\alpha\beta}\,\xi_\alpha\,\xi_\beta+h^{\alpha}(\xi_\alpha)^2,\quad (h^{\alpha\beta}=h^{\beta\alpha};\ h^{\alpha}\neq 0)
$$

where the function ξ_α transforms contragrediently to the function y^{α}.

To establish this theorem, let us adopt the following extensions to the notation[12] of (1.36)

$$
(3.4)\qquad \binom{\alpha}{\beta}=(K^{\alpha}\,\theta^{\alpha}_{\beta}+K^{\alpha}_{\beta});\quad \overline{\binom{\alpha}{\beta}}=(k_{\beta}\,\theta^{\alpha}_{\beta}+k^{\alpha}_{\beta})
$$

where θ^{α}_{β} is the same symbolic operator. With this notation the laws of transformation of the coefficients $g_{\alpha\beta}$ and $h^{\alpha\beta}$ respectively reduce to

$$
(3.5)\qquad \bar{g}_{\alpha\beta}=g_{\lambda\mu}\binom{\lambda}{\alpha}\binom{\mu}{\beta}+t^{\alpha}_{\beta},\quad \bar{h}^{\alpha\beta}=h^{\lambda\mu}\overline{\binom{\alpha}{\lambda}}\,\overline{\binom{\beta}{\mu}}+l^{\alpha}_{\beta}
$$

[12] From now on we shall use these extensions to the notation of (1.36) exclusively. Furthermore $h^{\alpha\beta}$ and h^{α} will have the significance of the coefficients of an *arbitrary* absolute form (3.3). The invariant (3.1) reduces to the length of the fundamental interval $(a,\ b)$ for the special form (2.18).

where[12]

$$(3.6)\qquad \begin{cases} t^{\alpha}_{\beta} = g_{(\alpha)}\, K^{\alpha}\, K^{\alpha}_{\beta} + g_{(\beta)}\, K^{\beta}\, K^{\beta}_{\alpha} + g_{\sigma}\, K^{\sigma}_{\alpha}\, K^{\sigma}_{\beta}, \\ l^{\alpha}_{\beta} = h^{(\alpha)}\, k_{\alpha}\, k^{\beta}_{\alpha} + h^{(\beta)}\, k_{\beta}\, k^{\alpha}_{\beta} + h^{\sigma}\, k^{\alpha}_{\sigma}\, k^{\beta}_{\sigma}. \end{cases}$$

Let us now form the expression

$$(3.7)\qquad \begin{aligned} &[\bar{g}_{\alpha\beta} + \bar{g}_{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}]\,[\bar{h}^{\alpha\beta} + \bar{h}^{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}] \\ &= \left[g_{\lambda\mu} \binom{\lambda}{\alpha} \binom{\mu}{\beta} + t^{\alpha}_{\beta} + g_{\sigma}\, K^{\sigma}\, K^{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}\right] \left[h^{\nu\tau}\, \overline{\binom{\alpha}{\nu}}\, \overline{\binom{\beta}{\tau}} + l^{\alpha}_{\beta} + h^{\varrho}\, k_{\varrho}\, k_{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}\right]. \end{aligned}$$

Because of the relations

$$(3.8)\qquad \begin{cases} [t^{\alpha}_{\beta} + g_{\sigma}\, K^{\sigma}\, K^{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}] = g_{\sigma} \binom{\sigma}{\alpha} \binom{\sigma}{\beta}, \\[2ex] [l^{\alpha}_{\beta} + h^{\varrho}\, k_{\varrho}\, k_{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}] = h^{\varrho}\, \overline{\binom{\alpha}{\varrho}}\, \overline{\binom{\beta}{\varrho}}, \end{cases}$$

$$(3.9)\qquad \binom{\alpha}{\sigma}\, \overline{\binom{\sigma}{\beta}} = K^{\alpha}\, k_{\beta}\, \theta^{\alpha}_{\beta},$$

(3.7) reduces to

$$(3.10)\qquad [\bar{g}_{\alpha\beta} + \bar{g}_{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}]\,[\bar{h}^{\alpha\beta} + \bar{h}^{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}] = [g_{\alpha\beta} + g_{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}]\,[h^{\alpha\beta} + h^{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}].$$

Expanding (3.10), we obtain

$$\bar{g}_{\alpha\beta}\, \bar{h}^{\alpha\beta} + \bar{g}_{\sigma\sigma}\, \bar{h}^{\sigma} + \bar{g}_{\sigma}\, \bar{h}^{\sigma\sigma} + \bar{g}_{\sigma}\, \bar{h}^{\sigma} = g_{\alpha\beta}\, h^{\alpha\beta} + g_{\sigma\sigma}\, h^{\sigma} + g_{\sigma}\, h^{\sigma\sigma} + g_{\sigma}\, h^{\sigma}$$

which proves our theorem.

One may derive the law of transformation of the coefficient of a general functional form

$$\Phi^{\alpha_1\alpha_2\ldots\alpha_\varrho}_{\beta_1\beta_2\ldots\beta_q}\, \xi_{\alpha_1}\, \xi_{\alpha_2} \cdots \xi_{\alpha_p}\, y^{\beta_1}\, y^{\beta_2} \cdots y^{\beta_q}$$

when it is an absolute form under the transformations (2.2) and (2.19). It is also interesting to note that we may form simultaneous invariants with the coefficient of this form and with the coefficients of the cogredient and contragredient forms (3.2) and (3.3). An example of such an invariant is the following

$$g_{\lambda_1\lambda_2}\, h^{\mu_1\mu_2}\, \Phi^{\lambda_1\lambda_2}_{\mu_1\mu_2} + g_{\lambda_1\lambda_2}\, h^{\tau}\, \Phi^{\lambda_1\lambda_2}_{\tau\tau} + g_{\sigma}\, h^{\mu_1\mu_2}\, \Phi^{\sigma\sigma}_{\mu_1\mu_2} + g_{\sigma}\, h^{\tau}\, \Phi^{\sigma\sigma}_{\tau\tau}.$$

With the coefficients of the two absolute quadratic functional forms (3.2) and

$$(3.11)\qquad G_{\alpha\beta}\, y^{\alpha}\, y^{\beta}, \qquad (G_{\alpha\beta} = G_{\beta\alpha}),$$

By the same reasoning as for (1.29), we may show that the quartic functional form

$$h^{\alpha_1\beta_1,\alpha_2\beta_2}\,\xi_{\alpha_1}\,\xi_{\beta_1}\,\eta_{\alpha_2}\,\eta_{\beta_2}+h^{\alpha\beta}\,h^{\gamma}\begin{vmatrix}\xi_\alpha, & \xi_\gamma\\ \eta_\alpha, & \eta_\gamma\end{vmatrix}\cdot\begin{vmatrix}\xi_\beta, & \xi_\gamma\\ \eta_\beta, & \eta_\gamma\end{vmatrix} \tag{2.20}$$
$$+h^{\gamma_1}\,h^{\gamma_2}\,\xi_{\gamma_1}\,\eta_{\gamma_2}\begin{vmatrix}\xi_{\gamma_1}, & \xi_{\gamma_2}\\ \eta_{\gamma_1}, & \eta_{\gamma_2}\end{vmatrix}$$

is an absolute form under the contragredient transformations (2.19). Theorem 1—V also holds for the coefficients of (2.20) with the understanding that the expressions $\overline{\binom{\alpha}{\beta}}$ and l^{α}_{β} are to be interpreted in accordance with the definitions (3.4) and (3.6) respectively.

3. **Simultaneous invariants.** Up to this point we have been concerned wholly with one quadratic functional form and its functional invariants. In this concluding paragraph we shall build up a class of simultaneous functional invariants of two quadratic functional forms.

THEOREM 3—I. *The expression*

$$g_{\alpha\beta}\,h^{\alpha\beta}+g_{\sigma\sigma}\,h^{\sigma}+g_{\sigma}\,h^{\sigma\sigma}+g_{\sigma}\,h^{\sigma} \tag{3.1}$$

is an absolute invariant of the coefficients $g_{\alpha\beta}$ and g_α of a form

$$g_{\alpha\beta}\,y^{\alpha}\,y^{\beta}+g_{\alpha}(y^{\alpha})^2, \quad (g_{\alpha\beta}=g_{\beta\alpha};\ g_\alpha\neq 0) \tag{3.2}$$

and $h^{\alpha\beta}$ and h^{α} of a form

$$h^{\alpha\beta}\,\xi_\alpha\,\xi_\beta+h^{\alpha}(\xi_\alpha)^2, \quad (h^{\alpha\beta}=h^{\beta\alpha};\ h^{\alpha}\neq 0) \tag{3.3}$$

where the function ξ_α transforms contragrediently to the function y^{α}.

To establish this theorem, let us adopt the following extensions to the notation[12] of (1.36)

$$\binom{\alpha}{\beta}=(K^{\alpha}\,\theta^{\alpha}_{\beta}+K^{\alpha}_{\beta});\quad \overline{\binom{\alpha}{\beta}}=(k_\beta\,\theta^{\alpha}_{\beta}+k^{\alpha}_{\beta}) \tag{3.4}$$

where θ^{α}_{β} is the same symbolic operator. With this notation the laws of transformation of the coefficients $g_{\alpha\beta}$ and $h^{\alpha\beta}$ respectively reduce to

$$\bar{g}_{\alpha\beta}=g_{\lambda\mu}\binom{\lambda}{\alpha}\binom{\mu}{\beta}+t^{\alpha}_{\beta},\quad \bar{h}^{\alpha\beta}=h^{\lambda\mu}\,\overline{\binom{\alpha}{\lambda}}\,\overline{\binom{\beta}{\mu}}+l^{\alpha}_{\beta} \tag{3.5}$$

[12] From now on we shall use these extensions to the notation of (1.36) exclusively. Furthermore $h^{\alpha\beta}$ and h^{α} will have the significance of the coefficients of an *arbitrary* absolute form (3.3). The invariant (3.1) reduces to the length of the fundamental interval (a, b) for the special form (2.18).

where[12]

$$
(3.6)\qquad \begin{cases} t^{\alpha}_{\beta} = g_{(\alpha)}\, K^{\alpha}\, K^{\alpha}_{\beta} + g_{(\beta)}\, K^{\beta}\, K^{\beta}_{\alpha} + g_{\sigma}\, K^{\sigma}_{\alpha}\, K^{\sigma}_{\beta}, \\ l^{\alpha}_{\beta} = h^{(\alpha)}\, k_{\alpha}\, k^{\beta}_{\alpha} + h^{(\beta)}\, k_{\beta}\, k^{\alpha}_{\beta} + h^{\sigma}\, k^{\alpha}_{\sigma}\, k^{\beta}_{\sigma}. \end{cases}
$$

Let us now form the expression

$$
(3.7)\qquad \begin{aligned} &[\bar{g}_{\alpha\beta} + \bar{g}_{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}]\, [\bar{h}^{\alpha\beta} + \bar{h}^{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}] \\ &= \left[g_{\lambda\mu} \binom{\lambda}{\alpha} \binom{\mu}{\beta} + t^{\alpha}_{\beta} + g_{\sigma}\, K^{\sigma}\, K^{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}\right] \left[h^{\nu\tau}\, \overline{\binom{\alpha}{\nu}}\, \overline{\binom{\beta}{\tau}} + l^{\alpha}_{\beta} + h^{\varrho}\, k_{\varrho}\, k_{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}\right]. \end{aligned}
$$

Because of the relations

$$
(3.8)\qquad \begin{cases} [t^{\alpha}_{\beta} + g_{\sigma}\, K^{\sigma}\, K^{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}] = g_{\sigma} \binom{\sigma}{\alpha} \binom{\sigma}{\beta}, \\ [l^{\alpha}_{\beta} + h^{\varrho}\, k_{\varrho}\, k_{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}] = h^{\varrho}\, \overline{\binom{\alpha}{\varrho}}\, \overline{\binom{\beta}{\varrho}}, \end{cases}
$$

$$
(3.9)\qquad \binom{\alpha}{\sigma}\, \overline{\binom{\sigma}{\beta}} = K^{\alpha}\, k_{\beta}\, \theta^{\alpha}_{\beta},
$$

(3.7) reduces to

$$
(3.10)\qquad [\bar{g}_{\alpha\beta} + \bar{g}_{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}]\, [\bar{h}^{\alpha\beta} + \bar{h}^{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}] = [g_{\alpha\beta} + g_{\sigma}\, \theta^{\sigma}_{\alpha}\, \theta^{\sigma}_{\beta}]\, [h^{\alpha\beta} + h^{\varrho}\, \theta^{\alpha}_{\varrho}\, \theta^{\beta}_{\varrho}].
$$

Expanding (3.10), we obtain

$$
\bar{g}_{\alpha\beta}\, \bar{h}^{\alpha\beta} + \bar{g}_{\sigma\sigma}\, \bar{h}^{\sigma} + \bar{g}_{\sigma}\, \bar{h}^{\sigma\sigma} + \bar{g}_{\sigma}\, \bar{h}^{\sigma} = g_{\alpha\beta}\, h^{\alpha\beta} + g_{\sigma\sigma}\, h^{\sigma} + g_{\sigma}\, h^{\sigma\sigma} + g_{\sigma}\, h^{\sigma}
$$

which proves our theorem.

One may derive the law of transformation of the coefficient of a general functional form

$$
\Phi^{\alpha_1\alpha_2\ldots\alpha_\varrho}_{\beta_1\beta_2\ldots\beta_s}\, \xi_{\alpha_1}\, \xi_{\alpha_2} \cdots \xi_{\alpha_p}\, y^{\beta_1}\, y^{\beta_2} \cdots y^{\beta_s}
$$

when it is an absolute form under the transformations (2.2) and (2.19). It is also interesting to note that we may form simultaneous invariants with the coefficient of this form and with the coefficients of the cogredient and contragredient forms (3.2) and (3.3). An example of such an invariant is the following

$$
g_{\lambda_1\lambda_2}\, h^{\mu_1\mu_2}\, \Phi^{\lambda_1\lambda_2}_{\mu_1\mu_2} + g_{\lambda_1\lambda_2}\, h^{\tau}\, \Phi^{\lambda_1\lambda_2}_{\tau\tau} + g_{\sigma}\, h^{\mu_1\mu_2}\, \Phi^{\sigma\sigma}_{\mu_1\mu_2} + g_{\sigma}\, h^{\tau}\, \Phi^{\sigma\sigma}_{\tau\tau}.
$$

With the coefficients of the two absolute quadratic functional forms (3.2) and

$$
(3.11)\qquad G_{\alpha\beta}\, y^{\alpha}\, y^{\beta}, \qquad (G_{\alpha\beta} = G_{\beta\alpha}).
$$

we may construct a third form

$$(g_{\alpha\beta} + \lambda G_{\alpha\beta}) y^\alpha y^\beta + g_\alpha (y^\alpha)^2, \qquad (\lambda = \text{parameter}) \tag{3.12}$$

which clearly will also be an absolute functional form. Hence

$$D\left[(\bar{g}_{\alpha\beta} + \lambda \bar{G}_{\alpha\beta}) / \sqrt{\bar{g}_\alpha \bar{g}_\beta}\,\right] = \{D[K_\tau^\sigma / K^{(\tau)}]\}^2 \, D\left[(g_{\alpha\beta} + \lambda G_{\alpha\beta}) / \sqrt{g_\alpha g_\beta}\,\right]. \tag{3.13}$$

Both sides of (3.13), however, may be expressed as an infinite power series in λ

$$\bar{\theta}_0 + \bar{\theta}_1 \lambda + \cdots = \{D[K_\tau^\sigma / K^{(\tau)}]\}^2 (\theta_0 + \theta_1 \lambda + \cdots) \tag{3.14}$$

where

$$\theta_0 = D[g_{\alpha\beta} / \sqrt{g_\alpha g_\beta}\,], \qquad \theta_i = \theta_i[g_{\alpha\beta}, G_{\alpha\beta}, g_\alpha],$$
$$\bar{\theta}_0 = D[\bar{g}_{\alpha\beta} / \sqrt{\bar{g}_\alpha \bar{g}_\beta}\,], \qquad \bar{\theta}_i = \theta_i[\bar{g}_{\alpha\beta}, \bar{G}_{\alpha\beta}, \bar{g}_\alpha].$$

In order that (3.14) be true, it is necessary and sufficient that the corresponding coefficients of λ be equal and so we have

$$\bar{\theta}_i = (D[K_\tau^\sigma / K^{(\tau)}])^2 \, \theta_i, \qquad (i = 0, 1, 2, \cdots).$$

In this manner we obtain an infinitude of relative scalar functional invariants of weight two of the coefficients of the two forms (3.2) and (3.11).

By expanding $D[(g_{\alpha\beta} + \lambda G_{\alpha\beta}) / \sqrt{g_\alpha g_\beta}\,]$, we find that θ_n has the form[13]

$$\begin{aligned} \theta_n = \sum \varepsilon_s \varepsilon_t \frac{G^{\sigma_{s_1}}_{\sigma_{t_1}} \cdots G^{\sigma_{s_n}}_{\sigma_{t_n}}}{g_{\sigma_1} \cdots g_{\sigma_n}} & \left\{ \frac{1}{n!} + \frac{1}{(n+1)!} \frac{1}{g_{\sigma_{n+1}}} g^{\sigma_{s_{n+1}}}_{\sigma_{t_{n+1}}} + \cdots \right. \\ & \left. + \frac{1}{(n+k)!} \frac{1}{g_{\sigma_{n+1}} \cdots g_{\sigma_{n+k}}} g^{\sigma_{s_{n+1}}}_{\sigma_{t_{n+1}}} \cdots g^{\sigma_{s_{n+k}}}_{\sigma_{t_{n+k}}} + \cdots \right\} \end{aligned} \tag{3.15}$$

or

$$\begin{aligned} \theta_n = \frac{1}{n!} \frac{1}{g_{\sigma_1} \cdots g_{\sigma_n}} \sum \varepsilon_s \varepsilon_t \, G^{\sigma_{s_1} \cdots \sigma_{s_n}}_{\sigma_{t_1} \cdots \sigma_{t_n}} & \left\{ \frac{1}{n!} + \frac{1}{1!\,(n+1)!} \frac{1}{g_{\sigma_{n+1}}} g^{\sigma_{s_{n+1}}}_{\sigma_{t_{n+1}}} + \cdots \right. \\ & \left. + \frac{1}{k!\,(n+k)!} \frac{1}{g_{\sigma_{n+1}} \cdots g_{\sigma_{n+k}}} g^{\sigma_{s_{n+1}} \cdots \sigma_{s_{n+k}}}_{\sigma_{t_{n+1}} \cdots \sigma_{t_{n+k}}} + \cdots \right\} \end{aligned}$$

where

$$f^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n} = \begin{vmatrix} f^{\alpha_1}_{\beta_1}, & \cdots, & f^{\alpha_1}_{\beta_n} \\ \cdot & \cdot & \cdot \\ f^{\alpha_n}_{\beta_1}, & \cdots, & f^{\alpha_n}_{\beta_n} \end{vmatrix}$$

[13] The first subscripts of $g_{\alpha\beta}$ and $G_{\alpha\beta}$ have been elevated in order to facilitate the following work.

for $G^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ and $g^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$, and where the summation is taken over all the permutations of $1, 2, \cdots, p$ ($p = n, n+1, \cdots$). For even permutations of $s_1, \cdots, s_p$ and $t_1, \cdots, t_p$ both ϵ_s and ϵ_t are respectively $+1$; for odd permutations they are -1; in all other cases, they are zero.

It is evident from (3.15) that θ_n is an homogeneous functional of degree n in the function $G_{\alpha\beta}$. From the manner in which the function $G_{\alpha\beta}$ was chosen, it is clear that $n!\,\theta_n$ is the nth partial differential (in the sense of Gateaux)[14] of $D[g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}\,]$ with respect to $g_{\alpha\beta}$ when $G_{\alpha\beta}$ is the increment in $g_{\alpha\beta}$. A similar relation holds good between $D[g_{\alpha\beta}]$ and θ_n for $g_\alpha \equiv 1$. Furthermore we can show that the functional $D[g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}\,]$ has an nth partial differential with respect to $g_{\alpha\beta}$ in the sense of Fréchet. Clearly this will be immediately forthcoming, if we show that the Fredholm determinant possesses a Fréchet differential. By a formula for the Fredholm determinant of the sum of two non-orthogonal kernels[15], we have

$$
\begin{aligned}
D[g^\lambda_\mu + G^\lambda_\mu] = {} & D[g^\lambda_\mu]\,D[G^\lambda_\mu] - D^\alpha_\beta[g^\lambda_\mu]\,D^\beta_\alpha[G^\lambda_\mu] + \cdots \\
& + \frac{(-1)^k}{(k!)^2}\,D^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}[g^\lambda_\mu]\,D^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}[G^\lambda_\mu] + \cdots
\end{aligned}
\tag{3.16}
$$

where $D^{\sigma_1\cdots\sigma_k}_{\tau_1\cdots\tau_k}[\omega_{\lambda\mu}]$ is the kth Fredholm minor of the function $\omega_{\lambda\mu}$.

By means of this formula we may calculate the first differential $\delta D[g^\lambda_\mu]$ in the sense of Fréchet. In fact, we have[16]

$$
D[g^\lambda_\mu + G^\lambda_\mu] - D[g^\lambda_\mu] = \{D[g^\lambda_\mu]\,G^\sigma_\sigma - D^\alpha_\beta[g^\lambda_\mu]\,G^\beta_\alpha\} + \epsilon \max|G^\lambda_\mu| \tag{3.17}
$$

where

$$
\begin{aligned}
\epsilon = \frac{1}{\max|G^\lambda_\mu|} \Big\{ & D[g^\lambda_\mu]\Big(\frac{1}{2!}G^{\sigma_1\sigma_2}_{\sigma_1\sigma_2} + \frac{1}{3!}G^{\sigma_1\sigma_2\sigma_3}_{\sigma_1\sigma_2\sigma_3} + \cdots\Big) \\
& - D^\alpha_\beta[g^\lambda_\mu]\Big(G^{\beta\sigma_1}_{\alpha\sigma_1} + \frac{1}{2!}G^{\beta\sigma_1\sigma_2}_{\alpha\sigma_1\sigma_2} + \cdots\Big) \\
& + \frac{1}{(2!)^2}D^{\alpha_1\alpha_2}_{\beta_1\beta_2}[g^\lambda_\mu]\,D^{\beta_1\beta_2}_{\alpha_1\alpha_2}[G^\lambda_\mu] + \cdots \Big\}.
\end{aligned}
\tag{3.18}
$$

In (3.18) it is quite evident that ϵ will be a quantity that approaches zero with $\max|G^\lambda_\mu|$.

Continuing in the same general manner, we may obtain the nth differential of $D[g^\lambda_\mu]$. After the usual procedure, we have

[14] Cf. P. Lévy, "*Leçons d'Analyse Fonctionelle*", Paris (1922), p. 51.

[15] Cf. Tr. Lalesco, "*Sur l'addition des noyaux non orthogonaux*", Bulletin des Sciences Mathématiques (2° Série), tome 42 (1918), pp. 195–199.

[16] For a further discussion on the *first* differentials of the Fredholm determinant and its minors, cf. G. C. Evans, American Mathematical Monthly, vol. 34 (1927), pp. 142–150.

$$D[g^\lambda_\mu + G^\lambda_\mu] - D[g^\lambda_\mu] - \cdots - \frac{1}{(n-1)!}\delta^{n-1} D[g^\lambda_\mu]$$

$$(3.19) = \frac{1}{n!}\left\{D[g^\lambda_\mu] G^{\sigma_1\cdots\sigma_n}_{\sigma_1\cdots\sigma_n} - \frac{n}{(1!)^2} D^\alpha_\beta[g^\lambda_\mu] G^{\beta\sigma_1\cdots\sigma_{n-1}}_{\alpha\sigma_1\cdots\sigma_{n-1}} + \cdots + \frac{(-1)^n}{n!} D^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}[g^\lambda_\mu] G^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n}\right\} + \varepsilon \max|G^\lambda_\mu|^n$$

where

$$\varepsilon = \frac{1}{\max|G^\lambda_\mu|^n}\left\{D[g^\lambda_\mu]\left(\frac{1}{(n+1)!} G^{\sigma_1\cdots\sigma_{n+1}}_{\sigma_1\cdots\sigma_{n+1}} + \cdots\right) - D^\alpha_\beta[g^\lambda_\mu]\left(\frac{1}{n!} G^{\beta\sigma_1\cdots\sigma_n}_{\alpha\sigma_1\cdots\sigma_n} + \cdots\right) + \cdots + \frac{(-1)^n}{(n!)^2} D^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}[g^\lambda_\mu]\left(G^{\beta_1\cdots\beta_n\sigma_1}_{\alpha_1\cdots\alpha_n\sigma_1} + \cdots\right) + \frac{(-1)^{n+1}}{[(n+1)!]^2} D^{\alpha_1\cdots\alpha_{n+1}}_{\beta_1\cdots\beta_{n+1}}[g^\lambda_\mu] D^{\beta_1\cdots\beta_{n+1}}_{\alpha_1\cdots\alpha_{n+1}}[G^\lambda_\mu] + \cdots\right\}$$

and clearly ε approaches zero with $\max|G^\lambda_\mu|$, since every term in the above brace contains G^λ_μ at least $(n+1)$ times.

THEOREM 3—II. *The nth differential of the Fredholm determinant $D[g^\lambda_\mu]$ in the sense of Fréchet has the form*

$$\delta^n D[g^\lambda_\mu] \\ (3.20) = D[g^\lambda_\mu] G^{\sigma_1\sigma_2\cdots\sigma_n}_{\sigma_1\sigma_2\cdots\sigma_n} - \frac{n}{(1!)^2} D^\alpha_\beta[g^\lambda_\mu] G^{\beta\sigma_1\cdots\sigma_{n-1}}_{\alpha\sigma_1\cdots\sigma_{n-1}} + \cdots + \frac{(-1)^n}{n!} D^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g^\lambda_\mu] G^{\beta_1\beta_2\cdots\beta_n}_{\alpha_1\alpha_2\cdots\alpha_n}.$$

This theorem coupled with the remarks made immediately preceding formula (3.16) leads us to the theorem.

THEOREM 3—III. *Each functional $I_n[g_{\alpha\beta}, G_{\alpha\beta}, g_\alpha]$, $(n = 1, 2, 3, \cdots)$ in the infinite sequence of functionals*

$$I_n = D\left[g^{(\lambda)}_\mu/\sqrt{g_\lambda g_\mu}\right]\frac{G^{\sigma_1\sigma_2\cdots\sigma_n}_{\sigma_1\sigma_2\cdots\sigma_n}}{g_{\sigma_1}g_{\sigma_2}\cdots g_{\sigma_n}} \\ (3.21) \quad - \frac{n}{(1!)^2}\frac{1}{\sqrt{g_\alpha g_\beta}} D^\alpha_\beta\left[g^{(\lambda)}_\mu/\sqrt{g_\lambda g_\mu}\right]\frac{G^{\beta\sigma_1\cdots\sigma_{n-1}}_{\alpha\sigma_1\cdots\sigma_{n-1}}}{g_{\sigma_1}\cdots g_{\sigma_{n-1}}} + \cdots \\ + \frac{(-1)^n}{n!}\frac{1}{\sqrt{g_{\alpha_1}g_{\beta_1}\cdots g_{\alpha_n}g_{\beta_n}}} D^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\left[g^{(\lambda)}_\mu/\sqrt{g_\lambda g_\mu}\right] G^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n},$$

where

$$(3.22) \qquad G^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_n} = \begin{vmatrix} G^{\lambda_1}_{\mu_1}, & \cdots, & G^{\lambda_1}_{\mu_n} \\ \cdot & \cdot & \cdot \\ G^{\lambda_n}_{\mu_1}, & \cdots, & G^{\lambda_n}_{\mu_n} \end{vmatrix}$$

is a simultaneous functional invariant of weight two of the functional forms $G_{\alpha\beta}\, y^\alpha y^\beta$, $g_{\alpha\beta}\, y^\alpha y^\beta + g_\alpha (y^\alpha)^2$, $(G_{\alpha\beta} = G_{\beta\alpha},\ g_{\alpha\beta} = g_{\beta\alpha},\ g_\alpha \neq 0)$, *that is*

$$I_n[\bar{g}_{\alpha\beta}, \bar{G}_{\alpha\beta}, \bar{g}_\alpha] = (D[K^\sigma_\tau / K^{(\tau)}])^2\, I_n[g_{\alpha\beta}, G_{\alpha\beta}, g_\alpha].$$

From the above theorem, it is clear, because of the relative invariance of the functional $D[g^{(\lambda)}_\mu / \sqrt{g_\lambda g_\mu}]$, that

$$(3.23) \qquad J_n = I_n / D[g^{(\lambda)}_\mu / \sqrt{g_\lambda g_\mu}]$$

is an absolute functional invariant subject to the conditions of the above theorem. From the mode of transformation of the function G^α_β, it follows immediately that if $G^\alpha_\beta = \xi_\alpha \xi_\beta$, then $\bar{G}^\alpha_\beta = \bar{\xi}_\alpha \bar{\xi}_\beta$ where

$$(3.24) \qquad \xi_\alpha = k_\alpha \bar{\xi}_\alpha + k^\sigma_\alpha \bar{\xi}_\sigma.$$

Generalizing this substitution by means of (3.22), we shall consider $G^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ to be of the form

$$\frac{1}{n!} \begin{vmatrix} \xi_{\alpha_1}, & \cdots, & \xi_{\alpha_n} \\ \cdot & \cdot & \cdot \\ \varrho_{\alpha_1}, & \cdots, & \varrho_{\alpha_n} \end{vmatrix} \cdot \begin{vmatrix} \xi_{\beta_1}, & \cdots, & \xi_{\beta_n} \\ \cdot & \cdot & \cdot \\ \varrho_{\beta_1}, & \cdots, & \varrho_{\beta_n} \end{vmatrix}$$

where in this case it follows directly from (3.22) that the n functions $\xi_\alpha, \cdots, \varrho_\alpha$ transform cogrediently to (3.24).

THEOREM 3—IV. *Each functional* $J_n[g_{\alpha\beta}, g_\alpha, \xi_\alpha, \cdots, \varrho_\alpha]$, $(n = 1, 2, 3, \cdots)$ *in the infinite sequence of functionals*

$$\begin{aligned} J_n = {} & h^{\alpha_1\beta_1,\cdots,\alpha_n\beta_n}\, \xi_{\alpha_1} \xi_{\beta_1} \cdots \varrho_{\alpha_n} \varrho_{\beta_n} \\ & + \frac{1}{[(n-1)!]^2} h^{\alpha_1\beta_1,\cdots,\alpha_{n-1}\beta_{n-1}}\, h^\sigma \begin{vmatrix} \xi_{\alpha_1}, & \cdots, & \xi_{\alpha_{n-1}}, & \xi_\sigma \\ \cdot & \cdot & \cdot & \cdot \\ \varrho_{\alpha_1}, & \cdots, & \varrho_{\alpha_{n-1}}, & \varrho_\sigma \end{vmatrix} \cdot \begin{vmatrix} \xi_{\beta_1}, & \cdots, & \xi_{\beta_{n-1}}, & \xi_\sigma \\ \cdot & \cdot & \cdot & \cdot \\ \varrho_{\beta_1}, & \cdots, & \varrho_{\beta_{n-1}}, & \varrho_\sigma \end{vmatrix} \\ & + \cdots \\ & + \frac{1}{n!} h^{\sigma_1} \cdots h^{\sigma_n} \begin{vmatrix} \xi_{\sigma_1}, & \cdots, & \xi_{\sigma_n} \\ \cdot & \cdot & \cdot \\ \varrho_{\sigma_1}, & \cdots, & \varrho_{\sigma_n} \end{vmatrix} \cdot \begin{vmatrix} \xi_{\sigma_1}, & \cdots, & \xi_{\sigma_n} \\ \cdot & \cdot & \cdot \\ \varrho_{\sigma_1}, & \cdots, & \varrho_{\sigma_n} \end{vmatrix} \end{aligned}$$

is an absolute simultaneous functional invariant of the coefficients of the absolute form (3.2) *and the n contragredient functions* $\xi_\alpha, \cdots, \varrho_\alpha$.

By the manner in which the coefficients of J_n are constructed, i. e.,

$$h^{\alpha_1\beta_1,\cdots,\alpha_n\beta_n} = (-1)^n \frac{D^{(\alpha_1)\cdots(\alpha_n)}_{\beta_1\cdots\beta_n}[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]}{\sqrt{g_{\alpha_1} g_{\beta_1} \cdots g_{\alpha_n} g_{\beta_n}}\, D[g_{\lambda\mu}/\sqrt{g_\lambda g_\mu}]},$$

$$h^\sigma = 1/g_\sigma,$$

and by special interpretation of the above theorem, it is clear that we may consider J_n as a $2n$-tic absolute contragredient functional form. In fact, for J_1 and J_2, we find functional forms with which we are already familar, namely (2.18) and (2.20). This last consideration leads us to the generalization of such forms.

ON GENERALISED COVARIANT DIFFERENTIATION.[1]

By A. W. Tucker.

The following attempt to present covariant differentiation ab initio in a highly generalised form develops an idea suggested by, and embracing, the *verallgemeinerte Riccidifferential* of W. Mayer.[2] The method of attack resembles in many respects the one adopted by M. R. Lagrange[3] in dealing with ordinary covariant differentiation. For the tensor terminology and notation to be used, the reader is referred to the well-known texts of L. P. Eisenhart.[4]

1. **Initial assumptions.** Under a non-singular coördinate transformation

$$x'^{a} = x'^{a}(x^1, x^2, \cdots, x^N) \qquad (a = 1, 2, \cdots, N) \tag{1.1}$$

the ordinary tensor law of transformation is

$$T'^{j_1 \cdots j_p}_{j_q \cdots j_r} = (T^{i_1 \cdots i_p}_{i_q \cdots i_r}) \frac{\partial x'^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{j_p}}{\partial x^{i_p}} \frac{\partial x^{i_q}}{\partial x'^{j_q}} \cdots \frac{\partial x^{i_r}}{\partial x'^{j_r}} \qquad (q = p+1).$$

More symmetrically stated this law reads

$$(T^{i_1 \cdots i_p}_{i_q \cdots i_r}) \frac{\partial x'^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{j_p}}{\partial x^{i_p}} = (T'^{j_1 \cdots j_p}_{j_q \cdots j_r}) \frac{\partial x'^{j_q}}{\partial x^{i_q}} \cdots \frac{\partial x'^{j_r}}{\partial x^{i_r}},$$

or as we shall put it

$$(T^{i_1 \cdots i_p}_{i_q \cdots i_r}) \frac{\partial x^{'i_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{'i_p}}{\partial x^{i_p}} = (T^{'i_1 \cdots 'i_p}_{'i_q \cdots 'i_r}) \frac{\partial x^{'i_q}}{\partial x^{i_q}} \cdots \frac{\partial x^{'i_r}}{\partial x^{i_r}} \tag{1.2}$$

where an accent at the upper left of an index is used to apply as a distinguishing mark *both* to the index and to the symbol to which the index is attached (i. e. $x^{'i_1} \equiv x'^{i'_1}$, $T^{'i_1} \equiv T'^{i'_1}$, etc.).

It has been assumed in the above equations that all the indices take the same range of values, viz. $1, 2, \cdots, N$, but we promptly forget this

[1] Received June 30, 1930.

[2] Duschek-Mayer, Lehrbuch der Differentialgeometrie, Bd. II (1930), Kap. VII.

[3] Lagrange, R., Mémorial des Sciences Mathématiques, Fascicule XIX, Calcul différentiel absolu (1926), Chap. II.

[4] Eisenhart, L. P., Riemannian Geometry (1926), Non-Riemannian Geometry (Ithaca Colloquium, 1925).

and give the equations a new significance by allowing the indices a *variety* of ranges. Relation (1.2) becomes something much more general than an expression of the ordinary tensor law of transformation. This added generality is expressly desired. To provide the explicit machinery for securing it we hypothesize that indices with subscript λ, such as i_λ, etc., take the range of values $n_\lambda+1, n_\lambda+2, \cdots, N_\lambda$ $(0 \leqq n_\lambda < N_\lambda \leqq N)$, where λ stands for a characteristic one of the numbers $1, 2, \cdots, p, q, \cdots, r, s, t$ $(s = r+1,\ t = r+2$ make their appearance as we proceed).

We do not require that the $(r+2)$ ranges just introduced be non-overlapping or even distinct. In fact $n_\lambda \gtreqless n_\mu$ $(\lambda \neq \mu)$, and similarly $N_\lambda \gtreqless N_\mu$, are all possibilities to be entertained, particularly that of equality since it serves to allow different indices attached to T to have the same range. Of course the main range $1, 2, \cdots, N$ can be split into "elements" out of which all the $(r+2)$ ranges can be compounded, for in the extreme case these "elements" could consist of the individual numbers $1, 2, \cdots, N$. Denote by

$$
\begin{aligned}
&1, 2, \cdots, m_1 \,|\, m_1+1, \cdots, m_2 \,|\, \cdots \,|\, m_{\sigma-1}+1, \cdots, m_\sigma \,|\, \cdots \\
&\qquad\qquad \cdots \,|\, m_{w-1}+1, \cdots, m_w = N
\end{aligned} \tag{1.3}
$$

the partition of $1, 2, \cdots, N$ into "elements" such that no two "elements" can be lumped into a single "element". Such a partition is unique.

Besides emancipating the indices we shall permit the x's some anomaly in that certain of them may be functions of certain others; we merely insist that each block of x's corresponding to one of the $(r+2)$ ranges be functionally independent. Accordingly we may, whenever we so desire, regard the x''s as additional blocks of x's by just imagining that the special accents introduced in (1.2) apply *solely* to the indices, so that an accented index with subscript λ, such as $'i_\lambda$, may take the range of values $N+n_\lambda+1$, $N+n_\lambda+2, \cdots, N+N_\lambda$. But it is not intended that this new interpretation of the special accents should be adopted to the exclusion of the original interpretation, or that, when used on occasion (merely to unify our assumptions), the new interpretation should cause the coördinate transformation significance of the functional relationship in (1.1) to be obscured in any way.

2. Ordinary differentiation inadequate. Differentiating (1.2) we obtain

$$
\left(\frac{\partial}{\partial x^{i_s}} T^{i_1\cdots i_p}_{i_q\cdots i_r}\right)\frac{\partial x^{'i_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{'i_p}}{\partial x^{i_p}} = \left(\frac{\partial}{\partial x^{'i_s}} T^{'i_1\cdots 'i_p}_{'i_q\cdots 'i_r}\right)\frac{\partial x^{'i_q}}{\partial x^{i_q}}\cdots\frac{\partial x^{'i_r}}{\partial x^{i_r}}\,\frac{\partial x^{'i_s}}{\partial x^{i_s}}, \tag{2.1}
$$

provided

$$
\frac{\partial}{\partial x^{i_s}} T^{'i_1\cdots 'i_p}_{'i_q\cdots 'i_r} = \frac{\partial}{\partial x^{'i_s}} T^{'i_1\cdots 'i_p}_{'i_q\cdots 'i_r}\frac{\partial x^{'i_s}}{\partial x^{i_s}}, \tag{2.2}
$$

which demands that the x''s not contained in the block associated with the subscript s be independent of the x's contained in the block associated with the subscript s, a requirement immediately satisfied by assuming from now on that (1.1) takes the form

$$x'^{a} = x'^{a}(x^{m_{\sigma-1}+1}, x^{m_{\sigma-1}+2}, \cdots, x^{m_\sigma}), \tag{2.3}$$

where σ takes the values $1, 2, \cdots, w$ (cf. (1.3)) in such a way that as a runs through the range $1, 2, \cdots, N$ we always have $m_{\sigma-1} < a \leqq m_\sigma$; *and provided*

$$\frac{\partial^2 x'^{i_\lambda}}{\partial x^{i_s}\, \partial x^{i_\lambda}} = 0 \qquad (\lambda = 1, 2, \cdots, r) \tag{2.4}$$

which amounts to requiring the functions in (2.3) to be linear.

The bracketed quantities in (2.1) may well be termed "offspring" of the corresponding quantities in (1.2) for they are derived from the latter and they satisfy the same law of transformation. Some "reproduction" of this sort is highly desirable, but condition (2.4) is much too restrictive. However, we shall show that by adding to the bracketed quantities in (2.1) certain corrective linear combinations of the corresponding quantities in (1.2) we achieve a "differentiation" yielding a method of "reproduction" untrammelled by such a severe proviso as (2.4).

3. **Another "differentiation"**. Instead of $\partial/\partial x^{i_s}$ consider

$$D_{x^{i_s}} = \frac{\partial}{\partial x^{i_s}} + E_{x^{i_s}}, \tag{3.1}$$

where $E_{x^{i_s}}$ operates for example on

$$T^{i_1 \cdots i_p}_{i_q \cdots i_r}$$

to produce

$$\sum_{\lambda=1}^{p} T^{i_1 \cdots h_\lambda \cdots i_p}_{i_q \cdots\cdots\cdots i_r} L^{i_\lambda}_{h_\lambda i_s} + \sum_{\lambda=q}^{r} T^{i_1 \cdots\cdots\cdots i_p}_{i_q \cdots h_\lambda \cdots i_r} M^{h_\lambda}_{i_\lambda i_s}. \tag{3.2}$$

Being a linear operator $E_{x^{i_s}}$ will obviously obey the sum and product rules of differentiation except possibly the inner product one. For simplicity let us test it on a scalar product. Operating on the product as a whole nothing should be obtained since there are no free indices. So we want

$$\begin{aligned} O &= E_{x^{i_s}}(U^{j_1} V_{j_1}) \\ &= (E_{x^{i_s}} U^{j_1}) V_{j_1} + U^{j_1}(E_{x^{i_s}} V_{j_1}) \\ &= U^{h_1} L^{j_1}_{h_1 i_s} V_{j_1} + U^{j_1} V_{h_1} M^{h_1}_{j_1 i_s} \\ &= U^{h_1} V_{j_1} (L^{j_1}_{h_1 i_s} + M^{j_1}_{h_1 i_s}). \end{aligned}$$

This will be satisfied if we assume that for all λ

$$M^{j\lambda}_{h_\lambda i_s} = -L^{j\lambda}_{h_\lambda i_s}. \tag{3.3}$$

In analogy with the "function of a function" rule of differentiation we also ask that

$$D_{x^{i_s}} = \frac{\partial x^{i_t}}{\partial x^{i_s}} D_{x^{i_t}} \tag{3.4}$$

whenever the functional dependence of the x's not included in the blocks associated with the subscripts s and t, on the x's of the s block can be completely expressed by means of their functional dependence on the x's of the t block and the set of equations

$$x^{i_t} = x^{i_t}(x^{n_s+1}, x^{n_s+2}, \cdots, x^{N_s}).$$

So qualified, (3.4) is satisfied immediately by the partial derivative portion of $D_{x^{i_s}}$, and by the remainder if for all λ

$$L^{j\lambda}_{h_\lambda i_s} = L^{j\lambda}_{h_\lambda i_t} \frac{\partial x^{i_t}}{\partial x^{i_s}}. \tag{3.5}$$

In particular, if i_λ and i_t have the same range, (3.5) states that

$$L^{j\lambda}_{h_\lambda i_s} = L^{j\lambda}_{h_\lambda i_\lambda} \frac{\partial x^{i_\lambda}}{\partial x^{i_s}}. \tag{3.6}$$

We shall assume that this latter relation, which is so convenient since the "mixed" L's on the left are expressed in terms of the "pure" L's on the right ("pure" because all indices have the same range), holds for $\lambda = 1, 2, \cdots, r$, and, what is prerequisite therefor, that all the partial derivatives $\partial x^{i_\lambda}/\partial x^{i_s}$ have a meaning (due to either functional dependence or independence). Of course (3.6) is to hold throughout when i_t replaces i_s in it, else we are discriminating too much between the operators in (3.4). On this basis (3.5) is a consequence of (3.6) for

$$L^{j\lambda}_{h_\lambda i_\lambda} \frac{\partial x^{i_\lambda}}{\partial x^{i_s}} = L^{j\lambda}_{h_\lambda i_\lambda} \frac{\partial x^{i_\lambda}}{\partial x^{i_t}} \frac{\partial x^{i_t}}{\partial x^{i_s}} = L^{j\lambda}_{h_\lambda i_t} \frac{\partial x^{i_t}}{\partial x^{i_s}}.$$

Qualified by (3.3) and (3.6) $D_{x^{i_s}}$ provides us with an operation sufficiently simulating actual differentiation to suit our purposes. When we pass to the x''s we introduce an analogous operator

$$D_{x'^{i_s}} = \frac{\partial}{\partial x'^{i_s}} + E_{x'^{i_s}}$$

subject to relations obtained by accenting the indices in (3.3) and (3.6). Also, regarding the x''s as additional blocks of x's, it is natural to take from (3.6) that

$$L'^{j_\lambda}_{'h_\lambda i_s} = L'^{j_\lambda}_{'h_\lambda 'i_\lambda} \frac{\partial x'^{i_\lambda}}{\partial x^{i_s}} = L'^{j_\lambda}_{'h_\lambda 'i_\lambda} \frac{\partial x'^{i_\lambda}}{\partial x^{i_\lambda}} \frac{\partial x^{i_\lambda}}{\partial x^{i_s}} = L'^{j_\lambda}_{'h_\lambda 'i_\lambda} \frac{\partial x'^{i_\lambda}}{\partial x'^{i_s}} \frac{\partial x'^{i_s}}{\partial x^{i_s}} \tag{3.7}$$

which entails, using (3.3) with accented indices, that

$$M'^{j_\lambda}_{'h_\lambda i_s} = -L'^{j_\lambda}_{'h_\lambda i_s}. \tag{3.8}$$

4. **New "differentiation" adequate.** We shall now show that operating with $D_{x^{i_s}}$ yields the desired "reproduction" (cf. § 2). From (1.2) we have

$$(D_{x^{i_s}} T^{i_1 \cdots i_p}_{i_q \cdots i_r}) \frac{\partial x'^{i_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{i_p}} = (D_{x'^{i_s}} T'^{'i_1 \cdots 'i_p}_{'i_q \cdots 'i_r}) \frac{\partial x'^{i_q}}{\partial x^{i_q}} \cdots \frac{\partial x'^{i_r}}{\partial x^{i_r}} \frac{\partial x'^{i_s}}{\partial x^{i_s}}, \tag{4.1}$$

provided

$$D_{x^{i_s}} T'^{'i_1 \cdots 'i_p}_{'i_q \cdots 'i_r} = D_{x'^{i_s}} T'^{'i_1 \cdots 'i_p}_{'i_q \cdots 'i_r} \frac{\partial x'^{i_s}}{\partial x^{i_s}}, \tag{4.2}$$

which is automatically satisfied because (3.6) with accented indices and (3.7) unite to give

$$L'^{j_\lambda}_{'h_\lambda i_s} = L'^{j_\lambda}_{'h_\lambda 'i_\lambda} \frac{\partial x'^{i_\lambda}}{\partial x'^{i_s}} \frac{\partial x'^{i_s}}{\partial x^{i_s}} = L'^{j_\lambda}_{'h_\lambda 'i_s} \frac{\partial x'^{i_s}}{\partial x^{i_s}};$$

and provided

$$D_{x^{i_s}} \left(\frac{\partial x'^{i_\lambda}}{\partial x^{i_\lambda}} \right) = 0 \qquad (\lambda = 1, 2, \cdots, r) \tag{4.3}$$

which using (3.3), (3.6) and (3.7) requires

$$\frac{\partial^2 x'^{i_\lambda}}{\partial x^{i_\lambda} \partial x^{j_\lambda}} + L'^{'i_\lambda}_{'h_\lambda 'k_\lambda} \frac{\partial x'^{h_\lambda}}{\partial x^{i_\lambda}} \frac{\partial x'^{k_\lambda}}{\partial x^{j_\lambda}} - L^{h_\lambda}_{i_\lambda j_\lambda} \frac{\partial x'^{i_\lambda}}{\partial x^{h_\lambda}} = 0. \tag{4.4}$$

Discarding the experimental operative symbols let us adopt the notation

$$\begin{aligned} T^{i_1 \cdots i_p}_{i_q \cdots i_r ; i_s} = \frac{\partial}{\partial x^{i_s}} T^{i_1 \cdots i_p}_{i_q \cdots i_r} &+ \sum_{\lambda=1}^{p} T^{i_1 \cdots h_\lambda \cdots i_p}_{i_q \cdots \cdots \cdots i_r} L^{i_\lambda}_{h_\lambda j_\lambda} \frac{\partial x^{j_\lambda}}{\partial x^{i_s}} \\ &- \sum_{\lambda=q}^{r} T^{i_1 \cdots \cdots \cdots i_p}_{i_q \cdots h_\lambda \cdots i_r} L^{h_\lambda}_{i_\lambda j_\lambda} \frac{\partial x^{j_\lambda}}{\partial x^{i_s}} \end{aligned} \tag{4.5}$$

not to be confused with other uses of the semi-colon such as *Eisenhart*: Non-Riemannian Geometry, p. 150. Then (4.1) reads

$$(4.6)\quad (T^{i_1\cdots i_p}_{i_q\cdots i_r;\,i_s})\frac{\partial x'^{i_1}}{\partial x^{i_1}}\cdots\frac{\partial x'^{i_p}}{\partial x^{i_p}} = (T'^{i_1\cdots i_p}_{i_q\cdots i_r;\,i_s})\frac{\partial x'^{i_q}}{\partial x^{i_q}}\cdots\frac{\partial x'^{i_r}}{\partial x^{i_r}}\frac{\partial x'^{i_s}}{\partial x^{i_s}}.$$

Since the introduction of L's subject to (3.6) and (4.4) does not seriously restrict the functions in (2.3), we have obtained a "differentiation" giving the desired "reproduction".

5. **Definitions.** If in all coördinate systems obtainable under non-singular analytic transformations (2.3) we have quantities of the type bracketed in (1.2) subject to relations (1.2), we say that these quantities are the components in the corresponding coördinate system of a *generalised tensor.* If we have a *generalised connection* L defined to have components in all coördinate systems subject to relations (3.6) and (4.4), we say that the bracketed quantities in (4.6) are the components in the corresponding coördinate system of a *generalised covariant derivative* of the generalised tensor just defined. The important observation is that a generalised covariant derivative is a generalised tensor with covariant order one higher than its "parent".

6. **Remarks.** The reader need hardly be reminded that ordinary covariant differentiation is a special case of the generalised, viz. when the ranges are all the same. Hence the above work is quite valid in the ordinary case.

For scalars, generalised covariant differentiation is just ordinary differentiation. Therefore

$$(6.1)\qquad \frac{\partial x'^{i_\lambda}}{\partial x^{i_\lambda}} = x'^{i_\lambda}_{;\,i_\lambda}.$$

Using (6.1), (4.4) reads, dropping the second semi-colon for short,

$$(6.2)\qquad 0 = x'^{i_\lambda}_{;\,i_\lambda;\,j_\lambda} = x'^{i_\lambda}_{;\,i_\lambda j_\lambda}.$$

By calculation it may be shown that, provided the conditions attached to (3.4) are reversible,

$$(6.3)\qquad \begin{aligned} &T^{i_1\cdots i_p}_{i_q\cdots i_r;\,i_s i_t} - T^{i_1\cdots i_p}_{i_q\cdots i_r;\,i_t i_s} \\ &\quad = -\sum_{\lambda=1}^{p} T^{i_1\cdots h_\lambda\cdots i_p}_{i_q\cdots i_r} L^{i_\lambda}_{h_\lambda j_\lambda k_\lambda}\frac{\partial x^{j_\lambda}}{\partial x^{i_s}}\frac{\partial x^{k_\lambda}}{\partial x^{i_t}} \\ &\qquad + \sum_{\lambda=q}^{r} T^{i_1\cdots i_p}_{i_q\cdots h_\lambda\cdots i_r} L^{h_\lambda}_{i_\lambda j_\lambda k_\lambda}\frac{\partial x^{j_\lambda}}{\partial x^{i_s}}\frac{\partial x^{k_\lambda}}{\partial x^{i_t}} \\ &\qquad - (T^{i_1\cdots i_p}_{i_q\cdots i_r;\,h_s} L^{h_s}_{i_s i_t} - T^{i_1\cdots i_p}_{i_q\cdots i_r;\,h_t} L^{h_t}_{i_t i_s}), \end{aligned}$$

where

$$(6.4)\quad L^{h\lambda}_{i_\lambda j_\lambda k_\lambda} = \frac{\partial}{\partial x^{j\lambda}} L^{h\lambda}_{i_\lambda k_\lambda} - \frac{\partial}{\partial x^{k\lambda}} L^{h\lambda}_{i_\lambda j_\lambda} + L^{g\lambda}_{i_\lambda k_\lambda} L^{h\lambda}_{g_\lambda j_\lambda} - L^{g\lambda}_{i_\lambda j_\lambda} L^{h\lambda}_{g_\lambda k_\lambda}.$$

If i_s and i_t have the same range the bracket in (6.3) becomes

$$2\, T^{i_1 \cdots i_p}_{i_q \cdots i_r; h_s}\, \Omega^{h_s}_{i_s i_t},$$

where

$$2\, \Omega^{h_s}_{i_s i_t} = L^{h_s}_{i_s i_t} - L^{h_s}_{i_t i_s}.$$

Since in obtaining (6.3) the second partial derivatives have been eliminated, (6.3) constitute conditions of integrability of (4.5), which are *generalised Ricci identities.*[5]

Applied to $\partial x'^{i\lambda}/\partial x^{i\lambda}$ (6.3) yield

$$0 = -\frac{\partial x'^{h\lambda}}{\partial x^{i\lambda}} L'^{i\lambda}_{h_\lambda j_\lambda k_\lambda} \frac{\partial x'^{j\lambda}}{\partial x^{j\lambda}} \frac{\partial x'^{k\lambda}}{\partial x^{k\lambda}} + \frac{\partial x'^{i\lambda}}{\partial x^{h\lambda}} L^{h\lambda}_{i_\lambda j_\lambda k_\lambda}$$

showing that (6.4) are components of a tensor — the *generalised curvature tensor.*

7. **Applications.** To take the concrete case which evoked this theory suppose there are just two ranges, viz. $1, 2, \cdots, n$ and $n+1, n+2, \cdots, n+m$ $(m \geqq n)$. For $x^{n+1}, x^{n+2}, \cdots, x^{n+m}$ write $y^1, y^2, \cdots, y^m$. Let the y's be functions of the x's. Geometrically speaking we are considering a space V_n immersed in a space V_m. The generalised tensor law of transformation now reads

$$(7.1)\quad \begin{aligned} &(T^{\cdots\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}) \cdots \frac{\partial y'^\alpha}{\partial y^\alpha} \cdots \frac{\partial x'^i}{\partial x^i} \cdots \\ &= (T'^{\cdots\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}) \cdots \frac{\partial y'^\beta}{\partial y^\beta} \cdots \frac{\partial x'^j}{\partial x^j} \cdots. \end{aligned}$$

(In this section Latin indices have the range $1, 2, \cdots, n$ and Greek indices the range $1, 2, \cdots, m$).

It is at once seen that the partial derivatives $\partial y^\alpha/\partial x^i$ are the components of a generalised tensor. Indeed they furnish a simple and yet highly fundamental example of such.

Let $a_{\alpha\beta}$ be the y components of the metric tensor in V_m and

$$(7.2)\quad g_{ij} = a_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

[5] Cf. Eisenhart, Non-Riemannian Geometry, p. 7.

the x components of the metric tensor in V_n. Using as "pure" (cf. § 3) L's the Christoffel symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$ and $\left\{ {i \atop jk} \right\}$ formed from $a_{\alpha\beta}$ and g_{ij} respectively, (4.5) becomes

$$
\begin{aligned}
T^{\cdots\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots;k} = {} & \frac{\partial}{\partial x^k} T^{\cdots\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots} + \sum_\alpha T^{\cdots\delta\cdots i\cdots}_{\cdots\beta\cdots j\cdots} \left\{ {\alpha \atop \delta\gamma} \right\} \frac{\partial y^\gamma}{\partial x^k} \\
& - \sum_\beta T^{\cdots\alpha\cdots i\cdots}_{\cdots\delta\cdots j\cdots} \left\{ {\delta \atop \beta\gamma} \right\} \frac{\partial y^\gamma}{\partial x^k} + \sum_i T^{\cdots\alpha\cdots h\cdots}_{\cdots\beta\cdots j\cdots} \left\{ {i \atop hk} \right\} \\
& - \sum_j T^{\cdots\alpha\cdots i\cdots}_{\cdots\beta\cdots h\cdots} \left\{ {h \atop jk} \right\}.
\end{aligned} \tag{7.3}
$$

In particular it is seen that

$$T^{\cdots\alpha\cdots}_{\cdots\beta\cdots;\gamma} = T^{\cdots\alpha\cdots}_{\cdots\beta\cdots,\gamma} \qquad \text{(no Latin indices),}$$

$$T^{\cdots\alpha\cdots}_{\cdots\beta\cdots;k} = T^{\cdots\alpha\cdots}_{\cdots\beta\cdots,\gamma} \frac{\partial y^\gamma}{\partial x^k} \qquad \text{(no Latin indices, except } k\text{),}$$

$$T^{\cdots i\cdots}_{\cdots j\cdots;k} = T^{\cdots i\cdots}_{\cdots j\cdots,k} \qquad \text{(no Greek indices),}$$

the commas indicating ordinary covariant differentiation. Hence

$$a_{\alpha\beta;\gamma} = 0, \qquad a_{\alpha\beta;k} = 0 \quad \text{and} \quad g_{ij;k} = 0.$$

Differentiating (7.2), which we now write

$$g_{ij} = a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;j}$$

as we well may, in the generalised covariant manner we obtain

$$0 = a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;jk} + a_{\alpha\beta}\, y^\alpha_{;ik}\, y^\beta_{;j}.$$

Subtracting from this the two corresponding relations obtained by cyclic permutation of i, j, k, we have, because of the symmetry of $y^\alpha_{;ij}$ in i and j, that

$$0 = a_{\alpha\beta}\, y^\alpha_{;ij}\, y^\beta_{;k}. \tag{7.4}$$

If

$$y^\alpha_{;ij} = 0,$$

i. e.

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} + \left\{ {\alpha \atop \beta\gamma} \right\} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} - \left\{ {k \atop ij} \right\} \frac{\partial y^\alpha}{\partial x^k} = 0,$$

the sub-space is totally geodesic.[6] Otherwise in general $y^\alpha_{;ij}$ can be con-

[6] Eisenhart, Riemannian Geometry, p. 185, Ex. 9.

sidered as the components of $n(n+1)/2$ contravariant vectors in V_m which, by (7.4), are normal to V_n.

Suppose V_n is a curve, $y^\alpha = y^\alpha(s)$, where s is the arc length (i. e. we take $n = 1$). Then $g_{ij} = 1$ and $\left\{ {i \atop jk} \right\} = 0$, while the Latin indices become unnecessary. Now (7.4) reads

(7.5) $$0 = a_{\alpha\beta}\, y^\alpha_{;(2)}\, y^\beta_{;}.$$

If

$$y^\alpha_{;(2)} = 0,$$

i. e.

$$\frac{d^2 y^\alpha}{ds^2} + \left\{ {\alpha \atop \beta\gamma} \right\} \frac{dy^\beta}{ds} \frac{dy^\gamma}{ds} = 0,$$

the curve is a geodesic. Otherwise $y^\alpha_{;(2)}$ are the components of a contravariant vector determining, as (7.5) shows, the direction of the principal normal to the curve. Under this notation a vector ζ^α defined along the curve is parallel to itself in V_m if

$$\zeta^\alpha_{;} = 0,$$

i. e.

$$\frac{d\zeta^\alpha}{ds} + \left\{ {\alpha \atop \beta\gamma} \right\} \zeta^\beta \frac{dy^\gamma}{ds} = 0.$$

And so we could go on developing the geometry of sub-spaces by the use of generalised covariant differentiation, thereby simplifying and clarifying the analytical work. W. Mayer[7] has initiated this, but unfortunately by the use of a notation rather off the beaten track.

As a rapid illustration of the power of the new differentiation as an analysing tool the following proof is offered. We shall use three ranges, viz. one for each of V_m, V_n and the curve (the latter being one-dimensional, requires no indices).

THEOREM.[8] *A vector ζ^α normal to a totally geodesic V_n remains normal when displaced parallel to itself in V_m along a curve in V_n.*

Set

(7.6) $$a_{\alpha\beta}\, \zeta^\alpha\, y^\beta_{;i} = \theta_i.$$

Since differentiating along the curve in the new manner

$$a_{\alpha\beta;} = a_{\alpha\beta;\gamma} \frac{dy^\gamma}{ds} = 0,$$

[7] Loc. cit.

[8] Cf. Cartan, E., Leçons sur la Géométrie des Espaces de Riemann (1928), p. 120.

$$\zeta^{\alpha}_{;} = 0 \qquad (\zeta^{\alpha} \text{ being self-parallel}),$$

$$y^{\beta}_{;i;} = y^{\beta}_{;ij} \frac{dx^j}{ds} = 0 \qquad (V_n \text{ being totally geodesic}),$$

we have by operating on (7.6) that

$$0 = \theta_{i;},$$

i. e.

$$0 = \frac{d\theta_i}{ds} - \left\{ {k \atop ij} \right\} \theta_k \frac{dx^j}{ds},$$

which means that $\theta_i = 0$ all along the curve if initially. Hence the theorem.

Grateful acknowledgment is due Professor L. P. Eisenhart for the encouragement and advice he has so freely given during the preparation of this paper, and also Dr. J. H. C. Whitehead for several helpful suggestions.

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NOTE ADDED IN PROOF. Already important strides have been taken in the study of subvarieties by the use of the type of generalised covariant differentiation exhibited in (7.3). Cf. J. A. Schouten and E. R. van Kampen, Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde, Math. Ann. 103 (1930) S. 752–783; Zur Krümmung einer V_m in V_n, to be published shortly (Professor Schouten kindly permitted me to read the manuscript), in which these authors develop the v. d. Waerden-Bortolotti symbol, as they call it (cf. S. 774 in the first paper just cited for a history of the idea), by ordinary covariant differentiation coupled with processes of projection. The use of the symbol is not confined to holonomic reference systems. This indicates the possibility of carrying through our above work with general mixed quantities replacing the partial derivatives in (1.2).

NOTE ON A SPECIAL PERSYMMETRIC DETERMINANT.[1]

By A. C. Aitken.

In a paper by H. T. Davis and V. V. Latshaw in these Annals of January 1930, on fitting polynomials to data by least squares, there is conjectured a general expression for the determinant of persymmetric type which has for its elements the sums of powers of the natural numbers. Actually, as Sir Thomas Muir's historical researches[2] show, the determinant had already been evaluated in 1903 by K. Petr, but it may be of interest to give a simple and rapid proof.

If $s_r = 1^r + 2^r + 3^r + \cdots + p^r$, the determinant in question is

$$P(s_0, s_1, \cdots, s_{2n}) = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}.$$

It is at once seen to be the square, in row-by-row multiplication, of the rectangular "alternant" array,

$$\begin{Vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & p \\ 1^2 & 2^2 & \cdots & p^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1^n & 2^n & \cdots & p^n \end{Vmatrix}.$$

Now an alternant whose elements in successive rows are polynomials, the same for each row but with different variables, and of degrees ascending from zero by unit steps from row to row, is readily seen to be unaltered in value if these polynomials be replaced by others of the same degree, provided only they have the same term of highest degree, e. g.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x-1 & y-1 & z-1 \\ x^2+x+1 & y^2+y+1 & z^2+z+1 \end{vmatrix}.$$

[1] Received May 2, 1930.

[2] Proc. Roy. Soc. Edin. XLVII (1926–27), p. 15.

Let us then replace the squared alternant array above by a product of two alternant arrays, the typical last column of the first array being taken to be $1, p-1, (p-1)(p-2), \cdots, (p-1)(p-2)\cdots(p-n)$, and that of the second array to be $1, p, p(p+1), \cdots, p(p+1)\cdots(p+n-1)$. Multiplying the new arrays in row-by-row fashion, and using familiar facts regarding sums of series whose terms are factorial polynomials, we find

$$P(s_0, s_1, \cdots, s_{2n})$$

$$= \begin{vmatrix} p & \frac{p(p+1)}{2} & \frac{p(p+1)(p+2)}{3} & \cdots & \frac{p(p+1)\cdots(p+n-1)}{n} \\ \frac{(p-1)p}{2} & \frac{(p-1)p(p+1)}{3} & \frac{(p-1)p(p+1)(p+2)}{4} & \cdots & \frac{(p-1)p\cdots(p+n-1)}{n+1} \\ \frac{(p-2)(p-1)p}{3} & \frac{(p-2)(p-1)p(p+1)}{4} & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(p-n+1)\cdots p}{n} & \frac{(p-n+1)\cdots p(p+1)}{n+1} & \cdots & \cdots & \frac{(p-n+1)(p-n+2)\cdots(p+n-1)}{2n-1} \end{vmatrix}$$

$$= p^n(p^2-1^2)^{n-1}(p^2-2^2)^{n-2}\cdots(p^2-\overline{n-1}^2)\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \cdots & \frac{1}{2n-1} \end{vmatrix}$$

$$= p^n(p^2-1^2)^{n-1}(p^2-2^2)^{n-2}\cdots$$

$$\cdots(p^2-\overline{n-1}^2)\,[1!\,2!\,3!\cdots(n-1)!]^4/[1!\,2!\cdots(2n-1)!],$$

since the persymmetric determinant whose elements are the reciprocals of integers is already known[3] as a case of Cauchy's "double alternant". This is Petr's result.

[3] Cf. Muir, History of Determinants, vol. III, p. 311, on Ligowski, 1861.

THE UNIVERSITY OF EDINBURGH.

LINEAR EQUATIONS IN NON-COMMUTATIVE FIELDS.*

To Professor JAMES PIERPONT on his 65-th Anniversary.

By Oystein Ore.

The problem of solving linear equations with coefficients in a non-commutative field (division-algebra) has recently been studied by A. R. Richardson,[1] Heyting[2] and Study.[3] The paper by Study is mainly confined to the case where the coefficients are contained in a quaternion field. In order to obtain the solutions of a system of simultaneous linear equations, certain expressions are introduced by Heyting and Richardson, which present numerous analogies to the determinants in the commutative case. Their usefulness for the solution of equations is however inconveniently limited by the fact, that they are not defined for all values of the coefficients and certain restrictions must be placed on the elements involved. This is particularly striking for the "designants" of Heyting which only exist if certain "principal minors" do not vanish. In his last paper A. R. Richardson obtains a general definition by means of recursion-formulas; there exists however in this formula a definite lack of symmetry depending on vanishing or non-vanishing of the coefficients, and this fact, it seems to me, makes the definition unsatisfactory.

Another type of expressions with various invariant properties has been investigated by MacDuffee.[4] These expressions do not seem to have any connection with the elimination in linear systems and are therefore outside the scope of this article.

I have proposed in this paper to determine the rings (algebras) in which elimination between linear systems can be performed. By adhering strictly to the elimination properties of determinants, a new definition of deter-

* Received December 8, 1930.

[1] A. R. Richardson, Hypercomplex determinants. Messenger of Math. 55 (1926), pp. 145–152. A. R. Richardson, Simultaneous linear equations over a division algebra. London Math. Soc. 28 (1928), pp. 395–420.

[2] A. Heyting, Die Theorie der linearen Gleichungen in einer Zahlenspezies mit nichtkommutativer Multiplikation. Math. Annalen 98 (1927), pp. 465–490.

[3] E. Study, Zur Theorie der linearen Gleichungen. Acta Mathematica 42 (1918), pp. 1–61. Certain older considerations on the same problem can be found by Caley, Philosophical Magazine 26 (1845), pp. 141–145, C. J. Joly in the second edition of Hamilton, Elements of quaternions.

[4] C. C. MacDuffee, Invariantive characterizations of linear algebras with the associative law not assumed. Transactions Am. Math. Soc. 23 (1922), pp. 135–150.

minants in a non-commutative field is introduced, which is not open to the criticisms made above. By special choices of the elements in this determinants it reduces to the expressions of Heyting or Richardson. By means of this definition a farreaching analogy to the commutative case is obtained. I shall only prove the most important properties of these determinants and give the principal results on the solution of equations. A series of further results can then be immediately implied from the commutative case. The rank of a system can be introduced and all the ordinary results on linear dependency can be derived.

If one had been primarily interested in the dependency of linear expressions, Toeplitz's[5] method for solving equations without determinants could have been generalized. Another method is given by Noether;[6] in this way one obtains existence theorems for solutions, but not a simple procedure to determine them.

In § 1 and § 2 I discuss the properties of rings in which the elimination can be performed; these rings must satisfy a certain axiom M_{V} and this is, as I show, equivalent to the fact, that the ring can be completed to a non-commutative field ("Quotientenkörper") by the introduction of formal quotients of elements in the ring. In the commutative case all domains of integrity (rings without divisors of zero) have a uniquely defined quotient-field, which is the least field containing the ring. For the non-commutative case v. d. Waerden[7] has recently indicated this problem as unsolved. The result mentioned above gives all rings for which a quotient-field can exist. For rings without quotient-field it might however, as I show by an example, be possible to construct by a different process a field that contains the given ring.

1. **The axioms.** We shall in the following consider a system S with *more than one element,* for which the following axioms are supposed to hold:

Equality. The equality of two elements a and b in S is defined by the following properties:

G_{I}. *Determination.* For two elements either $a = b$ or $a \neq b$.

G_{II}. *Reflexitivity.* $a = a$.

G_{III}. *Symmetry.* From $a = b$ follows $b = a$.

G_{IV}. *Transitivity.* From $a = b$, $b = c$ follows $a = c$.

Addition. For two arbitrary elements a and b a sum $a + b$ exists, having the properties:

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[5] O. Toeplitz, Über die Auflösung unendlich vieler linearen Gleichungen mit unendlich vielen Unbekannten. Rendiconti Palermo 28 (1909), pp. 88–96. Compare also H. Hasse, Algebra, vol. 1, Berlin 1926. O. Haupt, Einführung in die Algebra, vol. 1, Leipzig 1929.

[6] E. Noether, Hyperkomplexe Größen und Darstellungstheorie, Math. Zeitschr. 30 (1929), pp. 641–692.

[7] B. L. v. d. Waerden, Moderne Algebra, vol. 1, § 12. Berlin 1930.

A_{I}. *Uniqueness.* $a+b$ is a uniquely defined element of S.

A_{II}. *Equality.* From $a=b$ and $a_1=b_1$ follows $a+a_1=b+b_1$.

A_{III}. *Associative Law.* $a+(b+c)=(a+b)+c$.

A_{IV}. *Zero-element.* There exists an element 0 for which $0+a=a+0=a$.

A_{V}. *Commutative Law.* $a+b=b+a$.

A_{VI}. *Subtraction.* To every element a exists another $-a$ such that $a+(-a)=0$.

It is well known, that from these axioms follows, that 0 and $-a$ are uniquely defined.

Multiplication.

M_{I}. *Uniqueness.* $a\cdot b$ is a uniquely defined element in S.

M_{II}. *Equality.* From $a=b$ and $a_1=b_1$ follows $aa_1=bb_1$.

M_{III}. *Associative Law.* $a(bc)=(ab)c$.

M_{IV}. *Distributive Law.*

a) $(b+c)a=ba+ca$,

b) $a(b+c)=ab+ac$,

c) both a) and b).

Systems S satisfying these axioms are called (non-commutative) *rings* (or *algebras*). We shall in the following consider systems of linear equations with coefficients which are elements of such a ring. In order to perform an elimination to obtain a solution of a linear system, it seems necessary that the coefficients should satisfy the axioms mentioned. (Axiom M_{IV} possibly only in part.) The main operation for the usual elimination is however to multiply one equation by a factor and another equation by another factor to make the coefficients of one of the unknowns equal in the two equations. We must therefore also demand:

M_{V}. *Existence of common multiplum.* When $a\neq 0$, $b\neq 0$ are two arbitrary elements of S, then it is always possible to determine two other elements $m\neq 0$, $n\neq 0$ such that

$$an=bm. \tag{1}$$

For all rings satisfying these axioms the elimination-process can be carried out, and certain necessary conditions for the solvability can be established. In order to obtain necessary and sufficient conditions it is, as in the commutative rings, necessary to suppose that the ring does not contain any *divisors of zero,* i. e. from $ab=0$ follows $a=0$ or $b=0$ or both. This is equivalent to:

M_{VI}. *Converse equality axiom.* From $ab=ac$ or $ba=ca$, $a\neq 0$ follows $b=c$.

A ring satisfying the axioms M_{V} and M_{VI} shall be called a *regular* ring. It is not necessary that a regular ring contains a *unit-element,* i. e. that

M_{VII}. *Unit element.* There exists an element 1, for which $1 \cdot a = a \cdot 1 = a$ for an arbitrary element a in S.

is satisfied. If for a regular ring both M_{VII} and the axiom

M_{VIII}. *Division.* For every $a \neq 0$ exists an element a^{-1} such that $a \cdot a^{-1} = 1$.

are satisfied, then the domain is called a (non-commutative) *field* (or *division-algebra*).

Let us from now on suppose that the set S considered is a regular ring. From M_{V} follows by induction, that if $a_1 \neq 0 \cdots a_m \neq 0$, then such numbers $n_1 \neq 0, n_2 \neq 0 \cdots n_m \neq 0$ can be determined, that

$$a_1 n_1 = a_2 n_2 = \cdots = a_m n_m. \tag{2}$$

Furthermore if at the same time

$$an = bm, \qquad an_1 = bm_1, \qquad n_1 \neq 0, \qquad m_1 \neq 0, \tag{3}$$

then r and s can be so determined, that

$$nr = n_1 s, \tag{4}$$

from which one easily obtains

$$mr = m_1 s. \tag{5}$$

2. **Quotient fields.** We shall now prove the following theorem:

THEOREM 1. *All regular rings can be considered as subrings* (more exactly: *are isomorphic to a subring*) *of a non-commutative field.*

Let a and $b \neq 0$ be two arbitrary elements in S. We then introduce the symbol $\left(\frac{a}{b}\right) = (a \cdot b^{-1})$ and for these symbols (fractions) such rules of operation shall be defined, that their totality forms a field K.

Equality. Let $\left(\frac{a}{b}\right)$ and $\left(\frac{a_1}{b_1}\right)$ be two arbitrary fractions. According to M_{V} the elements $\beta \neq 0$, $\beta_1 \neq 0$ can be determined so that

$$b\beta_1 = b_1 \beta \tag{6}$$

and we say

$$\left(\frac{a}{b}\right) = \left(\frac{a_1}{b_1}\right) \tag{7}$$

when

$$a\beta_1 = a_1 \beta. \tag{8}$$

From the last remarks in § 1 it follows immediately that the equality of the two fractions (7) does not depend on the particular choice of the elements β and β_1 in (6). The axiom G_{I} is therefore satisfied. G_{II} and G_{III} are obviously satisfied. To derive G_{IV} let

$$\left(\frac{a}{b}\right) = \left(\frac{a_1}{b_1}\right), \qquad \left(\frac{a_1}{b_1}\right) = \left(\frac{a_2}{b_2}\right),$$

where

$$b\beta_1 = b_1\beta, \qquad a\beta_1 = a_1\beta; \tag{9}$$

$$b_1\beta_2 = b_2\beta_1', \qquad a_1\beta_2 = a_2\beta_1'. \tag{10}$$

We choose r and s such that

$$\beta r = \beta_2 s.$$

From (9) and (10) then it follows that

$$b(\beta_1 r) = b_2(\beta_1' s) \tag{11}$$

and from the second equations of (9) and (10) in the same way that

$$a(\beta_1 r) = a_2(\beta_1' s) \tag{12}$$

$\left(\frac{a_2}{b_2}\right)$

and (11) and (12) are equivalent to $\left(\frac{a}{b}\right) = \left(\frac{a_1}{b_1}\right)$. We note

$$\left(\frac{ac}{bc}\right) = \left(\frac{a}{b}\right) \tag{13}$$

for all $c \neq 0$.

Addition. We define

$$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a\beta_1 + a_1\beta}{b\beta_1}\right) = \left(\frac{a\beta_1 + a_1\beta}{b_1\beta}\right) \tag{14}$$

where β and β_1 satisfy (6). One easily sees that the sum (14) does not depend on the particular choice of β and β_1 and A_{I} is consequently satisfied. In order to prove A_{II}, we suppose

$$\left(\frac{a}{b}\right) = \left(\frac{a'}{b'}\right), \qquad \left(\frac{a_1}{b_1}\right) = \left(\frac{a_1'}{b_1'}\right)$$

i. e.:

$$b\beta' = b'\beta, \qquad a\beta' = a'\beta \tag{15}$$

and correspondingly for the second fraction. Then

$$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a\lambda + a_1\mu}{b_1\mu}\right), \qquad \left(\frac{a'}{b'}\right) + \left(\frac{a_1}{b_1'}\right) = \left(\frac{a'\lambda' + a_1\mu'}{b_1\mu'}\right), \tag{16}$$

where

$$b\lambda = b_1\mu, \qquad b'\lambda' = b_1'\mu'. \tag{17}$$

To compare the two fractions (16) we determine ϱ and σ by the condition

$$\mu\sigma = \mu'\varrho \tag{18}$$

and the two fractions are equal if

$$(a\lambda + a_1\mu)\sigma = (a'\lambda' + a_1\mu')\varrho$$

or according to (18)

$$a\lambda\sigma = a'\lambda'\varrho.$$

This is a simple consequence of (15), (17) and (18) and it is therefore

$$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a'}{b'}\right) + \left(\frac{a_1'}{b_1'}\right).$$

In the same way it follows that

$$\left(\frac{a'}{b'}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a'}{b'}\right) + \left(\frac{a_1'}{b_1'}\right)$$

and A_{II} must hold.

The associative law follows directly by assuming that the three fractions have the same denominator, which can always be obtained according to (2). A_{IV} follows from

$$\left(\frac{a}{b}\right) + \left(\frac{0}{c}\right) = \left(\frac{a}{b}\right), \qquad \left(\frac{0}{c}\right) = 0.$$

A_{V} is obvious, and A_{VI} is a consequence of

$$\left(\frac{a}{b}\right) + \left(\frac{-a}{b}\right) = 0.$$

Multiplication. We define

$$\left(\frac{a}{b}\right) \cdot \left(\frac{a_1}{b_1}\right) = \left(\frac{a\alpha_1}{b_1\beta}\right)$$

where

$$b\alpha_1 = a_1\beta, \; \beta \neq 0.$$

One easily proves that the product does not depend on the particular choice of α_1 and β, and M_{I} is fulfilled. M_{II} and M_{III} are derived by simple calculations. M_{IV} will hold to the extent which it holds in S. M_{V} follows easily and the unit-element is $\left(\frac{a}{a}\right) = 1$, which is independent of the choice of a according to (13). Finally M_{VIII} is satisfied since

$$\left(\frac{a}{b}\right) \cdot \left(\frac{b}{a}\right) = 1.$$

It is therefore proved, that the totality K of fractions $\left(\frac{a}{b}\right)$ forms a field; the regular ring S is, as one easily sees, isomorphic to the ring of all

elements in K of the form $\left(\frac{ac}{c}\right)$. As a corollary it follows, that every regular ring is a subring of a ring with unit elements.[8]

One can also easily prove the following theorem, which is, to a certain extent the converse of Theorem 1:

THEOREM II. *Let a and $b \neq 0$ run through all elements of a ring S without divisors of zero. If then the formal solutions of all equations*

$$xb = a \tag{19}$$

form a field, the ring S must be a regular ring.

If one supposes that all solutions $\left(\frac{a}{b}\right)$ of (19) form a field K, then as formerly the ring S must be isomorphic to the ring of elements of the form $\left(\frac{ac}{c}\right)$ in K. Furthermore in this field there must exist a b^{-1} and the equation

$$bx = a \tag{20}$$

must have a solution $x = \left(\frac{r}{s}\right)$ which is also contained in K. Then obviously

$$br = as$$

follows.

One can replace (19) by (20) in Theorem II or even in a slightly more general way by

$$axb = c.$$

A proper quotient-field can therefore only exist for regular rings. This result does however not exclude the possibility of rings, which are not regular, from still being subrings of fields; it might even be possible, as in the commutative case, for all rings without divisors of zero to be subrings in fields. A general construction of this kind seems to be difficult to define.

If for a ring S the axiom M_V is not satisfied, there must exist at least two elements A and B such that a relation

$$Aa + Bb = 0$$

[8] Every ring without divisors of zero is a subring of a ring with unit element; this follows immediately by considering the ring of elements $\left(\frac{ac}{c}\right)$ where $\left(\frac{ac}{c}\right) = \left(\frac{ad}{d}\right)$, $\left(\frac{ac}{c}\right)\left(\frac{bd}{d}\right) = \left(\frac{abc}{c}\right)$ etc. $\left(\frac{c}{c}\right)$ is then the unit element.

can only hold if $a = b = 0$. It would be natural to characterize such rings by the maximum number N of elements $A_1 \cdots A_N$ in the ring such that a relation

$$A_1 a_1 + \cdots + A_N a_N = 0$$

could only hold for $a_1 = \cdots = a_N = 0$, where the a_i are elements of S. The number N, which is finite or infinite, might suitably be called the *order of irregularity*. For an arbitrary element b in S the elements $b, A_1 \cdots A_N$ would be dependent with respect to S and a $c \neq 0$ could be found, such that

$$bc = A_1 a_1 + \cdots + A_N a_N.$$

Various interesting types of irregular rings exist, but I shall refrain from any further studies. Only one example will be given to show the existence of irregular rings and prove that even such rings can be contained in fields.

We consider all polynomials of the form

$$A(x) = a_1 x + \cdots + a_n x^n$$

where the coefficients are arbitrary complex numbers. Equality, addition and subtraction are defined the ordinary way. Multiplication is defined as composition

$$A(x) \times B(x) = A(B(x)).$$

In this way a non-commutative ring has been defined, in which M_{I}, M_{II}, M_{III} are satisfied; x is the unit-element. The distributive law holds for right-hand multiplication, but not for left-hand. From some interesting investigations by Ritt[9] on the composition of polynomials it follows easily that this ring is irregular. This ring is however obviously contained in the field of all algebraic functions that vanish for $x = 0$, when multiplication is defined as composition.

3. **Equations with 2 unknowns.** Let

(21)
$$\begin{aligned} x_1 a_{11} + x_2 a_{12} &= b_1, \\ x_1 a_{21} + x_2 a_{22} &= b_2 \end{aligned}$$

be two linear equations with coefficients in a regular ring S. We can then always find two elements A_{12} and A_{22} in S, such that

(22) $$a_{12} A_{22} = a_{22} A_{12}.$$

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[9] J. F. Ritt, Prime and composite polynomials. Trans. Am. Math. Soc. 23 (1922), pp. 51–66.

We make the assumption in the following, that by all solutions of equations of the form (22) $A_{22} \neq 0$, $A_{12} \neq 0$ when $a_{12} \neq 0$, $a_{22} \neq 0$. If $a_{12} = 0$, $a_{22} \neq 0$, then $A_{12} = 0$, $A_{22} \neq 0$, correspondingly $A_{12} \neq 0$, $A_{22} = 0$ if $a_{12} \neq 0$, $a_{22} = 0$; finally $A_{12} = A_{22} = 0$ if $a_{12} = a_{22} = 0$.

From (21) we then obtain

$$x_1 \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = \left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right\| \tag{23}$$

where

$$\Delta_{12}^{(12)} = \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = a_{11} A_{22} - a_{21} A_{12}, \quad \left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right\| = b_1 A_{22} - b_2 A_{12} \tag{24}$$

are called *right-hand determinants* of second order. By a different choice of the A_{12} and A_{22} in (22) we obtain different determinants, but by r. h. multiplication with elements in S they can all be obtained one from another. If we introduce the quotientfield K corresponding to S, all the different expressions (24) for the determinant can be obtained by multiplying one of them r. h. by an element $k \neq 0$ in K.

A determinant is therefore non-zero or zero. In most problems it is necessary only to decide, whether a determinant vanishes or not. We shall say that two determinants are *equivalent* (denoted by $\sim$) if they both vanish or do not vanish.

Obviously one obtains, by using the same values for A_{12} and A_{22} as in (22)

$$\Delta_{21}^{(12)} = \left|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right\| = a_{12} A_{21} - a_{22} A_{11} \sim \Delta_{12}^{(12)}$$

or by a different choice of A's

$$\Delta_{21}^{(12)} = -\Delta_{12}^{(12)} k, \quad k \neq 0, \quad \text{i. e. } \Delta_{21}^{(12)} \sim \Delta_{12}^{(12)}.$$

It can also be shown, that two columns can be interchanged

$$\Delta_{12}^{(21)} = \left|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right\| = a_{12} A_{21} - a_{22} A_{11} \sim \Delta_{12}^{(12)}. \tag{25}$$

It is sufficient to prove, that from $\Delta_{12}^{(12)} = 0$ follows $\Delta_{12}^{(21)} = 0$. Let us first suppose $a_{12} \neq 0$, $a_{11} \neq 0$, i. e. $A_{12} \neq 0$, $A_{11} \neq 0$. Then from (22) one obtains $a_{22} = a_{12} k$, and from $\Delta_{12}^{(12)} = 0$ in the same way $a_{21} = a_{11} k$. Substituting this in

$$a_{11} A_{21} = a_{21} A_{11}$$

it follows that $A_{21} = k A_{11}$ and (25) must also vanish. The same holds, as can easily be seen, when $a_{12} = 0$ or $a_{11} = 0$.

One therefore has the equivalences

$$\Delta_{12}^{(12)} \sim \Delta_{12}^{(21)} \sim \Delta_{21}^{(12)} \sim \Delta_{21}^{(21)}. \tag{26}$$

Through elementary calculations one also shows

$$\left|\begin{matrix} k a_{11}, & a_{12} \\ k a_{21}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & k a_{12} \\ a_{21}, & k a_{22} \end{matrix}\right\| \sim \Delta_{12}^{(12)}, \quad k \neq 0. \tag{27}$$

$$\left|\begin{matrix} a_{11} k, & a_{12} k \\ a_{21}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & a_{12} \\ a_{21} k, & a_{22} k \end{matrix}\right\| \sim \Delta_{12}^{(12)}, \quad k \neq 0. \tag{28}$$

$$\left|\begin{matrix} a_{11} + k a_{12}, & a_{12} \\ a_{21} + k a_{22}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & a_{12} + k a_{11} \\ a_{21}, & a_{22} + k a_{21} \end{matrix}\right\| \sim \Delta_{12}^{(12)}. \tag{29}$$

$$\left|\begin{matrix} a_{11} + a_{21} k, & a_{12} + a_{22} k \\ a_{21}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & a_{12} \\ a_{21} + a_{11} k, & a_{22} + a_{12} k \end{matrix}\right\| \sim \Delta_{12}^{(12)}. \tag{30}$$

From (21) one obtains analogously to (23)

$$x_2 \Delta_{12}^{(21)} = \left|\begin{matrix} b_1 & a_{11} \\ b_2 & a_{21} \end{matrix}\right\|. \tag{31}$$

Two determinants with the same second column are called *proportional*, if we agree, that in the calculation of the determinants the same set of A's shall be used in both cases. The determinants (23) are proportional; for two proportional determinants the quotient $\Delta' \Delta^{-1}$ always has the same value.

The relations (23) and (31) show that *the necessary and sufficient condition that the system* (21) *have a unique solution in the quotient-field K, is that the determinant $\Delta_{12}^{(12)}$ of* (21) *does not vanish. The solutions can then be found as quotients of proportional determinants.*

When the determinant vanishes, there must exist a linear relation

$$L_1(x_1, x_2) A_{22} - L_2(x_1, x_2) A_{12} = 0$$

where L_1 and L_2 denote the left-hand sides of the equations (21). In order that a solution then exist it is necessary that the same relation hold for b_1 and b_2 and the second determinant (24) therefore also vanishes. This can be expressed by the fact, that all second-order determinants in the matrix

$$\left|\begin{matrix} a_{11}, & a_{12}, & b_1 \\ a_{21}, & a_{22}, & b_2 \end{matrix}\right\|$$

vanish. In this case one of the equations (21) is a consequence of the other. The analogy to the commutative case is obvious.

Corresponding considerations give the same result for the solution of a left-hand system

$$(32)\qquad \begin{aligned} c_{11}\, x_1 + c_{12}\, x_2 &= d_1, \\ c_{21}\, x_1 + c_{22}\, x_2 &= d_2. \end{aligned}$$

The l. h. determinant is

$$\left\| \begin{matrix} c_{11}, & c_{12} \\ c_{21}, & c_{22} \end{matrix} \right| = C_{22}\, c_{11} - C_{12}\, c_{21},$$

where

$$C_{22}\, c_{12} = C_{12}\, c_{22}.$$

The notion of equivalence and proportionality are introduced in an analogous way, and corresponding conditions are obtained for the solvability of a system (32). The complete system of equivalences (27)–(30) also holds for l. h. determinants, when only l. h. and r. h. multiplication with k are interchanged. Finally one should observe the reciprocity between r. h. and l. h. determinants

$$(33)\qquad \left\| \begin{matrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{matrix} \right| \sim \left| \begin{matrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{matrix} \right\|$$

which can easily be derived.

4. **The general case.** After having studied the case $n = 2$ both Heyting and Richardson define the general determinant by induction-formulas. We shall also use an inductive definition, which is however selected on the principle of preserving as directly as possible the elimination properties of the determinant Δ.

Let

$$(34)\qquad \sum_{i=1}^{n} x_i\, a_{ji} = b_j \qquad (j = 1, 2, \cdots, n)$$

be a system of linear equations. We then define the right-hand determinant by

$$(35)\qquad |\, a_{ji} \,\| = a_{11}\, A_1^{(1)} + a_{21}\, A_1^{(2)} + \cdots + a_{n1}\, A_1^{(n)},$$

where the $A_1^{(j)}$ are a set of solutions of the homogeneous l. h. system

$$(36)\qquad \begin{aligned} a_{12}\, A_1^{(1)} + a_{22}\, A_1^{(2)} + \cdots + a_{n2}\, A_1^{(n)} &= 0, \\ \cdots\cdots\cdots\cdots & \\ a_{1n}\, A_1^{(1)} + a_{2n}\, A_1^{(2)} + \cdots + a_{nn}\, A_1^{(n)} &= 0. \end{aligned}$$

Let us now assume that the properties of l. h. and r. h. linear systems with $n-1$ unknowns have already been obtained as in the case $n = 2$.

By the choice of the solution

$$(37)\qquad A_1^{(j)} \qquad (j = 1, 2, \cdots, n)$$

of (36) we define $A_1^{(j)} = 0$ if all the l. h. determinants of order $n-1$ in the matrix

$$(38) \qquad \left\| \begin{matrix} a_{12}, & a_{22}, & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n}, & a_{2n}, & \cdots & a_{nn} \end{matrix} \right|$$

vanish. If one of the determinants in (38) is not equivalent to zero, the system (36) must have a fundamental solution (37) such that the most general solution can be obtained from it by r. h. multiplication of an arbitrary constant. If one supposes for example that the first determinant in (38) does not vanish, an arbitrary value $A_1^{(n)} \neq 0$ can be assigned to the last unknown in (36) and the others are then determined by equations of the type

$$(39) \qquad \| \Delta_j \| A_1^{(j)} = \| \Delta_j' \| A_1^{(n)}$$

where $\| \Delta_j \| \neq 0$ and $\| \Delta_j' \|$ are $(n-1)$-order proportional determinants from (38).

We multiply the equations (34) respectively by $A_1^{(1)}, A_1^{(2)} \cdots$ and add; considering (36) one obtains

$$(40) \qquad x_1 \cdot |a_{ij}\| = \left| \begin{matrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_n & a_{n2} & \cdots & a_{nn} \end{matrix} \right\| = b_1 A_1^{(1)} + \cdots + b_n A_1^{(n)}.$$

The right side determinant in (40) differs from $|a_{ij}\|$ only by the elements in the first column, and the same $A_1^{(j)}$ have been used in its calculation. Such determinants are called *proportional*. The equation (40) gives, when $|a_{ij}\| \neq 0$, a single value for x_1.

We shall now prove, chiefly by induction, a few of the properties of n-th order determinants. From the definition follows immediately:

An equivalent determinant is obtained when two rows are interchanged.

Such a change will only invert the order of a_{j1} and a_{i1} in (35), and then the corresponding $A_1^{(j)}$ and $A_1^{(i)}$ will also be interchanged.

A little more difficult to prove is:

When two columns are interchanged one obtains an equivalent determinant.

When the first column remains unchanged, the equations (36) will only change in order and their solutions are the same. Let us then suppose that the first column is interchanged with the second, and let us assume $|a_{ji}\| \neq 0$. We shall then prove that the new determinant

$$(41) \qquad \Delta' = a_{12} B_2^{(1)} + a_{22} B_2^{(2)} + \cdots + a_{n2} B_2^{(n)}$$

does not vanish. In (41) the $B_2^{(j)}$ must satisfy the same equations (36) as the $A_1^{(j)}$ except that the first equation must be replaced by

$$a_{11} B_2^{(1)} + a_{21} B_2^{(2)} + \cdots + a_{n1} B_2^{(n)} = 0. \tag{42}$$

If now $\Delta' = 0$, it follows from (41) that the $B_2^{(j)}$ would satisfy exactly the same $n-1$ equations as the $A_1^{(j)}$ and one must have $B_2^{(j)} = A_1^{(j)} k$. From (42) then follows $|a_{ji}\| = 0$ contrary to the supposition.

When all elements in a column in a r. h. determinant are multiplied l. h. with $k \neq 0$ one obtains an equivalent determinant.

Since the columns can be interchanged one can always suppose, that the first column is multiplied l. h. by k. From the definition (35) the theorem then follows immediately.

The following theorems are easily proved:

If one multiplies all elements of a row in a r. h. determinant r. h. by $k \neq 0$, one obtains an equivalent determinant.

When a row is added to another row or a column to a column, the resulting determinant is equivalent to the original.

Analogous theorems hold for l. h. determinants.

5. Let us now first consider the homogeneous system of equations

$$\sum_{i=1}^{n} x_i a_{ji} = 0 \tag{43}$$

where $|a_{ji}\| \neq 0$. As in (40) we then obtain relations

$$x_j \cdot |\Delta_j\| = 0 \tag{44}$$

where $|\Delta_j\|$ is obtained from $|a_{ji}\|$ by interchanging columns. All these determinants are therefore $\neq 0$ and we therefore have:

If the determinant of the system (43) *does not vanish, the only solution is $x_j = 0$ ($j = 1, 2, \cdots, n$).*

When the determinant vanishes, there exists a linear relation between the l. h. sides and one of the equations is redundant. If all $(n-1)$-order determinants in the corresponding matrix vanish, at least two of the equations are a consequence of the others etc. The notion of *rank* of a system can therefore be introduced, and all theorems on the solution of homogeneous systems can be derived as in the commutative case.

We shall now generalize the reciprocity relation (33) to determinants of order n; to prove

$$\| a_{ij} | \sim | a_{ji} \| \qquad (i, j = 1, 2, \cdots n) \tag{45}$$

it is sufficient to prove, that when one of these determinants does not vanish, the other must also be different from zero. Let us suppose

$|a_{ji}\| \neq 0$. Then according to definition not all determinants of order $n-1$ in (38) can vanish; by a rearrangement of rows and columns we can suppose

$$\| a_{ij} | \neq 0 \qquad (i, j = 2, 3, \cdots, n).$$

If we assume that (45) has been proved for all determinants of order $\leqq n-1$, we conclude

$$\| a_{ij} | \sim |a_{ji} \| \neq 0 \qquad (i, j = 2, 3, \cdots, n). \tag{46}$$

The determinant on the left in (45) has the expression

$$\| a_{ij} | = B_1^{(1)} a_{11} + B_2^{(1)} a_{12} + \cdots + B_n^{(1)} a_{1n} \tag{47}$$

where the $B_i^{(1)}$ must satisfy the equations

$$\sum_{i=1}^{n} B_i^{(1)} a_{ji} = 0 \qquad (j = 2, 3, \cdots, n). \tag{48}$$

From (46) follows, that at least one of the $B_i^{(1)}$ does not vanish. On the other hand, if (47) vanishes, the relations (48) must be satisfied even for $j = 1$, i. e. the $B_i^{(1)}$ must be a solution of (43) and since the determinant of this system does not vanish, we must have $B_i^{(1)} = 0$ $(i = 1, 2, \cdots, n)$ contrary to the supposition.

Let us now finally consider the complete system of equations (34) and suppose $|a_{ji}\| \neq 0$. Then one obtains analogously to (40)

$$x_i \cdot | \Delta_i \| = | \Delta_i' \| \tag{49}$$

where $|\Delta_i\| \neq 0$, since it is obtained from $|a_{ji}\|$ by interchanging columns. The relations (49) prove, that if the determinant of the system (34) is different from zero, there can be only a unique solution given by (49). In order to prove the existence of this solution, it is necessary to prove, that the x_i defined by (49) actually satisfy (34).

Let us put

$$\begin{aligned} L_j(x) &= x_1 a_{j1} + x_2 a_{j2} + \cdots + x_n a_{jn} - b_j, \\ M_j(x) &= x_j |\Delta_j\| - |\Delta_j'\|. \end{aligned} \qquad (j = 1, 2, \cdots, n).$$

From the way the relation (49) has been derived, it follows that

$$\sum_{j=1}^{n} L_j(x) A_i^{(j)} = M_i(x) \qquad (i = 1, 2, \cdots, n). \tag{50}$$

To show that the x_i defined by (49) satisfy (34), one must show that conversely the $L_j(x)$ can be homogeneously expressed by the $M_j(x)$. This is

according to (50) always possible, if the *complementary determinant* $|A_i^{(j)}||$ to $|a_{ji}||$ does not vanish. It is however simple to prove that

$$|\varDelta_i^{(j)}|| \sim ||A_j^{(i)}| \sim |a_{ji}||.$$

One observes easily that the complementary determinant to $|A_i^{(j)}||$ must be equivalent to $|a_{ji}||$ itself.

We have therefore proved:

A linear system (34) *has one and only one solution if its determinant does not vanish.*

Furthermore follows directly:

The necessary and sufficient condition, that a system (34) *have a solution is that any identical relation between the left-hand side linear forms must also be satisfied by the constants* b_i.

It is only necessary to indicate that a series of the properties of determinants in the commutative case can be extended by means of the principles used in this outline. Among the theorems of interest I shall only mention, that by a linear substitution one obtains also in the non-commutative case a *composed* determinant corresponding to the product of two determinants in the commutative case; it can be shown that this composed determinant is equivalent to the product of the two constituents.

It is perfectly obvious from the preceding results how matrices and rank of matrices can be introduced, and that the corresponding results from the commutative case can be derived in an analogous way. I shall therefore not go into the details of this theory.

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INVARIANTIVE ASPECTS OF A TRANSFORMATION ON THE BRIOSCHI QUINTIC.[1]

By Raymond Garver.

The transformation $y = (\mu w + \lambda)/(Z^{-1}w^2 - 3)$ as applied to the Brioschi normal quintic $w^5 - 10Zw^3 + 45Z^2w - Z^2 = 0$[2] is of some importance, inasmuch as it leads to a simple proof of the theorem that any sufficiently general quintic equation can be reduced to the Brioschi form. The proof consists of three steps, the first of which is the setting up of the transformed equation in y. This is found to be $y^5 + 5ay^2 + 5by + c = 0$, where a, b, c are defined by the following equations, in which V is written for $1728 - Z^{-1}$,

$$\begin{aligned} Va &= 8\lambda^3 + \lambda^2\mu + 72Z\lambda\mu^2 + Z\mu^3, \\ Vb &= -\lambda^4 + 18Z\lambda^2\mu^2 + Z\lambda\mu^3 + 27Z^2\mu^4, \\ Vc &= \lambda^5 - 10Z\lambda^3\mu^2 + 45Z^2\lambda\mu^4 + Z^2\mu^5. \end{aligned} \tag{1}$$

The second step is the identification of this transformed equation with a general principal quintic; that is, equations (1) must be solved for λ, μ, Z in terms of a, b, c. This solution is an interesting bit of algebraic manipulation; the only irrationality required is the square root of the discriminant of the general principal quintic. Details of the work are given by Dickson. Finally, the desired result follows, since any quintic can be reduced to principal form by a simple quadratic transformation, and since we know, on the basis of step two and a well known theorem on the reversibility of a rational transformation, that a transformation exists which will lead from any principal quintic, providing that it does not have a double root, to the Brioschi form.

The first of these steps is worthy of attention, since the transformed equation cannot be set up very easily by any of the methods usually employed. Perron seems to have been the first to obtain it in a direct, convenient manner.[3] Recently I gave a different presentation;[4] both it

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[1] Received August 28, 1930.

[2] The notation used here agrees with that of Dickson, Modern Algebraic Theories, 1926, pp. 242–247. In the Brioschi quintic, the constant term is not necessarily $-Z^2$, but such choice is convenient here. In fact, any Brioschi quintic with a non-zero constant term can be reduced easily to this one-parameter form.

[3] Algebra, vol. 2, 1927, pp. 209–216. Perron's transformation is not identical with the one used here.

[4] Bull. Am. Math. Soc., vol. 36 (1930), pp. 115–120. Here also will be found references to earlier articles on the same transformation, with brief comments.

and that of Perron require only a few simple, though perhaps novel, algebraic devices, in addition to procedure customarily used in transforming equations.

The present paper is concerned with this same matter, but treats it from a somewhat different standpoint. Two different methods for the determination of the transformed equation are developed, the first being based on a general theorem of Hermite[5] on Tschirnhaus transformations, the second on a similar theorem by Junker.[6] In the following statements, $f(x, y)$ means the binary quintic form $ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5$, and $f'(x, y)$ means the derivative of $f(x, y)$ with respect to x. The theorems are the following.

HERMITE'S THEOREM. *If $f(x, 1) = 0$ is transformed by*

$$z = \frac{t_1\varphi_1(x, 1) + t_2\varphi_2(x, 1) + t_3\varphi_3(x, 1) + t_4\varphi_4(x, 1)}{f'(x, 1)}, \tag{2}$$[7]

where $\varphi_i(x, y)$ is a cubic covariant of $f(x, y)$ of degree δ_i, and where t_i is a parameter, the transformed equation may be put in the form

$$A_0 z^5 + A_1 z^4 + A_2 z^3 + A_3 z^2 + A_4 z + A_5 = 0, \tag{3}$$

where A_0 is the discriminant of $f(x, y)$,[8] *A_1 is identically zero and the other A_i are homogeneous of the ith degree in t_1, t_2, t_3, t_4. Further, the coefficients of t_1^2, t_1t_2, t_1t_3, and so on, in A_2, of $t_1^3, t_1^2t_2$, and so on, in A_3, and the similar coefficients in the later terms, are all invariants of $f(x, y)$. Finally, the coefficient of t_it_j in A_2 is of degree $\delta_i + \delta_j + 6$, the coefficient of $t_it_jt_k$ in A_3 is of degree $\delta_i + \delta_j + \delta_k + 5$, and the coefficient of $t_it_jt_kt_l$ in A_4 is of degree $\delta_i + \delta_j + \delta_k + \delta_l + 4$.*[9]

JUNKER'S THEOREM. *If $f(x, 1) = 0$ is transformed by*

$$z = \frac{5g(x, 1)}{f'(x, 1)}, \tag{4}$$

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[5] When complete references are not given here they may be found easily from page 326 of my paper The Tschirnhaus transformation, published in these Annals, 2nd series, vol. 29 (1928), pp. 319–333.

[6] Die Verallgemeinerung der Hermiteschen Transformation in Zusammenhang mit der invariantentheoretischen Reduktion der Gleichungen, Köln, 1887, 32 pages. I have applied this theorem to a problem in connection with quartic equations, in a paper to be published in the Tôhoku Mathematical Journal.

[7] Henceforth z will be used to represent the new unknown, since y is commonly used with x in the notation for binary forms.

[8] This discriminant may be defined as $a^3 f'(x_1, 1) f'(x_2, 1) f'(x_3, 1) f'(x_4, 1) f'(x_5, 1)/5^5$, where the x_i are the roots of $f(x, 1) = 0$.

[9] The last two parts of this last sentence are due to Rahts, rather than to Hermite.

where $g(x, y)$ is a binary cubic form, the transformed equation may be put in form (3). *The first two coefficients are as in the above theorem, while the other A_i are simultaneous invariants of $f(x, y)$ and $g(x, y)$, of degree $8-i$ in the coefficients of f, and of degree i in the coefficients of g. In particular, as is easy to see, A_5 is the negative of the resultant of f and g.*

The application of these theorems will be made a little easier if we set $w = Z^{1/2}x$ in the Brioschi quintic $w^5 - 10Zw^3 + 45Z^2w - Z^2 = 0$. This gives

$$ax^5 - 10\,ax^3 + 45\,ax + f = 0, \tag{5}$$

where $a = 1$, $f = -Z^{-1/2}$. These substitutions, however, are not to be made at present. The transformation to be applied to (5) may be written

$$z = \frac{Ax + B}{a(x^2 - 3)}, \tag{6}$$

where $A = \mu Z^{1/2}$, $B = \lambda$, and z is written instead of the y of the first sentence of this article. We also note here that for the quintic form

$$ax^5 - 10\,ax^3y^2 + 45\,axy^4 + fy^5 \tag{7}$$

the invariants of degree indicated by the subscript reduce to

$$I_4 = a^2(f^2 - 1728\,a^2), \quad I_8 = 0, \quad I_{12} = a^8(f^2 - 1728\,a^2)^2,$$
$$I_{18} = a^{11}f(f^2 - 1728\,a^2)^3.\text{[10]}$$

All of these but the last can be computed easily from tables of invariants, which may be found in various places, while I_{18} may then be found by using the syzygy among the invariants, which takes a simple form when $I_8 = 0$. Also in this case the discriminant is known to equal I_4^2, and comparing this with the general value $a^8 \prod_{i=1}^{5} f'(x_i, 1)/5^5$ we obtain

$$\prod_{i=1}^{5} (x_i^2 - 3) = (f^2 - 1728\,a^2)/a^2,\text{[11]} \tag{8}$$

since $f'(x, 1) = 5\,a(x^2 - 3)^2$. Finally, we need to know the values taken by three of the covariants of a quintic form for (7). Indicating the degree by the first subscript and the order by the second, we find

[10] Numerical factors have been dropped in writing these values.

[11] That the sign has been chosen correctly in taking the square root can be seen by considering the case $f = 0$. For then $x_1^2 = 0$, $x_2^2 = x_3^2 = 5 + 2(-5)^{1/2}$, $x_4^2 = x_5^2 = 5 - 2(-5)^{1/2}$, and it is easy to see that the desired product must be negative, in fact, -1728. Formula (8) can also be verified easily in other ways.

$$
(9)\qquad
\begin{aligned}
C_{5,1} &= a^3(f^2-1728\,a^2)\,x,\\
C_{6,2} &= a^4(f^2-1728\,a^2)\,(x^2-3\,y^2),\\
C_{5,3} &= a^3(f^2-1728\,a^2)\,y(x^2-3\,y^2),
\end{aligned}
$$

where these are, except for sign, the covariants of corresponding degree and order given in Cayley's tables.

Let us now apply to (5) a transformation of type (2), with $t_1 = A$, $t_2 = B$, $t_3 = t_4 = 0$, $\varphi_1 = 5\,C_{5,1}\,C_{6,2}$, $\varphi_2 = 5\,C_{5,3}$, and therefore $\delta_1 = 11$, $\delta_2 = 5$. This transformation differs from (6) only in that $a^7(f^2-1728\,a^2)^2$, which we shall denote by k_1, and $a^3(f^2-1728\,a^2)$, which we shall denote by k_2, appear with the A and B, respectively, of the numerator. Hermite's theorem then tells us that the coefficients of A^2, AB, B^2, (in A_2), A^3, A^2B, AB^2, B^3, (in A_3), A^4, A^3B, A^2B^2, AB^3, B^4, (in A_4), are invariants of (7) of respective degrees 28, 22, 16, 38, 32, 26, 20, 48, 42, 36, 30, 24. This information is still too indefinite to determine the coefficients, but we may supplement it as follows. Keeping in mind that the transformation we are using is $z = (Ak_1x + Bk_2)(x^2-3)/a(x^2-3)^2$, and that $A_2/A_0 = \sum z_1 z_2$, $A_3/A_0 = -\sum z_1 z_2 z_3$, etc., we see that if A_0 is taken equal to the discriminant, or $a^4(f^2-1728\,a^2)^2$ then A_2 will equal

$$
(10)\qquad a^3 \sum (Ak_1x_1+Bk_2)\;(Ak_1x_2+Bk_2)\;a(x_3^2-3)\;a(x_4^2-3)\;a(x_5^2-3),
$$

multiplied by the product evaluated in (8). The summation is a symmetric function of degree not greater than two in any root of (5), and is therefore a quadratic polynomial in the elementary symmetric functions of (5). This apparently introduces a^2 in the denominator, but in the present case one a must cancel into the numerator, since the evaluation of (10) does not lead to a term f^2/a^2, since it obviously does not require the evaluation of $(x_1x_2x_3x_4x_5)^2$. Combining this last statement with the a^3 outside the summation in (10), which also enters in the later coefficients, the three a factors after the summation, which diminish in number in the later coefficients, and the product (8), which also enters in the later coefficients, we may say that A_2 involves $a^3(f^2-1728\,a^2)$. The expression for A_3 is similar to (10), the summation containing three factors linear in x, and two quadratic, each carrying an a. Hence A_3 will contain $a^2(f^2-1728\,a^2)$, and likewise A_4 will contain $a(f^2-1728\,a^2)$. Further, we see, from (10) and the expressions for A_3 and A_4, that the coefficient of A^iB^j, in A_2, A_3, or A_4, must contain $k_1^i k_2^j$.

Thus finally, the coefficient of A^2, in A_2, must contain $a^{17}(f^2-1728\,a^2)^5$, the coefficient of AB must contain $a^{13}(f^2-1728\,a^2)^4$, and so on; the reader may easily tabulate the other cases. Now using the values of the

invariants already listed, it requires only a small amount of calculation to conclude that the coefficients of A^2, AB, and so on, using the same order as the second sentence of the last paragraph, must be numerical multiples of $I_{12}^2 I_4$, $I_{18} I_4$, $I_{12} I_4$, $I_{18} I_{12} I_4^2$, $I_{12}^2 I_4^2$, $I_{18} I_4^2$, $I_{12} I_4^2$, $I_{12}^3 I_4^3$, $I_{18} I_{12} I_4^3$, $I_{12}^2 I_4^3$, $I_{18} I_4^3$, $I_{12} I_4^3$. The proper numerical factors must be determined by considering one or more special cases, suitably chosen.

However, the transformation we really wish to treat is (6), which does not have the k_1 and k_2 in the numerator. If (3) now represents the transformed equation resulting from (6) the A_i will be those of the last paragraph, except that the coefficient of $A^i B^j$ in A_2, A_3 or A_4 will no longer have $k_1^i k_2^j$ as a factor. Further, $(f^2 - 1728a^2)$ appears as a common factor, and if now put $a = 1$ we may conclude that, when $A_0 = f^2 - 1728$, the new A_2, A_3, A_4 must be of the forms $c_1 A^2 + c_2 f A B + c_3 B^2$, $c_4 f A^3 + c_5 A^2 B + c_6 f A B^2 + c_7 B^3$, $c_8 A^4 + c_9 f A^3 B + c_{10} A^2 B^2 + c_{11} f A B^3 + c_{12} B^4$, respectively, where the c_i are constants. To determine them, we take the special case $f = -24(3)^{1/2}$, which gives $3^{1/2}$ as a triple root of (5), the other two roots being those of the quadratic $x^2 + 3(3)^{1/2} x + 8 = 0$. For this case (6) is, to be sure, not valid, but we may apply the transformation $t = 1/z = (x^2 - 3)/(Ax + B)$, which leads to the transformed equation $A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_0 = 0$. Three of the values t takes on are zero, while for the other two the transformation reduces to $t = (-3(3)^{1/2} x - 11)/(Ax + B)$. It then requires only a minute's elementary algebra to find A_4/A_5 as the negative of the sum of the two non-vanishing t's, and A_3/A_5 as their product. The first of these quotients has the value

$$\frac{15(3)^{1/2} A - 5B}{8A^2 - 3(3)^{1/2} AB + B^2}, \tag{11}$$

the second, 40 divided by the denominator of (11). Also A_2/A_5 must vanish for all values of A and B, since there is no non-zero product of three t's; hence $c_1 = c_2 = c_3 = 0$.

To complete the determination, we need the value of A_5, which has not been mentioned until just before (11). Its value is obtainable directly from (6), since $-A_5/A_0 = \prod_{i=1}^{5} (Ax_i + B)/\prod_{i=1}^{5} (x_i^2 - 3)$. The denominator is found in (8). The numerator, by definition, is the resultant of the left side of (5)[12] and $Ax + B$. By a well-known theorem on resultants, this is equal to the negative of the resultant of the two polynomials taken in the opposite order, which gives at once the value

$$-B^5 + 10B^3A^2 - 45BA^4 + fA^5 \tag{12}$$

for A_5, if $A_0 = f^2 - 1728$.

[12] Taking $a = 1$, for convenience, but not making a substitution for f.

If we now put $f = -24(3)^{1/2}$ in (12) we have the correct A_5 for our special case. The denominator of (11) must be multiplied by

$$-[3(3)^{1/2}A^3 + 9A^2B + 3(3)^{1/2}AB^2 + B^3]$$

to give this value, hence the numerator of (11) must be multiplied by the same polynomial to give the special value of A_4. We find at once $c_8 = -135$, $c_9 = 5$, $c_{10} = -90$, $c_{11} = 0$, $c_{12} = 5$. In exactly the same way, $c_4 = 5$, $c_5 = -360$, $c_6 = 5$, $c_7 = -40$. This completes the determination of the transformed equation, and the coefficients are the same as those defined by (1) when A, B, and f are replaced by their values.

To use Junker's theorem, we apply (4) with $g(x, y) = (Ax + By)C_{6,2}$, which is transformation (6) with the right side multiplied by $a^4(f^2 - 1728a^2)$. If (3) denotes the transformed equation, with A_0 the discriminant, then A_2, A_3, A_4, are simultaneous invariants of $f(x, y)$ and $g(x, y)$ of degree 6, 5, 4 in a and f, and of degree 2, 3, 4 in the coefficients of $g(x, y)$. However, $g(x, y)$ involves A and B linearly, and a and f to the sixth degree; hence A_2, A_3, A_4 are of degree 18, 23, 28 in a and f, and of degree 2, 3, 4 in A and B. Further, and this explains our choice of $g(x, y)$, it is not difficult to show that, since $g(x, y)$ is the product of a linear form $Ax + By$ by a covariant of $f(x, y)$, A_2, A_3, A_4 must be covariants of the form $f(B, -A)$; the degree in A and B now becomes the order of the covariant. And by the same device used in the application of Hermite's theorem we may deduce easily that A_2 must contain as a factor $a^{11}(f^2 - 1728a^2)^3$, A_3 must contain $a^{14}(f^2 - 1728a^2)^4$, and A_4 must contain $a^{17}(f^2 - 1728a^2)^5$.

Knowing now that each coefficient is a covariant of certain order and degree, and knowing a great deal about its form, we might expect to obtain its value almost immediately. This seems impractical, however, since the covariants of a quintic form are, in part, very long, and since we should have to consider all possible ways of forming covariants of the desired order and degree, at least to see whether they contain the necessary factors

A satisfactory way of continuing is to employ the familiar theorem as to the weight of a covariant; A_2, A_3, A_4 must have the respective weights 46, 59, 72, where B is assigned the weight 1, and A the weight 0. Now any power of a multiplied by any power of $(f^2 - 1728a^2)$ is certainly of even weight, and we may conclude that A_2 must have the form

$$a^{11}(f^2 - 1728a^2)^3(c_3aB^2 + c_2fAB + c_1aA^2),$$

A_3 must have the form

$$a^{14}(f^2 - 1728a^2)^4(c_7aB^3 + c_6fAB^2 + c_5aA^2B + c_4fA^3),$$

and A_4 must have the form

$$a^{17}(f^2 - 1728a^2)^5(c_{12}aB^4 + c_{11}fAB^3 + c_{10}aA^2B^2 + c_9fA^3B + c_8aA^4).$$

The determination of the constants may be carried out just as in the application of Hermite's theorem, since, as in that case, we can now see that transformation (6) must lead to a transformed equation with its A_2 equal to $c_3 B^2 + c_2 f A B + c_1 A^2$, its A_3 equal to $c_7 B^3 + c_6 f A B^2 + c_5 A^2 B + c_4 f A^3$, and its A_4 equal to $c_{12} B^4 + c_{11} f A B^3 + c_{10} A^2 B^2 + c_9 f A^3 B + c_8 A^4$, provided $a = 1$, and A_0 is taken as $f^2 - 1728$. Junker's theorem refers specifically to the constant term of the transformed equation, and is essentially what was used in our previous determination of this coefficient.

It may be worth mentioning that the A_3 following from the transformation of type (4) can, after its determination in the last paragraph, be shown to equal the covariant $5 I_{12} I_4^2 C_{3,3}$, where $C_{3,3}$ is the cubic covariant of third degree of Cayley's tables, while A_4 is equal to $-5 I_{12} I_4^2/4$ multiplied by a quartic covariant of the eighth degree which can be obtained by adding the product of the quadratic covariants of degree 2 and 6 to the product of $C_{3,3}$ by the linear covariant of degree 5, the covariants being those of Cayley's tables.

ON THE IRREGULARITY OF CYCLIC MULTIPLE PLANES.[1]

By Oscar Zariski.

Introduction.

The investigation, to which this paper is devoted, has been instigated by the theorem, proved in sections 9 and 10, to the effect that if $f(x, y) = 0$ is a plane algebraic curve the fundamental group of which, with respect to its carrying complex plane, is cyclic, then the algebraic surface $z^n = f(x, y)$ is *regular* for any value of the positive integer n.[2] This suggests the problem of characterizing the plane algebraic curves $f = 0$, which give rise to *irregular* cyclic multiple planes $z^n = f(x, y)$ and which *eo ipso* can be considered as branch curves of *non-cyclic* multiple planes, since by the above theorem the fundamental group of such curves is not cyclic. This problem may be with reason restricted to the case of curves $f = 0$ possessing nodes and cusps only, because these curves offer a sufficient degree of generality for most questions of the theory of algebraic surfaces involving the consideration of the branch curve of a surface, either from the topological or from the algebro-geometric point of view. The problem, thus restricted, is treated in sections 1–5, regardless of whether the curve $f = 0$ is irreducible or not. In these sections algebro-geometric and transcendental methods are used exclusively. The result arrived at (section 5) allows us to evaluate the irregularity of the considered surfaces in terms of the superabundances of certain linear systems of curves determined by the cusps of f as basis points. This result is then applied in sections 6 and 7 to the case of an *irreducible* branch curve f, and leads to the complete classification of the corresponding irregular cyclic multiple planes. However, in the deduction a theorem proved in a previous paper of mine by means of purely topological considerations plays an essential rôle.

This introduction would not be complete, if we did not call attention to the applications, relative to curves with cusps and nodes, which are derived in the course of the investigation. Particulary noteworthy is the theorem of section 6, which affirms that certain linear systems of curves having

[1] Received February 5, 1931.

[2] We recall that the irregularity of an algebraic surface has a three-fold significance: (1) it is the difference $p_g - p_a$ between the geometric and the arithmetic genus of the surface; (2) it denotes the number of distinct exact differentials of the first kind attached to the surface; (3) if multiplied by 2 it gives the linear connection index of the surface. A surface is regular, if its irregularity is zero.

their basis points at the cusps of an irreducible curve are *regular*, thus excluding the cusps of such a curve from having certain special positions in the plane, although there is nothing to prevent the ordinary double points of the curve from having the same special positions. For instance, the mentioned theorem says in part, that the linear system of curves of order $m-4$, passing through the cusps of an irreducible curve of order m, is always regular, whenever $m>6$. Nothing of the sort holds for ordinary double points, as is shown by the examples of curves of any order m, for which the series g^2_m cut out by the lines is not complete.

As another application of the general results, we give for the first time several examples of Plücker numbers which do not correspond to any existent algebraic curve, thus solving in the negative the well-known question of the existence of plane algebraic curves with given Plückerian characters (see section 6).

The last sections (9, 10) deal with the fundamental group of the branch curves of irregular cyclic multiple planes. The result shows definitely, on the basis of this quite general class of curves, that there exists a connection between the structure of the fundamental group of a curve and the position of the cusps of the curve.

Throughout the paper the surface $z^n = f(x, y)$ is considered, and the order of f is denoted by m. The cases $n=m$, $n>m$, $n<m$ are treated separately.

I. First case: $m = n$.

1. We first consider the case $m = n$, and we proceed to evaluate the irregularity q of the surface F given by the equation

$$z^n = f(x, y),$$

where $f(x, y) = 0$ is a curve of order n possessing ordinary double points and cusps only. In order to evaluate q as the difference between the effective and the virtual dimensions of the complete linear system $|\varphi_{n-4}|$ of adjoint surfaces of F of order $n-4$, we must first determine the behaviour of the adjoint surfaces φ_{n-4} at the singular points of F. The surface F has only isolated singularities, namely, double points at the double points of the branch curve $f = 0$. Let 0 be a double point of the curve f, which we may suppose to coincide with the origin of coördinates of the (x, y)-plane. We must express the condition that the double integral

$$I = \int\!\!\int \frac{\varphi(x, y, z)}{z^{n-1}}\, dx\, dy \tag{1}$$

has a finite value for every analytical 2-cell containing the origin $x = y = 0$.

Let 0 be an ordinary double point of f. Since the required condition is of a differential character, it is permissible to replace $f(x, y)$ by the product xy. The integral (1) becomes

$$I = \int\int \frac{\varphi(x, y, z)}{x^{n-1/n} y^{n-1/n}} dx\, dy,$$

or putting $x = t^n$, $y = \tau^n$,

$$I = n^2 \int\int \varphi(t^n, \tau^n, t\tau)\, dt\, d\tau.$$

This integral is certainly finite in the neighborhood of the origin $t = \tau = 0$, φ being a polynomial in x, y, z. We thus conclude that *the singular points of the surface F arising from the ordinary double points of the curve f impose no conditions on the adjoint surfaces of F.*

Let 0 be a cusp of f. As above, it is now permissible to replace $f(x, y)$ by the expression $y^2 - x^3$. We have then to consider the integral (1) subject to the hypothesis that $z^n = y^2 - x^3$.

We introduce two new independent variables u and v, as follows:

$$y - x^{3/2} = u^n, \qquad y + x^{3/2} = v^n; \tag{2}$$

$$2^{2/3} \cdot x = (v^n - u^n)^{2/3}, \qquad 2y = u^n + v^n, \qquad z = uv. \tag{2'}$$

The transformed integral is, to within a constant factor, the following:

$$\int\int \varphi(x, y, z)/\sqrt{x} \cdot du\, dv,$$

and it is necessary to express the condition that this integral is finite in the neighborhood of $u = 0$, $v = 0$. Now this integral is the sum of integrals of the type

$$\int\int x^{k-1/2} y^l z^m\, du\, dv, \tag{3}$$

where k, l, m are non negative integers. If $k > 0$, the integral (3) is obviously finite, since, by (2′), the integrand is then zero at $u = v = 0$. If $k = 0$, but $l > 0$, the integral (3) is still finite. In fact, it is sufficient to prove that the integral

$$\int\int x^{-1/2}\, y\, du\, dv$$

is finite. We have

$$\int\int x^{-1/2}\, y\, du\, dv = \frac{1}{2}\int\int x^{-1/2} (v^n - u^n)\, du\, dv + \int\int x^{-1/2} u^n\, du\, dv.$$

The first integral is of the form $c\int\int x\, du\, dv$, where c is a constant, and therefore is finite. In the second integral we set $v = ut$. The integral is transformed, to within a constant factor, into the following integral:

$$\iint (t^n - 1)^{-1/3} u^{2n+3/3} \, du \, dt.$$

It is obvious that this new integral is finite at $u = 0$, t arbitrary, not excepting the values of t for which $t^n - 1 = 0$. For the direction $t = v/u = \infty$ we arrive at the same conclusion by using the transformation $u = vt$.

There remains to consider the integral

$$\iint x^{-1/2} \varphi(0, 0, z) \, du \, dv,$$

which is the sum of integrals of the type

$$\iint (v^n - u^n)^{-1/3} u^m v^m \, du \, dv, \quad m \geqq 0.$$

We put again $v = ut$, and we obtain the following integral

$$\iint (t^n - 1)^{-1/3} u^{2m+1-n/3} t^m \, dt \, du.$$

In order that this last integral be finite at $u = 0$, t arbitrary and finite, it is necessary and sufficient that $2m + 1 - n/3 > -1$, or $m > n/6 - 1$, or finally $m \geqq [n/6]$, where $[n/6]$ denotes the maximum integer which is not greater than $n/6$. We thus conclude: *the adjoint surfaces $\varphi(x, y, z) = 0$ of the surface F must satisfy at each cusp $x = x_0$, $y = y_0$ the condition that the polynomial $\varphi(x_0, y_0, z)$ possess a zero of order $[n/6]$ at $z = 0$.*

In geometric terms the last condition can be stated as follows: *The system of adjoint surfaces of F of a given order is characterized by the requirement that it possess at each cusp $(x_0\, y_0)$ of the curve f $[n/6]$ infinitely near basis points in the direction of the line $x = x_0$, $y = y_0$.*

2. It follows from the preceding section that the postulation formula of the system $|\varphi_\nu|$ of the adjoint surfaces of F of a given order ν is the following:

$$r'_\nu = \binom{\nu + 3}{3} - 1 - k[n/6], \tag{4}$$

where k is the number of cusps of the branch curve $f = 0$, and r' is the virtual dimension of the system. In particular, for $\nu = n - 4$, we obtain the value of the arithmetic genus p_a of F:

$$p_a = \binom{n-1}{3} - k[n/6].$$

The effective dimension of the system $|\varphi_{n-4}|$ is given by the formula

$$r_{n-4} = p_g - 1 = \binom{n-1}{3} - 1 - k[n/6] + q,$$

where q is the irregularity of F.

We observe that the system $|\varphi_\nu|$ of adjoint surfaces of order ν of F cuts out in the plane (x, y) the *complete* system $|C_\nu|$ of curves of order ν passing through the cusps of the curve f. In fact, given any curve C_ν of the system, the cone which projects C_ν from the point at infinity on the z-axis is obviously an adjoint surface φ_ν of F. The adjoint surfaces φ_ν which are constrained to contain the (x, y) plane break up into this plane and into a residual surface $\varphi^{(1)}_{\nu-1}$ of order $\nu-1$ having at each cusp (x_0, y_0) of f, $[n/6]-1$ (instead of $[n/6]$) coincident intersections with the line $x = x_0$, $y = y_0$. Hence, if we denote by r_ν and by $r^{(1)}_{\nu-1}$ the effective dimensions of the systems $|\varphi_\nu|$ and $|\varphi^{(1)}_{\nu-1}|$ respectively, the dimension ϱ_ν of the complete system $|C_\nu|$ is given by the formula:

$$\varrho_\nu = r_\nu - r^{(1)}_{\nu-1} - 1. \tag{5}$$

Denoting by δ_ν and by $\delta^{(1)}_{\nu-1}$ the differences between the effective and virtual dimensions of the systems $|\varphi_\nu|$ and $|\varphi^{(1)}_{\nu-1}|$ respectively, we have

$$\left\{\begin{aligned} r_\nu &= \binom{\nu+3}{3} - 1 - k[n/6] + \delta_\nu, \\ r_{\nu-1} &= \binom{\nu+2}{3} - 1 - k\{[n/6]-1\} + \delta^{(1)}_{\nu-1}. \end{aligned}\right. \tag{6}$$

Substituting into (5) we obtain:

$$\varrho_\nu = \binom{\nu+2}{2} - 1 - k + \delta_\nu - \delta^{(1)}_{\nu-1}. \tag{7}$$

Denoting the superabundance of the system $|C_\nu|$ by s_ν, we deduce from (7) the following relation:

$$\delta_\nu = s_\nu + \delta^{(1)}_{\nu-1}. \tag{8}$$

We now may apply to the system $|\varphi^{(1)}_{\nu-1}|$ the same considerations as those which we have applied to the system $|\varphi_\nu|$. As before, the system $|\varphi^{(1)}_{\nu-1}|$ cuts out in the (x, y) plane the complete system $|C_{\nu-1}|$ determined by basis points at the cusps of f, provided $[n/6]-1>0$. The surfaces $\varphi^{(1)}_{\nu-1}$, which are constrained to contain the (x, y) plane, break up into this plane and into a residual surface $\varphi^{(2)}_{\nu-2}$ of order $\nu-2$, having at each cusp (x_0, y_0) of the curve f, $[n/6]-2 \geqq 0$ coincident intersections with the line $x = x_0$, $y = y_0$. Denoting by $\delta^{(2)}_{\nu-2}$ the difference between the effective and the virtual dimensions of the system $|\varphi^{(2)}_{\nu-2}|$ and by $s_{\nu-1}$ the superabundance of the system $|C_{\nu-1}|$, we shall have

$$\delta^{(1)}_{\nu-1} = s_{\nu-1} + \delta^{(2)}_{\nu-2},$$

and, by (8),

$$\delta_\nu = s_\nu + s_{\nu-1} + \delta^{(2)}_{\nu-2}. \tag{8'}$$

Repeating the same reasoning over again, we shall finally arrive at the relation:

$$\delta_\nu = s_\nu + s_{\nu-1} + \cdots + s_{\nu+1-[n/6]} + \delta^{([n/6])}_{\nu-[n/6]}, \tag{9}$$

where the last term denotes the difference between the two dimensions of the system of the surfaces of order $\nu - [n/6]$, possessing no basis points at all, and hence vanishes when $\nu - [n/6] \geqq -3$. The formula (9) holds also if some of the systems $|C_\nu|, |C_{\nu-1}|, \cdots, |C_{\nu+1-[n/6]}|$ cease to exist, provided the effective dimension of each missing system is put equal to -1, while the virtual dimension is always evaluated according to the postulation formula. The same rule should be applied to the evaluation of the last term of (9), whenever $\nu - [n/6]$ becomes negative.

Let us now apply the formula (9) to the case $\nu = n - 4$. In this case $\delta_\nu = \delta_{n-4} = q$, and $\delta^{([n/6])}_{n-4-[n/6]} = 0$, since $n - 4 - [n/6] \geqq -3$, if $n > 0$. Hence

$$q = s_{n-4} + s_{n-5} + \cdots + s_{n-3-[n/6]}, \tag{10}$$

i. e., the irregularity of the surface F is given by the sum of the superabundances of the systems $|C_{n-4}|, |C_{n-5}|, \cdots, |C_{n-3-[n/6]}|$, where $|C_\nu|$ is the system of the curves of order ν in the (x, y)-plane having the cusps of f as basis points.

An application. I have proved in a previous paper,[3] that if the curve $f(x, y) = 0$ is irreducible and if n is a power of a prime number, then the surface $z^n = f(x, y)$ is regular. In view of the theorem proved above, we can affirm that *if $f(x, y) = 0$ is an irreducible algebraic curve possessing ordinary double points and k cusps, and if the order of f is the power of a prime number, then the k cusps of f impose k independent conditions on the curves of any order $\nu \geqq n - 3 - [n/6]$ constrained to contain them.* This is a particular case of a more general theorem, which will be proved in section 6.

II. Second case: $n > m$.

3. If $n > m$, then the line at infinity in the (x, y) plane is an $(n-m)$-fold line of the surface F. Moreover the $n - m$ tangent planes at a generic point of that line coincide with the plane at infinity. Hence in the first place the adjoint surfaces of F, apart from satisfying the conditions at the cusps derived in the preceding section, must pass through that line with the multiplicity $n - m - 1$. But this additional condition is not sufficient to characterize the adjoint surfaces. In general it will happen that the adjoint surfaces will have to pass with certain multiplicities through other

[3] "On the linear connection index of the algebraic surfaces $z^n = f(x, y)$", Proceedings of the National Academy of Sciences, vol. 15, no. 6, June 1929.

base lines lying in the plane at infinity and infinitely near to the line at infinity in the (x, y) plane, according to a scheme which can be derived from the nature of the singularity at infinity of a generic plane section of F. At any rate the behaviour of the adjoint surfaces φ of F at the line at infinity in the (x, y) plane is completely determined—except perhaps at a finite number of isolated singularities on that line—by the condition of cutting out on a generic plane of the space adjoint curves of the section of F by the same plane. This condition is expressed by the vanishing of several coefficients in the equation of an adjoint surface. We proceed to write down explicitly these relations.

As far as the isolated singularities of the multiple line of F are concerned, they are the intersections of that line with the branch curve f. It will be shown below that if—*as we will suppose from now on*—the branch curve f is in a generic position with respect to the line at infinity, namely if it intersects that line in m distinct points, then these points do not impose additional conditions on the adjoint surfaces of F.

It will considerably simplify matters if we interchange the roles of the line at infinity and of the x-axis in the (x, y) plane. This can be done by simply interchanging the corresponding homogeneous coördinates. With the same notation $f(x, y) = 0$ for the transformed equation of the branch curve, the transformed equation of the surface F is the following:

$$z^n = y^{n-m} f(x, y). \tag{11}$$

Let $\varphi(x, y, z) = 0$ be the equation of an adjoint surface of F. We must express the condition that *the curve* $\varphi(\bar{x}, y, z) = 0$, where $\bar{x}$ is constant, is an adjoint curve of the curve

$$z^n = y^{n-m} f(\bar{x}, y),$$

or, what is the same, that the abelian integral

$$\int \frac{\varphi(\bar{x}, y, z)}{z^{n-1}}\, dy$$

is finite at $y = z = 0$.

Let $c_{\alpha\beta\gamma}\, x^\alpha y^\beta z^\gamma$ be the general term of the polynomial $\varphi(x, y, z)$. For a generic value of $\bar{x}$ we shall have $f(\bar{x}, 0) \neq 0$, and hence the first term of the expansion of z in terms of y will contain y to the power $(n-m)/n$. In order that the integral

$$\int \frac{\bar{x}^\alpha y^\beta z^\gamma}{z^{n-1}}\, dy$$

be finite, it is therefore necessary and sufficient that $\beta + \gamma(n-m)/n > (n-1)(n-m)/n - 1$, or

$$(12) \qquad n\beta + (n-m)\gamma > (n-m-1)(n-1) - 1.$$

From this it follows in the first place that $\beta + \gamma \geqq n - m - 1$, which merely says that the x-axis is an $(n-m-1)$-fold line of the adjoint surface φ. Let

$$(13) \qquad \beta + \gamma = n - m - 1 + \delta, \qquad \delta \geqq 0.$$

Replacing γ by $n - m - 1 + \delta - \beta$, we deduce from (12) that

$$(14) \qquad \beta > n - m - (\delta + 1)(n-m)/m - 1.$$

All the coefficients $c_{\alpha\beta\gamma}$, *in which* β *does not satisfy* (14), *must vanish.* Hence, if for a given value (13) of the sum $\beta + \gamma$ we denote by ϱ_δ the minimum value of β, then ϱ_δ is given by the expression

$$\varrho_\delta = n - m - [(\delta + 1)(n-m)/m],$$

where $[a]$ denotes the integral part of a, provided the right-hand member is $\geqq 0$. In the contrary case $\varrho_\delta = 0$. Let

$$(15) \qquad n - m = hm - m_1$$

where h and m_1 are integers, and $0 \leqq m_1 < m$. Then

$$(15') \qquad \varrho_\delta = n - m - (\delta + 1)h + [(\delta + 1)m_1/m].$$

It will be noted that $h > 0$, since $n > m$.

It is now easily shown that the above conditions are sufficient in order that the double integral

$$\iint \varphi(x, y, z)/z^{n-1}\, dx\, dy$$

be finite at *every* point of the x-axis, including the points in which the x-axis meets the curve $f(x, y) = 0$. In fact, let us suppose that $x = 0$ is such a point, so that $f(0, 0) = 0$. By hypothesis, the tangent to the curve f at the origin is distinct from the x-axis. Hence we may suppose that the tangent coincides with the axis $x = 0$. Dealing with a question of an essentially differential character, it is permissible to replace the polynomial $f(x, y)$ by x. We may then consider the integral

$$\iint \frac{\varphi(x, y, z)\, dx\, dy}{x^{1/n}\, y^{n-m/n}}.$$

If conditions (12) are satisfied, $\varphi/y^{n-m/n}$ contains y in the denominator to a power less than 1. It is obvious then that, if we put $x = t^n$, $y = \tau^n$, the transformed integral will be of the form

$$\iint \bar{\varphi}(t, \tau)\, dt\, d\tau,$$

where $\bar{\varphi}(t, \tau)$ is regular at $t = \tau = 0$. The original integral is therefore finite at $x = y = 0$.

The adjoint surfaces of F are thus completely characterized by the condition (12), or (14), and by the additional conditions relative to the cusps of the curve f, which require that at a cusp (x_0, y_0) of f the adjoint surfaces of F should have $[n/6]$ coincident intersections with the line $x = x_0,\ y = y_0$.

4. Having thus established the conditions for the adjoint surfaces of F, let us now consider the complete system $|\varphi_\nu|$ of the adjoint surfaces of F, of a given order ν. Together with the system $|\varphi_\nu|$ we shall have to consider in the sequel the system of all surfaces of order $\nu - d$ (d a positive integer), which together with the plane $z = 0$ counted d times constitute adjoint surfaces φ_ν. This system, necessarily complete, will be denoted by $|\varphi_\nu - d\pi|$, where π denotes the plane $z = 0$. This definition of the system $|\varphi_\nu - d\pi|$ completely determines its basis. In particular, if $d < [n/6]$, then the surfaces $\varphi_\nu - d\pi$ pass through the cusps of the curve f and have at each cusp (x_0, y_0) $[n/6] - d$ coincident intersections with the line $x = x_0,\ y = y_0$. Let r'_d denote the virtual dimension of the system $|\varphi_\nu - d\pi|$. For $d = 0$, we will have

$$r'_0 = \binom{\nu + 3}{3} - 1 - k_1 \nu + k'_1 - k[n/6], \tag{16}$$

where k is the number of the cusps of f, and k_1, k'_1 are certain constants, independent of ν.[4] As far as r'_d is concerned, $d > 0$, we observe that when the order of the surfaces $\varphi_\nu - d\pi$ is sufficiently high, r'_d and r'_{d+1} coincide with the effective dimensions of the systems $|\varphi_\nu - d\pi|$, $|\varphi_\nu - (d+1)\pi|$, and that then $r'_d - r'_{d+1} - 1$ gives the dimension of the system of curves

[4] It is easily seen that $k_1 = \frac{1}{2}[(n - m - 1)(n - 1) + \mu - 1]$, where μ is the h. c. d. of n and m. In fact, denoting by C a generic plane section of F, and by A the (multiple) point of C on the x-axis, k_1 is the number of conditions imposed by the multiple point A of C on the adjoint curves of C of a sufficiently high order. This number k_1 is equal to the diminution of the genus of C caused by the singularity at the point A. We have now only to observe that the function z defined by an equation of the type $z^n = y^{n-m}\psi(y)$ has at $y = 0$ a critical point, in the neighborhood of which the n branches of z are permuted in μ cycles of order n/μ (and hence this critical point absorbs $n - \mu$ simple branch points of the function z), and that on the other hand this critical point must be considered as the limit of $n - m$ critical points of order n, which were brought into coincidence.

The same value for k_1 is found by observing that $k_1 = \frac{1}{2}(n - m - 1)(n - m) + \varrho_0 + \varrho_1 + \cdots + \varrho_{m-2}$, where ϱ_δ is defined by formula (15′).

cut out by the system $|\varphi_\nu - d\pi|$ on the plane π, outside the line $y=0$. Let us denote this system by $|C^{(d)}|$. If σ_d denotes the multiplicity, with which the generic surface of the system $|\varphi_\nu - d\pi|$ passes through the line $y=0$, we will have that

$$\text{order of } c^{(d)} = \nu - d - \sigma_d.$$

The k cusps of the curve f are, or are not, basis points of the curves $C^{(d)}$, according as $d < [n/6]$ or $d \geqq [n/6]$. Now it is easily shown *that in both cases the system* $|C^{(d)}|$ *is complete.* In fact, let $C^{(d)}$ be any curve of the corresponding complete system, i. e. any curve of order $\nu - d - \sigma_d$, if $d \geqq [n/6]$, or any curve of order $\nu - d - \sigma_d$ passing through the k cusps, if $d < [n/6]$. It will be sufficient to show that there exists a surface $\varphi_\nu - d\pi$, which degenerates into the cone which projects the curve $C^{(d)}$ from the point at infinity on the z-axis and into the plane $y=0$, counted σ_d times. In order to show this, let us recall the conditions for an adjoint surface φ_ν, expressed by formula (14). This formula assigns a lower limit for the exponent of y in a term $C_{\alpha\beta\gamma}\, x^\alpha y^\beta z^\gamma$ of the equation of φ_ν. This lower limit depends on the sum $\beta + \gamma$, and as (14) shows, *does not increase as* $\beta+\gamma$ *increases.*

If now in the equation $\varphi(x, y, z) = 0$ of the surface φ_ν every term contains z^d as a factor, so that $\varphi = z^d \varphi_1$, where $\varphi_1 = 0$ is the equation of a surface $\varphi_\nu - d\pi$, the conditions on the coefficients of the polynomial φ_1 will be of the same nature as those on the coefficients of φ. Since the terms of lowest degree in φ_1 are of degree σ_d, these conditions will be certainly satisfied if all terms of φ_1 contain y^{σ_d} as a factor. Hence any surface of order $\nu - d$ is a $\varphi_\nu - d\pi$, if it degenerates into the plane $y=0$ counted σ_d times, provided of course that the residual component of the surface satisfies the conditions at the cusps of the branch curve f. Since the conditions at the cusps are certainly satisfied, if this residual component is a cone, which projects the curve $C^{(d)}$ in the plane (x, y) from the point at infinity of the z-axis, our assertion is proved.

It is important to notice that this proof always holds, whenever the system $|C^{(d)}|$ exists. We have then the following result, which will be of use in the sequel: *every system* $|\varphi_\nu - d\pi|$ *cuts out on the plane* $\pi (z=0)$ *the complete system of curves* $|C^{(d)}|$ *of order* $\nu - d - \sigma_d$, *defined above, provided the curves* $C^{(d)}$ *exist.*

The dimension of the system $|C^{(d)}|$, supposed regular, is $\binom{\nu-d-\sigma_d+2}{2}-k-1$, if $d<[n/6]$, and $\binom{\nu - d + \sigma_d + 2}{2} - 1$, if $d \geqq [n/6]$. We thus deduce the following relations:

(17a) $$r'_d - r'_{d+1} - 1 = \binom{\nu - d - \sigma_d + 2}{2} - 1 - k, \quad \text{if} \quad d < [n/6],$$

and

(17b) $$r'_d - r'_{d+1} - 1 = \binom{\nu - d - \sigma_d + 2}{2} - 1, \quad \text{if} \quad d \geqq [n/6].$$

Let now r_d and r_{d+1} denote the effective dimensions of the systems $|\varphi_\nu - d\pi|$, $|\varphi_\nu - (d+1)\pi|$ respectively, and let s_d denote the superabundance of the system $|C^{(d)}|$. Supposing that $d < [n/6]$, we will have

(18) $$r_d - r_{d+1} - 1 = \binom{\nu - d - \sigma_d + 2}{2} - 1 - k + s_d,$$

since by the previous theorem the system $|C^{(d)}|$ is complete. From (17a) and (18) we deduce

(19) $$r_d - r'_d = r_{d+1} - r'_{d+1} + s_d.$$

Adding the relations (19) for $d = 0, 1, 2, \cdots, [n/6] - 1$, we arrive at the following formula:

(20) $$r_0 - r'_0 = s_0 + s_1 + s_2 + \cdots + s_{[n/6]-1} + (r_{[n/6]} - r'_{[n/6]}).$$

It should be noticed that the formula (18) and hence also (20) still hold if some of the linear systems whose characters occur in these formulas cease to exist, provided the actual dimension of a non-existent system is taken to be -1. For instance, if both systems $|\varphi_\nu - d\pi|$, $|\varphi_\nu - (d+1)\pi|$ exist, but the system $|C^{(d)}|$ does not exist, the superabundance of $|C^{(d)}|$ will be, by definition, a number such that the right-hand member of (18) is equal to -1, and therefore (18) becomes $r_d = r_{d+1}$, which is in agreement with the fact that in this case all the surfaces of the system $|\varphi_\nu - d\pi|$ must contain the plane π as a component. Similarly, if the system $|\varphi_\nu - (d+1)\pi|$ does not exist, i. e. if $r_{d+1} = -1$, then the formula (18) merely says that the dimension of the system $|\varphi_\nu - d\pi|$ is equal to the dimension of the system $|C^{(d)}|$, which is obviously true.

5. In order that formula (20) could be utilized it is necessary to evaluate the orders of the curves of the systems $|C^{(0)}|$, $|C^{(1)}|, \cdots, |C^{([n/6]-1)}|$, and also to evaluate the difference $r_{[n/6]} - r'_{[n/6]}$. We must also bear in mind that for each of the above systems of curves $|C^{(d)}|$ in the (x, y) plane, the basis is composed of the cusps of the curve f, and that the surfaces of the system $|\varphi_\nu - [n/6]\pi|$ are not constrained to pass through the cusps of f and are defined solely by their behaviour along the line $y = z = 0$.

We consider the system of curves $|C^{(d)}|$ cut out on the plane π $(z = 0)$ by the surfaces of the system $|\varphi_\nu - d\pi|$ and we proceed to determine the order of the curves $C^{(d)}$. We consider two cases: (a) $d \leqq n - 2$; (b) $d > n - 2$.

(a) $d \leqq n-2$. We determine a positive integer β such that

$$(21) \quad (h+1)\beta - \left[\frac{\beta m_1}{m}\right] - 2 < d \leqq (h+1)(\beta+1) - \left[\frac{(\beta+1)m_1}{m}\right] - 2,$$

where h and m_1 are integers defined in section 3(15). We also introduce the following expression, defined for every non-negative integer δ_i:

$$(22) \qquad \varrho(\delta) = n - m - (\delta+1)h + \left[\frac{(\delta+1)m_1}{m}\right].$$

In view of formula (15′) section 3, $\varrho(\delta) = \varrho_\delta$, if $\varrho(\delta) \geqq 0$, and $\varrho_\delta = 0$, if $\varrho(\delta) < 0$. Since $h \geqq 1$, $m_1 < m$, the function $\varrho(\delta)$ decreases as δ increases. Since obviously

$$\varrho(m-1) = 0,$$

it follows that for any $\delta \leqq m-1$, $\varrho(\delta) \geqq 0$ and consequently $\varrho(\delta) = \varrho_\delta$. If $\delta > m-1$, then $\varrho(\delta) < 0$ and $\varrho_\delta = 0$.

The inequalities (21) can now be written as follows:

$$(23\text{a}) \qquad d + \varrho(\beta-1) > n - m - 1 + (\beta-1);$$

$$(23\text{b}) \qquad d + \varrho(\beta) \leqq n - m - 1 + \beta.$$

Since, by hypothesis, $d \leqq n-2$, it follows from (23a) that

$$\beta - 1 - \varrho(\beta-1) < m-1.$$

Since $\delta - \varrho(\delta)$ is an increasing function of δ, and since its value for $\delta = m-1$ is $m-1$, the last inequality shows that $\beta \leqq m-1$, and that consequently $\varrho(\beta) = \varrho_\beta$. We will have *a fortiori*

$$(23\text{c}) \qquad d + \varrho_{\alpha-1} > n - m - 1 + (\alpha-1),$$

for any $\alpha \leqq \beta$. Now, recalling that $\varrho_{\alpha-1}$ is the lowest power to which y may occur in any term $cx^a y^b z^c$ of the polynomial $\varphi_\nu(x, y, z)$ for which $b+c = n-m-1+(\alpha-1)$, we deduce from (23c) that if φ_ν contains z^d as a factor it cannot contain terms in which $b+c$ is less than $n-m-1+\beta$. On the other hand (23b) shows that φ_ν will contain terms of the lowest degree $n-m-1+\beta$ in y and z, and that consequently φ_ν/z^d will contain terms of lowest degree $n-m-1+\beta-d$ in y and z. It follows that the surfaces of the system $|\varphi_\nu - d\pi|$ pass through the line $y = z = 0$ with the multiplicity $\sigma_d = n-m-1+\beta-d$, and that consequently

$$(24) \qquad \text{order of } C^{(d)} = \nu - d - \sigma_d = \nu - n + m + 1 - \beta.$$

(b) $d > n-2$. It is easily seen that if $d = n-2$, the surfaces $\varphi_\nu - d\pi$ do not pass through the line $y = z = 0$ and are, as a matter of fact, the most general surfaces of their order, since for the above value of d these surfaces are not constrained any more to pass through the cusps of the curve f ($n-2 \geqq [n/6]$, if $n > 1$). In fact, if the polynomial φ_ν contains z^{n-2} as a factor, then, for any term of φ_ν we will have $b+c = n-m-1+\delta$, where necessarily $\delta \geqq m-1$. Since ϱ_δ is zero for $\delta \geqq m-1$, it follows that the polynomial φ_ν/z^d can be taken arbitrarily.

It follows that, for $d \geqq n-2$,

$$\text{order of } C^{(d)} = \nu - d.$$

The preceding considerations show that the systems

$$|C^{(0)}|,\quad |C^{(1)}|,\quad |C^{(2)}|,\quad \cdots,\quad |C^{(n-2)}| \tag{25}$$

distribute themselves into sets of consecutive systems all of order $\nu-n+m+1-\beta$, where β runs from 0 to $m-1$. The system $|C^{(0)}|$ has the maximum order $\nu-n+m+1$, the last system $|C^{(n-2)}|$ has the minimum order $\nu-n+2$. How many consecutive systems of a given order $\nu-n+m+1-\beta$ are there in the above set of systems? Obviously as many as there are non negative integers d satisfying the inequalities (21) for a given value of β. This number is evidently equal to $h+1$, if $\left[\frac{\beta m_1}{m}\right] = \left[\frac{(\beta+1)m_1}{m}\right]$ and $\beta \neq 0$, and is equal to h, if $\left[\frac{\beta m_1}{m}\right] = \left[\frac{(\beta+1)m_1}{m}\right] - 1$, or if $\beta = 0$. If we define a symbol ε_β, $0 \leqq \beta \leqq m-1$, as follows:

$$\begin{aligned} \varepsilon_\beta &= 1, \text{ if } \left[\frac{\beta m_1}{m}\right] = \left[\frac{(\beta+1)m_1}{m}\right] \text{ and } \beta \neq 0; \\ \varepsilon_\beta &= 0, \text{ if } \left[\frac{\beta m_1}{m}\right] = \left[\frac{(\beta+1)m_1}{m}\right] - 1, \text{ or if } \beta = 0, \end{aligned} \tag{26}$$

we can say that the first $h+\varepsilon_0 = h$ systems of the set (25) are all of order $\nu-n+m+1$, the following $h+\varepsilon_1$ systems are all of order $\nu-n+m$, etc., and that in general the number of systems in the set, of order $\nu-n+m+1-\beta$, is $h+\varepsilon_\beta$. Since $\varepsilon_{m-1} = 0$, the last h systems of the set are of order $\nu-n+2$.

We now come back to the formula (20). Since $[n/6]-1$ is never greater than $n-2$, it follows that the $[n/6]$ systems $|C^{(d)}|$, the superabundances of which occur in the formula (20), are members of the set (25). We change slightly our notation in the sense that we will denote the super-

abundance of the system $|C^{(d)}|$ not by s_d but by s_j, where the subscript j is the order of the curves $C^{(d)}$. *We will prove in a while that if* $\nu \geqq n-4$, *then* $r_{[n/6]} - r'_{[n/6]} = 0$.

In view of this, and with the above change of notation, we can reassert our result in the following statement:

The difference $r_0 - r'_0$ *between the effective and the virtual dimensions of the system* $|\varphi_\nu|$ *of the adjoint surfaces of* F *of a given order* $\nu \geqq n-4$, *is equal to the sum of the first* $[n/6]$ *terms of the series:*

$$(27) \quad h\, s_{\nu-n+m+1} + (h+\varepsilon_1)\, s_{\nu-n+m} + (h+\varepsilon_2)\, s_{\nu-n+m-1} + \cdots + h\, s_{\nu-n+2},$$

where each term $(h+\varepsilon_\beta)\, s_{\nu-n+m+1-\beta}$ *counts for* $h+\varepsilon_\beta$ *terms of the series.*

We now have to prove the above assertion: $r_{[n/6]} - r'_{[n/6]} = 0$. We consider the *subadjoint surfaces* $\bar{\varphi}_\nu$ of F of order ν. The surfaces $\bar{\varphi}_\nu$ are defined by the condition that they behave as the adjoint surfaces φ_ν along the multiple line $y = z = 0$ of F; they are however not constrained to pass through the isolated singularities of F, i. e. through the double points and the cusps of f. Since likewise the surfaces of the system $|\varphi_\nu - [n/6]\pi|$ do not pass through the isolated singularities of F, it follows that this system coincides with the system $|\bar{\varphi}_\nu - [n/6]\pi|$. Now the results of section 4, relative to the adjoint surfaces φ_ν and to the surfaces $\varphi_\nu - d\pi$, hold *a fortiori* for the subadjoint surfaces $\bar{\varphi}_\nu$ and for the surfaces $\bar{\varphi}_\nu - d\pi$. Thus, if $|\bar{C}^{(d)}|$ denotes the system of curves cut out on the plane π by the system $|\bar{\varphi}_\nu - d\pi|$, $|\bar{C}^{(d)}|$ is a complete system. Also, if we denote by $\bar{r}_d$ and $\bar{r}'_d$ the effective and the virtual dimension respectively of $|\bar{\varphi}_\nu - d\pi|$, and by $\bar{s}_d$ the superabundance of $|\bar{C}^{(d)}|$, we will have a formula analogous to (19):

$$\bar{r}_d - \bar{r}'_d = (\bar{r}_{d+1} - \bar{r}'_{d+1}) + \bar{s}_d.$$

But now the system $|\bar{C}^{(d)}|$ is in effect the system of all curves of order $\nu - d - \sigma_d$ (the same as the order of the curve $C^{(d)}$), and hence $\bar{s}_d = 0$, if $\nu - d - \sigma_d \geqq -2$. It follows that for any d, such that the order of the curves $\bar{C}^{(d)}$ (or $C^{(d)}$) is $\geqq -2$,

$$(28) \qquad \bar{r}_0 - \bar{r}'_0 = \bar{r}_{d+1} - \bar{r}'_{d+1}.$$

We saw above that for any $d \leqq [n/6] - 1$, the system $|C^{(d)}|$ is a member of the set (25), hence the order of $C^{(d)}$ is $\geqq \nu - n + 2$. If then $\nu \geqq n-4$, the formula (28) will certainly hold for $d = [n/6] - 1$. Hence, recalling that the systems $|\varphi_\nu - [n/6]\pi|$, $|\bar{\varphi}_\nu - [n/6]\pi|$ coincide, we deduce that if $\nu \geqq n-4$,

$$\bar{r}_0 - \bar{r}'_0 = r_{[n/6]} - r'_{[n/6]}.$$

To show that $\bar{r}_0 = \bar{r}_0'$ it is sufficient, by a well-known theorem of Castelnuovo on linear systems of surfaces,[5] to show that the systems $|\bar{\varphi}_{\nu+1}|$, $|\bar{\varphi}_{\nu+2}|$, etc. cut out on a generic plane of the space *complete and regular* linear systems of curves. Now, if $\nu \geqq n-4$, the above systems cut out on a generic plane ω systems of adjoint curves of order $\geqq n-3$ of the corresponding plane section of F, and these systems, as it is well known, are regular. The completeness of these systems is obvious, since given in the plane ω an arbitrary adjoint curve $C_{\nu'}$ of the plane section (F, ω), a subadjoint surface $\bar{\varphi}_{\nu'}$ of F, which cuts out on ω the curve $C_{\nu'}$, is immediately obtained—for instance, by projecting the curve $C_{\nu'}$ from the point at infinity on the x-axis.

Our assertion: $r_{[n/6]} - r'_{[n/6]} = 0$, if $\nu \geqq n-4$, is thus proved.

The previous result, which gives the value of the difference between the effective and the virtual dimensions of the adjoint surfaces φ_ν of F, leads for $\nu = n-4$ to the following theorem:

The irregularity of the surface F is equal to the sum of the first $[n/6]$ *terms of the sequence*

$$h s_{m-3}, \quad (h+\varepsilon_1) s_{m-4}, \quad (h+\varepsilon_2) s_{m-5}, \cdots, \quad (h+\varepsilon_\beta) s_{m-3-\beta}, \cdots,$$

where *each product $(h+\varepsilon_\beta) s_{m-3-\beta}$ is the symbol of $h+\varepsilon_\beta$ consecutive terms all equal to $s_{m-3-\beta}$, and where $s_{m-3-\beta}$ is the superabundance of the system of curves of order $m-3-\beta$ passing through the cusps of the branch curve f; h and ε_β are the numbers defined in* (15) *and* (26).

III. The conditions for the irregularity of F in the case of an irreducible branch curve f.

6. In this section we will discuss the case in which the branch curve $f(x, y) = 0$ is irreducible. The result of the previous section, combined with the theorem of my Proceedings Note, quoted in section 2, will enable us to characterize completely the irreducible curves f possessing only nodes and cusps, which give rise to irregular cyclic multiple planes F.

If the curve f is irreducible, it is well known that $s_{m-3} = 0$, so that the first h terms in the sequence (29) vanish. Let s_{m-3-j} be the first superabundance in the sequence (29) which is not zero: $s_{m-3-j} > 0$, but for any $\beta < j$, $s_{m-3-\beta} = 0$. Then the surface F is regular or irregular, according as the number of the terms of the sequence (29) which precede

[5] G. Castelnuovo, Alcune proprietà fondamentali dei sistemi lineari di curve tracciate sopra una superficie algebrica, Annali di Matematica pura ed applicata, series II, vol. XXV, pp. 14–17 (1897).

$(h+\epsilon_j)\, s_{m-3-j}$ is not or is less than $[n/6]$. Denote the difference between these two members by λ. Then it follows from the definition of the numbers ϵ_β (26) that λ can be written in the following form

$$\lambda = j(h+1) - [j m_1/m] - 1 - [n/6], \tag{30}$$

and, as we have just observed, $q = 0$, if $\lambda \geqq 0$, and $q > 0$, if $\lambda < 0$. Let

$$n = 6\varrho + \sigma \qquad (0 \leqq \sigma < 6); \tag{31a}$$

$$j m_1 = h_1 m + m_2 \qquad (0 \leqq m_2 < m). \tag{31b}$$

If we multiply both members of (30) by 6 and recall that, by (15), $n = (h+1)\,m - m_1$, we deduce the following expression for 6λ:

$$6\lambda = (h+1)\,(6j - m) - 6h_1 + m_1 + \sigma - 6. \tag{32}$$

As n increases indefinitely, the curve f being given, h increases indefinitely, while all the other elements in the expression (32) of 6λ are fixed, except σ, which by (31a) is < 6. It follows that for large values of n the sign of 6λ will be the same as the sign of the difference $6j - m$. Let us suppose that $m > 6j$. Then, as n is sufficiently large, 6λ will be always negative, and the surface $z^n = f(x, y)$ will be always irregular. This contradicts the theorem, quoted in section 2, to the effect that if n is a power of a prime number, the surface $z^n = f(x, y)$, (f irreducible) is regular. *It follows that* $m \leqq 6j$.

The geometric significance of this inequality can hardly be missed. It says in fact that, *if $f(x, y) = 0$ is an irreducible curve of order m, possessing nodes and cusps only, and if β denotes the maximum integer such that $6\beta < m$, then the complete systems of curves of orders $m-4$, $m-5$, $\cdots$ $\cdots$, $m-3-\beta$, passing through the cusps of f, are regular.* This theorem excludes in the first place the possibility of an irreducible curve f possessing such a large number of cusps that the virtual dimension of any of the above mentioned systems becomes less than -1. In the second place, it precludes the cusps of an irreducible curve from having such special positions in the plane that they do not impose independent conditions on the curves of orders listed above.

Hence in the first place the number k of cusps of an irreducible curve of order m must satisfy the inequality

$$k < \frac{(m-3-\beta)(m-\beta)}{2} + 2, \tag{33}$$

because otherwise the virtual dimension of the system $|C_{m-3-\beta}|$ would have been -2. It is easily seen that when m is sufficiently large, the upper

limit for k given by (33) is much greater than the upper limit for k which can be derived arithmetically from the condition that the Plückerian characters of the curve f should not be negative. Hence for large values of m the inequality (33) is trivial. But for certain small values of m, this inequality allows us to assign sets of 6 non-negative integers, which satisfy the Plücker relations, and yet which are not the Plückerian characters of any effectively existent curve, thus resolving in the negative the known question of the existence of curves with given Plückerian characters.[6] For instance, there do not exist curves of order $m = 7$ with $k = 11$ cusps and no nodes $(d = 0)$,[7] although the remaining Plückerian characters are not negative: m' (class) $= 9$, i (number of flexes) $= 17$, τ (number of double tangents) $= 7$. This is an immediate consequence of the above general result. However, we give here in full the proof for this particular example, in order to illustrate the general considerations of the previous section. Suppose then that a curve $f(x, y) = 0$ of order 7, possessing 11 cusps and no nodes, exists. We consider the surface $z^7 = f(x, y)$. By my theorem, quoted above, this surface is regular. Now let us evaluate the irregularity of this surface as the difference between the geometric and the arithmetic genus of the surface. The adjoint surfaces of order 3 are cubic surfaces passing through the 11 cusps of f. Hence $p_a = 19 - 11 = 8$. Since among these adjoint cubic surfaces there are those which degenerate into the plane (x, y) and into an arbitrary quadric surface, it follows that $p_g \geqq 9$. Hence $p_g - p_a > 0$, which is impossible, since the surface is regular.

There also does not exist a self-dual curve corresponding in the following values of the Plücker numbers: $m = m' = 8$, $k = i = 16$, $d = \tau = 0$.[8] Other examples of Plücker numbers, which do not correspond to existent curves, are the following:

$$\begin{array}{llllllll}
(1) & n = 13, & m = 18; & k = 46, & i = 61; & d = 0, & \tau = 55; \\
(2) & n = 19, & m = 24; & k = 106, & i = 121; & d = 0, & \tau = 85; \\
(3) & n = 25, & m = 27; & k = 191, & i = 197; & d = 0, & \tau = 86.
\end{array}$$

[6] See, for instance, S. Lefschetz, On the existence of loci with given singularities, Transactions of the American Mathematical Society, vol. 14 (1913), pp. 23–41, where a detailed and interesting discussion of this question may be found. See also, F. Enriques and O. Chisini, Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, vol. 1, Chap. II, pp. 267-268.

[7] It will be noticed that such curves could not be reducible. The same remark applies to the other examples given below.

[8] If thus appears very natural that this and the preceding example should be the first instances of Plückerian characters, for which the existence of corresponding curves is questioned by Lefschetz in his quoted paper.

As an application of the second consequence of the inequality $m < 6j$, we quote the following examples: an irreducible curve of order 7 cannot have 9 cusps at the basis points of a pencil of cubics, or 10 cusps on a cubic; an irreducible curve of order 8 cannot have 12 cusps at the intersection of a cubic and an irreducible quartic curve, or 13 cusps on a cubic.

It is convenient to express the above general result in the following form: If the cusps of an irreducible curve f of order m do not impose independent conditions on the curves of order $m-4$, then $m \leqq 6$; if the cusps do not impose independent conditions on the curves of order $m-5$, then $m \leqq 12$; etc. That the upper limits 6, 12 etc. are attained, is shown by the examples of the curves $f_3^2 + f_2^3 = 0$ (sectics with 6 cusps on a conic), $f_6^2 + f_4^3 = 0$ (C_{12}'s with 24 cusps on a C_4 and a C_6), etc.

7. We now consider the two cases: $m < 6j$ and $m = 6j$.

Let $m < 6j$. We have from (31b), $m = (jm_1 - m_2)/h_1$, and hence

$$6j - m = \frac{j(6h_1 - m_1) + m_2}{h_1}. \tag{34}$$

If $6h_1 - m_1 \leqq 0$, then (32) shows that $6\lambda > -6$, and hence $\lambda \geqq 0$. Let now $6h_1 - m_1 > 0$. We observe in the first place that since $m_1 < m$, it follows that $h_1 < j$. Hence, from (34), $6j - m > 6h_1 - m$ and from (32),

$$6\lambda > h(6h_1 - m_1) + \sigma - 6 \geqq -6,$$

so that again $\lambda \geqq 0$. *Thus, if $m < 6j$, the surface F is necessarily regular, for any $n \geqq m$.* It is then evident that, under the same hypothesis, the surface F will also be regular for any $n < m$, because if, for a given n, say $n = n_1$, the above surface is irregular, the surface will be necessarily irregular for any n which is a multiple of n_1. In fact, if $n = \nu n_1$, and if we denote by F_1 and by F_2 the two surfaces $z^{n_1} = f(x, y)$, $z^{\nu n_1} = f(x, y)$ respectively, we see that F_2 can be represented upon the ν-fold surface F_1, and that hence the linear connection index of F_2 cannot be less than the linear connection index of F_1. Since the linear connection index of a surface is twice its irregularity, our assertion follows.

We see that the only possible irregular surfaces F correspond to the case $m = 6j$. Then $h_1 = [jm_1/m] = [m_1/6]$, and (32) becomes

$$6\lambda = m_1 - 6[m_1/6] - 6 + \sigma.$$

This shows that $\lambda < 0$, and hence that the surface F is irregular if, and only if, m_1 is divisible by 6. Since $n = (h+1)m - m_1$, n must also be

divisible by 6. If this condition is satisfied, $\sigma = 0$ and we find that $\lambda = -1$, and hence that the irregularity of F does not depend on n and is equal to the superabundance s_{m-3-j}, which, by hypothesis, is different from zero. This result holds subject to the hypothesis $n \geqq m$. We will prove below that it holds also for $n < m$. Hence we have the following theorem:

If $f(x, y) = 0$ is an irreducible curve of order m, possessing nodes and cusps only, the necessary and sufficient condition in order that the surface $z^n = f(x, y)$ be irregular, is that m and n should be divisible by 6, and that, putting $m = 6j$, the system $|C_{m-3-j}|$ of the curves of order $m-3-j$ passing through the cusps of the curve f should be superabundant. If these conditions are satisfied, the irregularity of the surface does not depend on n and is equal to the superabundance of the above system $|C_{m-3-j}|$.

8. We have proved the theorem, stated at the end of the previous section, subject to the hypothesis $n \geqq m$. We now complete the proof, showing that it also holds when $n < m$. If $n < m$, and m is not divisible by 6, then the surface F,

$$z^n = f(x, y),$$

is regular: this was proved already in the preceding section. Now let m be divisible by 6. If n is not divisible by 6, F is regular. In fact, let n_1 be a number such that nn_1 is not less than m and is not divisible by 6. Then by the above theorem, the surface F_1,

$$z^{nn_1} = f(x, y),$$

is regular, and hence also the surface F, which is in $(1, n_1)$ correspondence with F_1, is regular.

There remains to be considered the case in which n is divisible by 6. We know already that, if n_1 is any integer such that $6n_1 \geqq m$, the irregularity q of the surface $z^{6n_1} = f(x, y)$ is the same for all values of n_1, namely, $q = s_{m-3-j}$. What we have to prove is that the same is true when $6n_1 < m$.

It is obvious that it is sufficient to prove this for $n_1 = 1$. In fact, let n_1 be any integer such that $6n_1 < m$, and let n_2 be any integer such that $6n_1n_2 \geqq m$. Then the irregularity of the surface $z^{6n_1} = f(x, y)$ is $\leqq q$ and is not less than the irregularity of the surface $z^6 = f(x, y)$. If we prove then that the irregularity of this last surface is exactly equal to q, it will follow that the irregularity of the surface $z^{6n_1} = f(x, y)$ is also equal to q.

In the following proof it is convenient to take as a starting point the transcendental definition of the irregularity of an algebraic surface as the

number of distinct (independent) exact differentials of the first kind attached to the surface. In order to make proper use of this significance of the irregularity, we first proceed to prove the following lemma:

An algebraic surface F,

$$z^n = f(x, y),$$

of irregularity q, *possesses* q *exact differentials of the first kind of the reduced type*

$$\frac{P(x, y)\,dx + Q(x, y)\,dy}{z^\alpha},$$

where α *is an integer which satisfies the inequalities* $0 < \alpha \leqq n-1$, *and where* $P(x, y)$ *and* $Q(x, y)$ *are polynomials in* x *and* y.

Proof. It is known[9] that an exact differential of the first kind attached to the surface F is necessarily of the type

$$\frac{R_0(x, y, z)\,dx + S_0(x, y, z)\,dy}{z^{n-1}}, \tag{35}$$

where $R_0(x, y, z)$ and $S_0(x, y, z)$ are *polynomials* in x, y and z. We may suppose that R_0 and S_0 are of degree $n-1$ at most in the variable z. We put

$$\begin{aligned} R_0(x, y, z) &= P_0(x, y) + P_1(x, y)z + \cdots + P_{n-1}(x, y)z^{n-1}, \\ S_0(x, y, z) &= Q_0(x, y) + Q_1(x, y)z + \cdots + Q_{n-1}(x, y)z^{n-1}, \end{aligned}$$

where P_i and Q_i are polynomials in x and y. The surface F goes into itself by the $n-1$ transformations $x' = x, y' = y, z' = \omega^k z$ $(k = 1, 2, \cdots, n-1)$, where ω is a primitive root of the equation $x^n = 1$. It follows that

$$\frac{R_k(x, y, z)\,dx + S_k(x, y, z)\,dy}{z^{n-1}},$$

where

$$\begin{aligned} R_k(x, y, z) &= P_0(x, y) + \omega^k P_1(x, y)\,z + \cdots + \omega^{(n-1)k} P_{n-1}(x, y)\,z^{n-1}, \\ S_k(x, y, z) &= Q_0(x, y) + \omega^k Q_1(x, y)\,z + \cdots + \omega^{(n-1)k} Q_{n-1}(x, y)\,z^{n-1}, \end{aligned}$$

is also an exact differential of the first kind. Hence also, for a fixed j,

$$\frac{\sum_{0}^{n-1}{}_k\, \omega^{-kj} R_k(x, y, z)\,dx + \sum_{0}^{n-1}{}_k\, \omega^{-kj} S_k(x, y, z)\,dy}{z^{n-1}}$$

[9] E. Picard et G. Simart, Théorie des fonctions algébriques de deux variables indépendantes, vol. I, ch. V, p. 116.

is an exact differential of the first kind. Since

$$1+\omega^{(i-j)}+\omega^{2(i-j)}+\cdots+\omega^{(n-1)(i-j)}$$

is 0 or n according as $i \neq j$ or $i=j$, the above linear combination of the n original differentials reduces to

$$n\cdot\frac{P_j(x,y)\,dx+Q_j(x,y)\,dy}{z^{n-1-j}} \qquad (j=0,1,\cdots,n-1), \tag{35'}$$

which is of the reduced type. It follows in the first place that necessarily $P_{n-1}(x,y)$ and $Q_{n-1}(x,y)$ vanish identically, because otherwise we would have an exact differential of the first kind, $P_{n-1}(x,y)\,dx+Q_{n-1}(x,y)\,dy$, involving the independent variables only, which is impossible. Since the original exact differential (35) is a combination of the $n-1$ exact differentials (35′), it follows that any exact differential of the first kind attached to the surface F depends on similar differentials of the reduced type, whence the Lemma follows.

Let n_1 and n_2 be two *relatively prime* integers such that $6n_1 \geqq m$, $6n_2 \geqq m$. Let us consider the surfaces F_1, F_2, given by the equations

$$F_1:\quad z_1^{6n_1}=f(x,y);$$
$$F_2:\quad z_2^{6n_2}=f(x,y).$$

The two surfaces have the same irregularity q. There will be q distinct exact differentials of the first kind and of the reduced type attached to the surface F_1:

$$\frac{P_i^{(1)}(x,y)\,dx+Q_i^{(1)}(x,y)\,dy}{z_1^{\alpha_i}} \qquad \begin{pmatrix} i=1,2,\cdots,q; \\ 0<\alpha_i\leqq 6n_1-1\end{pmatrix}. \tag{36}$$

Similarly there will be q distinct exact differentials of the first kind and of the reduced type attached to the surface F_2:

$$\frac{P_j^{(2)}(x,y)\,dx+Q_j^{(2)}(x,y)\,dy}{z_2^{\beta_j}} \qquad \begin{pmatrix} j=1,2,\cdots,q; \\ 0<\beta_j\leqq 6n_2-1\end{pmatrix}. \tag{36'}$$

Now consider the surface F, given by the equation

$$z^{6n_1n_2}=f(x,y).$$

If we make in (36) the substitution $z_1=z^{n_2}$, we evidently obtain a set of q distinct exact differentials of the first kind attached to the surface F.

Similarly, if in (36′) we make the substitution $z_2 = z^{n_1}$, we obtain again a set of q distinct exact differentials of fhe first kind attached to the same surface F. Since the irregularity of F is still equal to q, the differentials of one set must be linear homogeneous combinations of the differentials of the other set. Equating the coefficients of dx and of dy we arrive at the following *identities on the surface F:*

$$\frac{P_j^{(2)}(x,y)}{z^{\beta_j n_1}} - \left[c_{1j}\frac{P_1^{(1)}(x,y)}{z^{\alpha_1 n_2}} + c_{2j}\frac{P_2^{(1)}(x,y)}{z^{\alpha_2 n_2}} + \cdots + c_{qj}\frac{P_q^{(1)}(x,y)}{z^{\alpha_q n_2}}\right] = 0 \tag{37}$$
$$(j = 1, 2, \cdots, q),$$

$$\frac{Q_j^{(2)}(x,y)}{z^{\beta_j n_1}} - \left[c_{1j}\frac{Q_1^{(1)}(x,y)}{z^{\alpha_1 n_2}} + c_{2j}\frac{Q_2^{(1)}(x,y)}{z^{\alpha_2 n_2}} + \cdots + c_{qj}\frac{Q_q^{(1)}(x,y)}{z^{\alpha_q n_2}}\right] = 0 \tag{37′}$$
$$(j = 1, 2, \cdots, q),$$

where the c_{ij}'s are constants. At least one of the two polynomials $P_j^{(2)}$, $Q_j^{(2)}$ does not vanish identically. Suppose, for instance, that $P_j^{(2)}$ does not vanish identically. If we multiply (37) by $z^{6n_1n_2}$ we obtain an algebraic equation in x, y, z, of degree less than $6n_1 n_2$ in z, and since this equation must be satisfied by the function z, defined by the irreducible equation $z^{6n_1n_2} = f(x, y)$, the first member of the equation must vanish identically in x, y and z.

It follows that the exponent $\beta_j n_1$ of z in the first term of (37) must be divisible by n_2, since the exponents of z in each of the remaining terms are divisible by n_2. Since by hypothesis n_1 and n_2 are relatively prime, it follows that β_j is divisible by n_2. Putting $\beta_j = \lambda_j n_2$ $(0 < j \leqq 5)$ and $Z = z_2^{n_2}$, we see immediately that the differentials (36′), which now can be written in the form

$$\frac{P_j^{(2)}(x, y)\,dx + Q_j^{(2)}(x, y)\,dy}{Z^{\lambda_j}}$$

constitute a set of q distinct exact differentials of the first kind attached to the surface

$$Z^6 = f(x, y).$$

The irregularity of this surface is then at least q, and since it cannot be greater than q, it is exactly equal to q, which is what we had to prove.

An example. A branch curve f of order $m = 6j$ given by an equation of the type

$$[\varphi_{3j}(x, y)]^2 + [\psi_{2j}(x, y)]^3 = 0,$$

where φ and ψ are arbitrary polynomials of order $3j$ and $2j$ respectively, gives rise to irregular cyclic multiple planes F,

$$z^{6n_1} = [\varphi_{3j}(x, y)]^2 + [\psi_{2j}(x, y)]^3,$$

having the same irregularity $q = 1$. In fact, the curve f possesses $6j^2$ cusps, which are the intersections of the curves $\varphi = 0$, $\psi = 0$. Consider the system Σ of curves C_{m-3-j} of order $m-3-j = 5j-3$ passing through the cusps of f. The superabundance of this system is equal to the index of speciality of the complete linear series cut out by the curves of the system on the curve ψ of order $2j$ outside the basis points of the system. This linear series is nothing else than the canonical series g_{2p-2}^{p-1}, where p is the genus of the curve ψ, for the system Σ contains curves which degenerate into the curve φ and into an arbitrary curve of order $2j-3$. Hence the superabundance of Σ is equal to 1, which proves that the irregularity of the surface F is 1. An exact differential of the first kind attached to F is easily obtained. Putting

$$t = \frac{\psi}{z^{2n_1}}, \quad u = \frac{\varphi}{z^{3n_1}},$$

we will have

$$u^2 = 1 - t^3. \tag{38}$$

The differential of the first kind dt/u, attached to the elliptic curve (38), is evidently an exact differential of the first kind attached to the surface F.

IV. A topological property of the branch curves of irregular cyclic multiple planes.

9. The irreducible plane algebraic curves f distribute themselves, from the topological point of view, into two classes, according to the structure of the fundamental (Poincaré) group of the residual space with respect to f of its carrying complex projective plane (briefly: the fundamental group of f).[10] The curves f of one class are those whose fundamental group is cyclic (in which case the group is necessarily finite, of order equal to the order of f); the curves of the second class are those whose fundamental group is not cyclic (in which case the group may be either a finite or an infinite discrete group).

This classification of the plane algebraic curves is quite analogous to the classification of the closed curves of ordinary three-dimensional space into two types: unknotted and knotted curves. The cyclic multiple planes are the only algebraic surfaces which admit as branch curve a curve of

[10] See my paper "On the problem of existence of algebraic functions of two variables possessing a given branch curve", American Journal of Math., Vol. LI, 2, April, 1929.

the first class, while the curves of the second class can always be taken as branch curves of non-cyclic multiple planes.

As an application of the theorem of the section 7, we want to show that the irreducible branch curves f of irregular cyclic multiple planes, characterized in that theorem, possess a non-cyclic fundamental group. We shall thus have the following theorem:

A sufficient condition in order that an irreducible algebraic curve f of order $m = 6j$, possessing only nodes and cusps, should possess a non-cyclic fundamental group, is that the linear system $|C_{m-3-j}|$ of curves of order $m - 3 - j$ passing through the cusps of f should be superabundant.

This theorem puts in evidence for the first time, by means of a quite general example, the connection which exists between the structure of the fundamental group of a plane algebraic curve and the special position of its cusps, as reflected in the value of the superabundance of the mentioned system of curves.

Obviously, all we have to show, in order to prove the above theorem, is that:

If the fundamental group of a plane irreducible algebraic curve $f(x, y) = 0$, in general position with respect to the line at infinity, is cyclic, then for every value of the positive integer n the surface

$$z^n = f(x, y)$$

is regular.

We devote the next section to the proof of this theorem.

10. We point out that the total branch curve of the above surface consists, for a generic value of n, of the curve f *and* of the line at infinity. If we knew the fundamental group of this total (reducible) branch curve, we could easily determine the fundamental group of the surface. By adding then the commutativity relations between any pair of generators of the group, we should obtain a fundamental set of homologies for the surface, whence the value of the linear connection index and hence also of the irregularity of the surface would follow immediately.

Let a be generic line in the plane (line at infinity). By hypothesis the curve f is in generic position with respect to a. It will be sufficient to require that a should meet f in m distinct points, where m is the order of f. Let G be the fundamental group of f. For the sake of generality we drop the hypothesis that G is cyclic. We choose as generators of G m non-intersecting loops $g_1, g_2, \cdots, g_m$ in a line $x = \bar{x} =$ constant, which start from a fixed point 0 and which surround the m intersections of f with the line $x = \bar{x}$. Then one generating relation is

$$g_1 g_2 \cdots g_m = 1. \tag{39}$$

The other generating relations of G are all of the type[11]

$$g_i = g_{ij}^{-1} g_j g_{ij}, \tag{40}$$

where $i, j = 1, 2, \cdots, m$, and where g_{ij} is some element of the group G. There may be more than one relation (40) in a complete set of generating relations, corresponding to a given pair of indices i and j.

Let us now denote by G_1 the fundamental group of the reducible curve $f + a$. As generators of G_1 we may choose the generators g_i of the group G and a loop γ in the line $x = \bar{x}$ starting from the point 0 and surrounding the intersection of the line a with $x = \bar{x}$. As for the structure of G_1 we prove the following theorem:

The generating relations (40) *are also generating relations of* G_1. *A complete set of generating relations of* G_1 *is obtained by adding to the relations* (40) *the following relations*:

$$\gamma g_i = g_i \gamma, \qquad (i = 1, 2, \cdots, m) \tag{41}$$

and by replacing (39) *by the following*:

$$\gamma g_1 g_2 \cdots g_m = 1.$$

We may suppose that the line a is parallel to the axis x. Let the equation of a be $y = b$, and let $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m$ be the ordinates of the intersections of the curve f with the line $x = \bar{x}$. We mark in the plane of the complex variable y the $m+1$ points $b, \bar{y}_1, \cdots, \bar{y}_m$. As $\bar{x}$ varies, the points $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m$ vary, while b remains fixed. Let A be a common point of a and f. The point A is an ordinary double point of the curve $f + a$, at which one of the points $\bar{y}_i$ coincides with b. Let for instance $\bar{y}_1 = b$. The double point A yields the following generating relation:

$$\gamma g_1 = g_1 \gamma.$$

Now, instead of varying the line $x = \bar{x}$ and keeping fixed the line a, we keep the line $x = \bar{x}$ fixed and we vary the line a, starting from its original position and returning to it, keeping a always parallel to the axis of x. Then, as long as in the course of the variation of the line a the variable point b in the plane of the variable y does not coincide with any of the fixed points $\bar{y}_i$, we can follow by continuity the variable loop γ and arrange the deformation of the loops g_i so that they never intersect. Let γ and g_1 be deformed into γ' and g_1' at the end of the variation of the line a. Then

$$\gamma' g_1' = g_1' \gamma'.$$

[11] See my paper quoted in the preceding section, p. 309.

the first class, while the curves of the second class can always be taken as branch curves of non-cyclic multiple planes.

As an application of the theorem of the section 7, we want to show that the irreducible branch curves f of irregular cyclic multiple planes, characterized in that theorem, possess a non-cyclic fundamental group. We shall thus have the following theorem:

A sufficient condition in order that an irreducible algebraic curve f of order $m = 6j$, possessing only nodes and cusps, should possess a non-cyclic fundamental group, is that the linear system $|C_{m-3-j}|$ of curves of order $m - 3 - j$ passing through the cusps of f should be superabundant.

This theorem puts in evidence for the first time, by means of a quite general example, the connection which exists between the structure of the fundamental group of a plane algebraic curve and the special position of its cusps, as reflected in the value of the superabundance of the mentioned system of curves.

Obviously, all we have to show, in order to prove the above theorem, is that:

If the fundamental group of a plane irreducible algebraic curve $f(x, y) = 0$, in general position with respect to the line at infinity, is cyclic, then for every value of the positive integer n the surface

$$z^n = f(x, y)$$

is regular.

We devote the next section to the proof of this theorem.

10. We point out that the total branch curve of the above surface consists, for a generic value of n, of the curve f *and* of the line at infinity. If we knew the fundamental group of this total (reducible) branch curve, we could easily determine the fundamental group of the surface. By adding then the commutativity relations between any pair of generators of the group, we should obtain a fundamental set of homologies for the surface, whence the value of the linear connection index and hence also of the irregularity of the surface would follow immediately.

Let a be generic line in the plane (line at infinity). By hypothesis the curve f is in generic position with respect to a. It will be sufficient to require that a should meet f in m distinct points, where m is the order of f. Let G be the fundamental group of f. For the sake of generality we drop the hypothesis that G is cyclic. We choose as generators of G m non-intersecting loops $g_1, g_2, \cdots, g_m$ in a line $x = \bar{x} =$ constant, which start from a fixed point 0 and which surround the m intersections of f with the line $x = \bar{x}$. Then one generating relation is

$$g_1 g_2 \cdots g_m = 1. \tag{39}$$

The other generating relations of G are all of the type[11]

$$g_i = g_{ij}^{-1} g_j g_{ij}, \tag{40}$$

where $i, j = 1, 2, \cdots, m$, and where g_{ij} is some element of the group G. There may be more than one relation (40) in a complete set of generating relations, corresponding to a given pair of indices i and j.

Let us now denote by G_1 the fundamental group of the reducible curve $f + a$. As generators of G_1 we may choose the generators g_i of the group G and a loop γ in the line $x = \bar{x}$ starting from the point 0 and surrounding the intersection of the line a with $x = \bar{x}$. As for the structure of G_1 we prove the following theorem:

The generating relations (40) *are also generating relations of* G_1. *A complete set of generating relations of* G_1 *is obtained by adding to the relations* (40) *the following relations*:

$$\gamma g_i = g_i \gamma, \qquad (i = 1, 2, \cdots, m) \tag{41}$$

and by replacing (39) *by the following*:

$$\gamma g_1 g_2 \cdots g_m = 1.$$

We may suppose that the line a is parallel to the axis x. Let the equation of a be $y = b$, and let $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m$ be the ordinates of the intersections of the curve f with the line $x = \bar{x}$. We mark in the plane of the complex variable y the $m+1$ points $b, \bar{y}_1, \cdots, \bar{y}_m$. As $\bar{x}$ varies, the points $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m$ vary, while b remains fixed. Let A be a common point of a and f. The point A is an ordinary double point of the curve $f + a$, at which one of the points $\bar{y}_i$ coincides with b. Let for instance $\bar{y}_1 = b$. The double point A yields the following generating relation:

$$\gamma g_1 = g_1 \gamma.$$

Now, instead of varying the line $x = \bar{x}$ and keeping fixed the line a, we keep the line $x = \bar{x}$ fixed and we vary the line a, starting from its original position and returning to it, keeping a always parallel to the axis of x. Then, as long as in the course of the variation of the line a the variable point b in the plane of the variable y does not coincide with any of the fixed points $\bar{y}_i$, we can follow by continuity the variable loop γ and arrange the deformation of the loops g_i so that they never intersect. Let γ and g_1 be deformed into γ' and g_1' at the end of the variation of the line a. Then

$$\gamma' g_1' = g_1' \gamma'.$$

[11] See my paper quoted in the preceding section, p. 309.

Given any product Γ of the generators g_i, *which does not involve the generator* g_1, it is always possible to determine a closed path of the point b (and hence a corresponding cyclic variation of the line a), which does not cross the loop g_1 and such that γ should be deformed into $\gamma' = \Gamma\gamma\Gamma^{-1}$. Since in this case we have $g_1' = g$, it follows:

$$\Gamma\gamma\Gamma^{-1}\cdot g_1 = g_1\cdot\Gamma\gamma\Gamma^{-1},$$

or

$$\gamma\cdot\Gamma^{-1}g_1\Gamma = \Gamma^{-1}g_1\Gamma\cdot\gamma, \tag{42}$$

i. e., γ *is commutative with the transform of* g_1 *by any element* Γ *which is a product of the generators* $g_2, g_3, \cdots, g_m$.

It is now easy to prove that if Γ' is *any* element of G_1, then γ is commutative with $(\Gamma')^{-1}g_1\Gamma'$. In fact:

(a) First let Γ' be an element of G_1 involving the generators $g_2, g_3, \cdots, g_m$ and γ. Then Γ' can be written in the form

$$\Gamma' = \Gamma_1\gamma^{k_1}\Gamma_2\gamma^{k_2}\cdots\Gamma_i,$$

where $\Gamma_1, \Gamma_2, \cdots, \Gamma_i$ involve only the generators $g_2, g_3, \cdots, g_m$, and where $k_1, k_2, \cdots, k_i$ are positive or negative integers. Then observing that, by (42), $\Gamma^{-1}g_1\Gamma$ is commutative with any power of γ, we deduce

$$(\Gamma')^{-1}g_1\Gamma' = \Gamma_i^{-1}\cdots\Gamma_2^{-1}\Gamma_1^{-1}g_1\Gamma_1\Gamma_2\cdots\Gamma_i,$$

i. e., $(\Gamma')^{-1}g_1\Gamma'$ is equivalent to an element $\Gamma^{-1}g_1\Gamma$, where Γ involves only the elements $g_2, g_3, \cdots, g_m$, and hence is commutative with γ.

(b) In order to prove that γ is commutative with $(\Gamma')^{-1}g_1\Gamma'$, even if Γ' involves the generator g_1, it is sufficient to prove that if this is true for a given element Γ', then it is also true for $g_1\Gamma'$ and $\Gamma' g_1$. For $g_1\Gamma'$ this is evident, because $(g_1\Gamma')^{-1}g_1(g_1\Gamma') = (\Gamma')^{-1}g_1\Gamma'$. Now let $\Gamma'' = \Gamma' g_1$. Then, recalling that g_1 is commutative with γ, we have

$$\begin{aligned}(\Gamma'')^{-1}g_1\Gamma''\cdot\gamma &= g_1^{-1}(\Gamma')^{-1}g_1\Gamma' g_1\cdot\gamma = g_1^{-1}(\Gamma')^{-1}g_1\Gamma'\gamma g_1\\ &= g_1^{-1}\gamma(\Gamma')^{-1}g_1\Gamma' g_1 = \gamma g^{-1}(\Gamma')^{-1}g_1\Gamma' g_1 = \gamma(\Gamma'')^{-1}g_1\Gamma'', \quad \text{q. e. d.}\end{aligned}$$

Now since f is an irreducible curve, the generators $g_1, g_2, \cdots, g_m$ are conjugate elements of G_1.[12] Since we have just proved that γ is commutative with any element of G_1 which is the transform of g_1 by some element of G_1, we deduce that γ is commutative not only with g_1 but with all the generators g_i, which proves (41).

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[12] See may paper, quoted above, p. 309.

It remains to be proved that the relations (40) are also generating relations of G_1. A relation of type (40) arises when the independent variable x turns around a critical point of the function y, so that the point $\overline{y}_i$ is carried into the point y_j. Now if we consider the group G_1, instead of G, the considered relation will have to be modified in the sense that factors γ will have to be introduced in g_{ij}, which in (40) is a product of the generators g_i only, *without changing the number and the order of the factors* g_i. But since γ is commutative with all the generators g_i, the factors γ will ultimately drop out, leaving the original relation unaltered.

Let us now suppose that G is a cyclic group. Then G_1 is generated by 2 generators λ, $g (= g_1 = g_2 = \cdots = g_m)$, satisfying the two generating relations:

$$\lambda g = g\lambda, \qquad \lambda g^m = 1.$$

Since $\lambda = g^{-m}$, we deduce that G_1 *is an infinite cyclic group*

$$\{\cdots g^{-2}, g^{-1}, g, g^2 \cdots\}.$$

But then it is obvious that the fundamental group of any cyclic multiple plane F, $z^n = f(x, y)$, possessing f and the line at infinity as total branch curve, is the identity. It follows that the surface F is regular, which proves the theorem stated at the end of Section 9.

CRITICAL SETS OF AN ARBITRARY REAL ANALYTIC FUNCTION OF n VARIABLES.[1]

By Arthur B. Brown.[2]

1. Notations and hypotheses. Notations are as in an earlier paper.[3] Notations in analysis situs are as in Lefschetz's Colloquium Publication.[4] In particular, $\approx$ denotes homology with division allowed, and $R_i(H)$ denotes the ith Betti number of H.

In this paper we treat the case where the critical points may form an arbitrary complex. This condition is satisfied by the critical points of an analytic function. A more general problem has been considered independently by Morse,[5] but with a different kind of treatment. In earlier papers containing results, for a function of n variables, similar to those in Morse I, the broadest class of critical points treated is that of isolated critical points.[6]

The hypotheses of the paper follow.

A closed region R of the euclidean space of the variables $x_1, x_2, \cdots, x_n$ is given, bounded by regular $(n-1)$-spreads of class $C^{(3)}$. A single-valued function f is given, real and analytic over R. Its inner normal derivative, at any boundary point, is negative.

These hypotheses may be modified by dropping the condition that f be analytic, but requiring that f be of class C''; and that R can be covered by a complex of which the locus of all critical points is a sub-complex, and of which the locus $f=c$, for any critical value c, is a sub-complex.

As in B I, page 259, we replace f by a function (again denoted by f) which is constant on the boundary of R, and greater there than at any interior point.[7]

[1] Received November 12, 1930.—Presented to the Amer. Math. Society, October 25, 1930.

[2] Part of the work on this paper was done while the author was a National Research Fellow.

[3] A. B. Brown, Relations between the critical points of a real analytic function of n independent variables, Amer. Journ. Math., vol. 52 (1930), pp. 251–270. (B I.) Prof. Morse called my attention to an omission in this paper. In the proof of Lemma 14 a motion is described which is said to be a deformation, but is not proved to be one. The results of this paper were originally presented in different form, with the proofs completely worked out. Morse has since obtained results regarding the neighborhood of an isolated critical point, which make available the material needed at the point in question. (Proc. Nat. Acad. Sci., vol. 15 (1930), pp. 777–779.)

[4] Solomon Lefschetz, Topology, New York, 1930. (Lefschetz I.)

[5] See end of note 13. The first significant paper on the subject was Marston Morse's, Relations between the critical points of a real function of n independent variables, Trans. Amer. Math. Soc., vol. 27 (1925), pp. 345–396. (Morse I.)

[6] References to papers by Poincaré, Birkhoff, Morse and W. M. Whyburn are given in the author's paper, Relations between the critical points and curves of a real analytic function of two independent variables, these Annals, vol. 31 (1930), pp. 449–456. (B II.)

[7] Cf. Morse I, pp. 394–396.

We shall assume that certain loci can be covered by complexes. For proofs, we refer to van der Waerden,[8] and Chap. VIII in Lefschetz I, for loci defined by analytic functions.[8a] Proofs for regular spreads, with certain singularities allowed, have been completed by S. S. Cairns.[9]

The proofs and results in the present paper are valid when Betti numbers are considered either absolute or modulo p, p any prime greater than unity.

2. **Critical sets and their type numbers.** Since the critical points are defined by the vanishing of the function $\sum_{i=1}^{n} (\partial f/\partial x_i)^2$, which is analytic near the critical points, the locus of critical points is a complex, whose cells are regular and analytic.[10] Hence f is constant over any connected part of the complex. An immediate consequence is the following.

THEOREM 1. *The number of critical values is finite.*

A *critical set* is a set of critical points all with the same critical value, having a neighborhood in which there are no other critical points.[11]

If f is greater, on a given critical set K, than at all points of a certain neighborhood of K, K is called a *locus of maximum*. A *locus of minimum* is similarly defined.

Let K be any critical set, with critical value c, and D the locus of points in R at which $f \leqq c$. *The ith type number of* $K (i = 0, 1, \cdots, n)$ *is the number of i-chains in a maximal set of i-chains on D whose boundaries do not meet K and which are independent*[12] *on D of i-chains which do not meet* K[13]. It is denoted by $M_i(K)$. The following theorem follows at once from the definition.

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[8] B. L. van der Waerden, Topologische Begründung des Kalküls der abzählenden Geometrie, Math. Annalen, vol. 102 (1929), pp. 337–362.

[8a] Detailed proofs will be given by B. O. Koopman and the writer, in a later paper.

[9] S. S. Cairns, The cellular division and approximation of regular spreads, Proc. Nat. Acad. Sci., vol. 16 (1930), pp. 488–491.

[10] See Lefschetz I, Chap. VIII.

[11] This notation was used by W. M. Whyburn under much broader hypotheses. His critical sets are connected. (Bull. Amer. Math. Soc., vol. 35 (1929), pp. 701–708.)

[12] Independence of chains will always refer to homologies.

[13] My original definition (cf. Theorem 3 and note 15) depended indirectly on the entire locus $f \leqq c$. Prof. Morse called my attention to this defect, and himself was the first to state and prove that under that definition (slightly modified), the type numbers are topological invariants of the critical set and its neighborhood only in the locus $f \leqq c$. The definition given here was suggested independently by Prof. Lefschetz, to replace the original one. Cf. the definition of pseudocycle in Lefschetz I, Ch. VI, No. 36 ff. A definition not essentially different from this was sent to me by Morse prior to the suggestion by Lefschetz, and applied by Morse to a more general case than that considered in this paper. (ADDED IN PROOF:) Cf. M. Morse, "The critical points of a function of n variables", Trans. Amer. Math. Soc., vol. 33 (1931), pp. 72–91.

THEOREM 2. *The type numbers of K are topological invariants of K and any neighborhood of K in D (the locus $f \leqq c$).*

3. Betti numbers of certain related complexes. The following lemma will be used a number of times. It is a slight extension of Theorem 2 in B I.

LEMMA A. *Let K_1, K_2 be point sets, with $K_2 \supset K_1$, such that there exists a deformation T of K_2 over itself keeping K_1 on K_1 and deforming both into subsets of K_1. Then any minimal base for $\approx$ and the i-cycles on K_1 is also one for K_2; and any minimal base for $\approx$ and the i-cycles on K_2, composed of i-cycles on K_1, is also such a base for K_1.*

Proof. The existence of T implies that every cycle on K_2 is $\approx$ one one K_1. On the other hand let $K_1 \supset \Gamma_i \approx 0$ on K_2. Then $K_2 \supset C_{i+1} \rightarrow \Gamma_i$. Hence $K_1 \supset TC_{i+1} \rightarrow T\Gamma_i \approx \Gamma_i$; that is, $\Gamma_i \approx 0$ on K_1. Therefore, if $\Gamma_i \not\approx 0$ on K_1, it $\not\approx 0$ on K_2, which implies the first conclusion of the lemma. The second is proved in similar manner.

The lemma is true also mod. m, m any integer greater than unity; and holds for $\sim$ as well as $\approx$.

4. Another evaluation of the type numbers.

THEOREM 3. *Let K be a critical set, with critical value c. Let D be a simplicial complex covering the locus $f \leqq c$, and D' and D'' regular subdivisions of D and D' respectively. Let N be the D''-neighborhood of K,*[14] *$\bar{N}$ the closure of N, and $C = \bar{N} - N$. Then $M_i(K) = R_i(\bar{N}; C)$.*[15]

Proof. The members of the set of i-chains used in defining $M_i(K)$ can be deformed, by the Alexander-Veblen process, so as to consist of cells coinciding with cells of D'', and still have their boundaries on $D - K$. From the hypotheses on the original set of i-chains, and the theorems about deformation,[16] it is easily proved that a set of i-cycles on $\bar{N} \bmod C$ is thus obtained which is a minimal base for $\approx$ and such i-cycles. Hence the theorem is true.

THEOREM 4. *If K_1 and K_2 are critical sets having the same critical value, but having no points in common, then*

$$M_i(K_1) + M_i(K_2) = M_i(K_1 + K_2), \qquad i = 0, 1, \cdots, n.$$

Proof. This follows from Theorem 3, since the $\bar{N}$'s for two unconnected K's are unconnected.

5. LEMMA 1. *Let e be a positive constant such that there is no critical value in the interval $c - e \leqq f < c$; K the set of all critical points satis-*

[14] The D''-neighborhood of K is the locus of all cells of D'' that have a vertex on K.

[15] In the earlier version this was the definition of type number. It was then proved only to be a topological invariant of K and the entire locus $f \leqq c$. Cf. note 13.

[16] Cf. Lefschetz I, Chap. II.

fying $f = c$; E the locus of points in R satisfying $f \leqq c - e$. Let D'' be taken so that $N \subset D'' - E$, and B denote $D'' - N$. Then any minimal base for $\approx$ and i-cycles on E is also one for B.

Proof. The locus $D'' - K$ can be deformed, by use of the orthogonal trajectories, onto E so as to satisfy Lemma A. Hence the minimal base in question is likewise one for $D'' - K$. Now $D'' - K$ can be deformed onto $D'' - N$, again so as to satisfy Lemma A, since N is a normal neighborhood of K.[16] Therefore, by Lemma A, the minimal base is also one for $D'' - N$, which completes the proof.[17]

LEMMA 2. *Let K be any critical set; D, D'' and N as defined in Theorem 3, and $B = D'' - N$. Then any minimal base for $\approx$ and i-cycles on B is also one for $D - K$.*

Proof. The result follows from Lemma A, as in the last proof.

LEMMA 3. *The Betti numbers of K, N and $\bar{N}$ are equal.*

Proof. The result follows from Lemma A, by use of the normal neighborhood.

6. **Changes in Betti numbers; new and dropped i-cycles.** For an arbitrary critical set K, we use the notations of the previous sections. By the number of *dropped i-cycles* in passing from B to D we mean the number in a maximal set of independent i-cycles on B each of which bounds on D. The number of *new i-cycles* denotes the number in a maximal set of i-cycles on D that are independent, on D, of the i-cycles on B. These numbers are easily proved to be uniquely determined, hence are topological invariants.

THEOREM 5. *The type number $M_i(K)$ equals the sum of the number of dropped $(i-1)$-cycles and the number of new i-cycles, in passing from* $D - K$ to D.

Proof. Let $A^i, \cdots, C^i_d, D^i$ be defined as maximal sets of i-cycles $\cdots$, as in B I, p. 253 bottom and p. 256 top, but with $\bar{N}$ in this paper replacing A in B I. Then, in the notations of B I,

$$M_i(K) = R_i(\bar{N}; C) = a^i + c_d^{i-1} + c_a^{i-1}, \quad (6.1)$$

[18] as may be proved easily by the methods of B I. The proofs of Lemmas 2 and 6 in B I show that the following are maximal independent sets of i-cycles:

(6.2) B^i, C^i_1 and C^i_a constitute such a set on B;

(6.3) A^i, B^i, C^i_1 and D^i constitute such a set on D.

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[17] Under the alternative hypotheses (§ 1) the existence of certain of these minimal bases must be proved. This is done easily by use of Lemma A.

[18] Any undefined symbol, as c_d^{-1}, is understood to equal zero.

THEOREM 2. *The type numbers of K are topological invariants of K and any neighborhood of K in D (the locus $f \leqq c$).*

3. **Betti numbers of certain related complexes.** The following lemma will be used a number of times. It is a slight extension of Theorem 2 in B I.

LEMMA A. *Let K_1, K_2 be point sets, with $K_2 \supset K_1$, such that there exists a deformation T of K_2 over itself keeping K_1 on K_1 and deforming both into subsets of K_1. Then any minimal base for $\approx$ and the i-cycles on K_1 is also one for K_2; and any minimal base for $\approx$ and the i-cycles on K_2, composed of i-cycles on K_1, is also such a base for K_1.*

Proof. The existence of T implies that every cycle on K_2 is $\approx$ one one K_1. On the other hand let $K_1 \supset \Gamma_i \approx 0$ on K_2. Then $K_2 \supset C_{i+1} \rightarrow \Gamma_i$. Hence $K_1 \supset TC_{i+1} \rightarrow T\Gamma_i \approx \Gamma_i$; that is, $\Gamma_i \approx 0$ on K_1. Therefore, if $\Gamma_i \not\approx 0$ on K_1, it $\not\approx 0$ on K_2, which implies the first conclusion of the lemma. The second is proved in similar manner.

The lemma is true also mod. m, m any integer greater than unity; and holds for $\sim$ as well as $\approx$.

4. **Another evaluation of the type numbers.**

THEOREM 3. *Let K be a critical set, with critical value c. Let D be a simplicial complex covering the locus $f \leqq c$, and D' and D'' regular subdivisions of D and D' respectively. Let N be the D''-neighborhood of K,[14] $\bar{N}$ the closure of N, and $C = \bar{N} - N$. Then $M_i(K) = R_i(\bar{N}; C)$.*[15]

Proof. The members of the set of i-chains used in defining $M_i(K)$ can be deformed, by the Alexander-Veblen process, so as to consist of cells coinciding with cells of D'', and still have their boundaries on $D - K$. From the hypotheses on the original set of i-chains, and the theorems about deformation,[16] it is easily proved that a set of i-cycles on $\bar{N}$ mod C is thus obtained which is a minimal base for $\approx$ and such i-cycles. Hence the theorem is true.

THEOREM 4. *If K_1 and K_2 are critical sets having the same critical value, but having no points in common, then*

$$M_i(K_1) + M_i(K_2) = M_i(K_1 + K_2), \qquad i = 0, 1, \cdots, n.$$

Proof. This follows from Theorem 3, since the $\bar{N}$'s for two unconnected K's are unconnected.

5. LEMMA 1. *Let e be a positive constant such that there is no critical value in the interval $c - e \leqq f < c$; K the set of all critical points satis-*

[14] The D''-neighborhood of K is the locus of all cells of D'' that have a vertex on K.

[15] In the earlier version this was the definition of type number. It was then proved only to be a topological invariant of K and the entire locus $f \leqq c$. Cf. note 13.

[16] Cf. Lefschetz I, Chap. II.

fying $f = c$; *E the locus of points in R satisfying* $f \leqq c - e$. *Let* D'' *be taken so that* $N \subset D'' - E$, *and B denote* $D'' - N$. *Then any minimal base for* $\approx$ *and i-cycles on E is also one for B.*

Proof. The locus $D'' - K$ can be deformed, by use of the orthogonal trajectories, onto E so as to satisfy Lemma A. Hence the minimal base in question is likewise one for $D'' - K$. Now $D'' - K$ can be deformed onto $D'' - N$, again so as to satisfy Lemma A, since N is a normal neighborhood of K.[16] Therefore, by Lemma A, the minimal base is also one for $D'' - N$, which completes the proof.[17]

LEMMA 2. *Let K be any critical set*; *D*, D'' *and N as defined in Theorem* 3, *and* $B = D'' - N$. *Then any minimal base for* $\approx$ *and i-cycles on B is also one for* $D - K$.

Proof. The result follows from Lemma A, as in the last proof.

LEMMA 3. *The Betti numbers of K, N and* $\bar{N}$ *are equal.*

Proof. The result follows from Lemma A, by use of the normal neighborhood.

6. **Changes in Betti numbers; new and dropped *i*-cycles.** For an arbitrary critical set K, we use the notations of the previous sections. By the number of *dropped i-cycles* in passing from B to D we mean the number in a maximal set of independent i-cycles on B each of which bounds on D. The number of *new i-cycles* denotes the number in a maximal set of i-cycles on D that are independent, on D, of the i-cycles on B. These numbers are easily proved to be uniquely determined, hence are topological invariants.

THEOREM 5. *The type number* $M_i(K)$ *equals the sum of the number of dropped* $(i-1)$*-cycles and the number of new i-cycles, in passing from* $D - K$ to D.

Proof. Let $A^i, \cdots, C_d^i, D^i$ be defined as maximal sets of i-cycles $\cdots$, as in B I, p. 253 bottom and p. 256 top, but with $\bar{N}$ in this paper replacing A in B I. Then, in the notations of B I,

$$M_i(K) = R_i(\bar{N}; C) = a^i + c_d^{i-1} + c_a^{i-1},\text{[18]} \tag{6.1}$$

as may be proved easily by the methods of B I. The proofs of Lemmas 2 and 6 in B I show that the following are maximal independent sets of i-cycles:

(6.2) B^i, C_1^i and C_a^i constitute such a set on B;

(6.3) A^i, B^i, C_1^i and D^i constitute such a set on D.

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[17] Under the alternative hypotheses (§ 1) the existence of certain of these minimal bases must be proved. This is done easily by use of Lemma A.

[18] Any undefined symbol, as c_a^{-1}, is understood to equal zero.

From (6.2), (6.3), the relation $c_d^{i-1} = d^i$, and the definition of C_a^i we infer that in passing from B to D the number of dropped i-cycles is c_a^i, and the number of new i-cycles is $(a^i + c_d^{i-1})$. The theorem follows from Lemma 2, the preceding statement, and (6.1).

7. LEMMA 4. *If there is no critical value in the interval $c < f \leqq c + e$, any minimal base for $\approx$ and i-cycles on the locus $f \leqq c$ is likewise such a base for the locus $f \leqq c + e$.*

Proof. Let e_1 be a positive constant such that the locus $c \leqq f \leqq c + e_1$ is in a normal neighborhood of the locus $f = c$ in the locus $f \leqq c + e$. We deform the locus $c + e_1 < f \leqq c + e$ onto the locus $f = c + e_1$ by use of the orthogonal trajectories, then deform the locus $c < f \leqq c + e_1$ onto the locus $f = c$ by means of the normal neighborhood. The result of the two deformations is one which carries the locus $f \leqq c + e$ onto the locus $f \leqq c$, while satisfying the hypotheses of Lemma A. Hence Lemma 4 is true.

8. Relations between the critical points.

THEOREM 6. *Let c and $e > 0$ be constants such that c is the only critical value satisfying $c - e \leqq f \leqq c + e$. Let J and E be the loci of points satisfying $f \leqq c + e$ and $f \leqq c - e$ respectively, and $\Delta R_i = R_i(J) - R_i(E)$. Let ΔM_i denote the ith type number of the set K of all critical points on the locus $f = c$. Let ΔL_i and ΔH_i denote the ith Betti numbers of the loci of minimum and maximum, respectively, for which $f = c$. Then the relations of Theorem 7, below, hold where each symbol is replaced by the same symbol preceded by Δ.*

Proof. We shall use (6.1), which gives us

$$\Delta M_i = a^i + c_d^{i-1} + c_a^{i-1}. \tag{8.1}$$

From Lemmas 1 and 4 we infer that $\Delta R_i = R_i(D) - R_i(B)$, and thus from (6.2), (6.3), and the relation $c_d^{i-1} = d^i$, we have

$$\Delta R_i = a^i + c_d^{i-1} - c_a^i. \tag{8.2}$$

For a locus of minimum taken as K, the corresponding C is vacuous, and $\bar{N} = K$. It follows from the definition of a^i that for the K of our hypotheses,

$$a^i \geqq \Delta L_i, \qquad i = 0, 1, \cdots, n. \tag{8.3}$$

Finally we shall prove that

$$c_a^i \geqq \Delta H_{n-i-1}, \qquad i = 0, 1, \cdots, n-1. \tag{8.4}$$

Let A denote the locus of maximum, $N(A)$ its D'-neighborhood, $\bar{N}(A)$ the closure of $N(A)$. The ith Betti number of $\bar{N}(A)$ is ΔH_i, by Lemma 3.

It follows from the Alexander duality theorem[19] that if S denotes the n-space in which R is immersed (considered closed by a point at infinity), there are $\Delta H_{n-i-1} + \delta_0^i - \delta_{n-1}^i$[20] independent non-bounding i-cycles on $S - \bar{N}(A)$; and they may be taken composed of cells of S providing S is regularly subdivided once. We assume that that is done, and keep the same notations. Since the Betti numbers of S are δ_0^i, $i = 0, 1, \cdots, n-1$, we infer that there are just $\Delta H_{n-i-1} - \delta_{n-1}^i$ independent i-cycles on $S - \bar{N}(A)$ each of which bounds on S, and these can be taken composed of cells of S. We denote such a set of i-cycles by S_1^i.

Since A is a locus of maximum, $N(A)$ contains a neighborhood of A in S, hence C contains the point set boundary of $\bar{N}(A)$ in S. Therefore if we take $(i+1)$-chains composed of cells of S, bounded by the cycles of S_1^i, the boundaries of the parts in $\bar{N}(A)$ of these chains will be i-cycles of C. We denote the set of these i-cycles by C_2^i. They are likewise $\Delta H_{n-i-1} - \delta_{n-1}^i$ in number. Each cycle of C_2^i bounds on $\bar{N}$, but all are independent on $S - N(A)$ and also on B. For if they satisfied any homology on $S - N(A)$, or on B, the corresponding combination of i-cycles of S_1^i would bound an $(i+1)$-chain on $S - N(A)$. Since the cycles of S_1^i are on $S - \bar{N}(A)$, we could deform part of the $(i+1)$-chain along the projecting curves of the normal neighborhood into $S - \bar{N}(A)$, without moving the cycles of S_1^i. Hence the combination of cycles of S_1^i in question would bound on $S - \bar{N}(A)$, a contradiction. We conclude that the cycles of C_2^i are independent on $S - N(A)$ and on B.

Since they are independent on B and each of them bounds on $\bar{N}$, they can be counted in the set C_a^i. Thus (8.4) is proved, except that for $i = n-1$ we must establish the existence of one more cycle of the set C_a^i. Now the boundary of $\bar{N}(A)$ is easily shown to determine an $(n-1)$-cycle which bounds on $\bar{N}$ and on $S - N(A)$, but not on B. Since the new $(n-1)$-cycle bounds on $S - N(A)$, and the cycles of C_2^{n-1} are independent on $S - N(A)$, it must be independent of them on B; for $S - N(A)$ contains B. Hence (8.4) is valid for $i = n-1$, as well as for $i = 0, 1, \cdots, n-2$.

We can now verify the relations of Theorem 7, with the symbols preceded by Δ's, very easily by use of (8.1), (8.2), (8.3) and (8.4), and the relations $a^n = c_a^n = c_d^{n-1} = 0$. That $a^n = c_a^n = 0$ follows from the facts that $\bar{N}$ does not cover S and that C contains no n-cells. If c_d^{n-1} were not zero, an $(n-1)$-cycle composed of cells of C would bound both on $\bar{N}$

[19] J. W. Alexander, A proof and extension of the Jordan-Brouwer separation theorem, Trans. Amer. Math. Soc., vol. 23 (1922), pp. 333–349. Other proofs have been given. The theorem holds for Betti numbers absolute or mod. m, as well as mod. 2 (Lefschetz I, Chap. III).

[20] δ_j^i is one or zero according as $i = j$ or $i \neq j$.

and on B. The difference of two such bounded chains composed of cells of $\bar{N}$ and of B respectively, would then be a non-vacuous n-cycle composed of cells of D. That is impossible, since D does not cover S. Therefore $c_d^{n-1}=0$, and the proof of Theorem 6 is complete.

THEOREM 7. *Let M_i denote the sum of the ith type numbers of all the connected critical sets of f in R. Let R_i, H_i, L_i denote the ith Betti numbers of R, of the sum of all loci of maximum, and of the sum of all loci of minimum, respectively. Then the following relations* (8.5), $\cdots$, (8.10) *are valid.*

$$M_i \geqq R_i + H_{n-i} + H_{n-i-1}, \qquad i = 0, 1, \cdots, n-1; \tag{8.5}$$

$$M_i \geqq L_i + L_{i-1} - R_{i-1}, \qquad i = 0, 1, \cdots, n; \tag{8.6}$$

$$M_i \geqq L_i + H_{n-i}, \qquad i = 0, 1, \cdots, n; \tag{8.7}$$

$$(-1)^k \sum_{i=0}^{k} (-1)^i M_i \geqq (-1)^k \sum_{i=0}^{k} (-1)^i R_i + H_{n-k-1}, \qquad k = 0, 1, \cdots, n-1; \tag{8.8}$$

$$(-1)^k \sum_{i=0}^{k} (-1)^i M_i \geqq (-1)^k \sum_{i=0}^{k-1} (-1)^i R_i + L_k, \qquad k = 0, 1, \cdots, n-1; \tag{8.9}$$

$$\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i R_i. \tag{8.10}$$

Proof. Let us consider the changes in Betti numbers occuring when we pass from a locus $f \leqq a$ to a locus $f \leqq b$, where $a < b$ and neither is a critical value. According to Lemma 4 no changes occur unless there is a critical value between a and b. Therefore the ith Betti number of R is the sum of the ΔR_i appearing in Theorem 6, for all the critical values of f. Consequently relations (8.5), $\cdots$, (8.10) can be obtained by adding the corresponding relations with every symbol preceded by Δ (Theorem 6). Thus Theorem 7 is proved.

9. The type numbers of the loci of minimum and of maximum.

THEOREM 8. *If K is a locus of minimum, its ith type number is $R_i(K)$, $i = 0, 1, \cdots, n$.*

Proof. By Theorem 3 it equals $R_i(\bar{N}; C)$, with $\bar{N}$ and C as defined in § 4. Since K is a locus of minimum, $\bar{N} = K$ and C is vacuous. Hence $R_i(\bar{N}; C) = R_i(K)$, and the theorem follows.

THEOREM 9. *The type number M_0 equals the number of parts in the sum of all loci of minimum.*

Proof. According to Theorem 8 it will be sufficient to show that a critical set K which does not contain a locus of minimum can contribute nothing to M_0. We may assume that K is connected. According to

(6. 1.), $M_0(K) = a^0$, the number of 0-cycles on $\bar{N}$ independent of those on C. Since K is not a locus of minimum, C is not vacuous, and is connected to K. Hence any point of $\bar{N}$ can be connected with C by a 1-cell on $\bar{N}$. Therefore $a^0 = 0$, and the theorem is proved.

THEOREM 10. *If K is a locus of maximum, $M_i(K) = R_{n-i}(K)$, $i = 0, 1, \cdots, n$.*

Proof. In this case C is the complete point-set boundary of $\bar{N}$ in S, since N is a neighborhood of K in S. It follows that N and $S-D$ are not connected, so that $R_i(S-B) = R_i(N+S-D) = R_i(N) + R_i(S-D)$. By the Alexander duality theorem we have

$$\left.\begin{aligned} R_i(D) &= R_{n-i-1}(S-D) + \delta_0^i - \delta_{n-1}^i, \\ R_i(B) &= R_{n-i-1}(S-D) + R_{n-i-1}(N) + \delta_0^i - \delta_{n-1}^i, \end{aligned}\right\} \quad i = 0, 1, \cdots, n-1;$$

where the relations are seen to hold for $i = n$, under the convention that $R_{-1} = 0$ for any configuration. In the proof of Theorem 6 we showed that $\Delta R_i = R_i(D) - R_i(B)$. From the relations above we infer that $\Delta R_i = -R_{n-i-1}(N)$. By Lemma 3, N and K have the same Betti numbers. Therefore

$$\Delta R_i = -R_{n-i-1}(K), \qquad i = 0, 1, \cdots, n. \tag{9.1}$$

We shall now prove that $a^i = c_d^i = 0$, $i = 0, 1, \cdots, n$. That $a^i = 0$ follows easily from the facts that C is the complete boundary of $\bar{N}$ and that n-space (equals S minus point at infinity) contains no non-bounding cycles other than a single 0-cell. Also $c_d^i = 0$, for if an i-cycle, say C_3^i, composed of cells of C, bounds both on $\bar{N}$ and on B, the difference of two corresponding bounded chains composed of cells of D must form an $(i+1)$-cycle, which in turn bounds an $(i+2)$-chain composed of cells of S. The boundary of the part of the latter in $\bar{N}$, less the $(i+1)$-chain of $\bar{N}$ bounded by C_3^i, is then an $(i+1)$-chain of C bounded by $(-C_3^i)$. Thus C_3^i bounds on C. From the definition of c_d^i it follows that $c_d^i = 0$.

Since $a^i = c_d^i = 0$, from (6.1) and (8.2) we obtain the following relations.

$$M_i(K) = c_a^{i-1}; \tag{9.2}$$

$$\Delta R_i = -c_a^i. \tag{9.3}$$

(The proof of (8.2) is valid for any critical set K.) We can now verify Theorem 10 very easily by use of (9.1), (9.2) and (9.3).

THEOREM 11. *The type number M_n equals the number of parts in the sum of all loci of maximum.*

and on B. The difference of two such bounded chains composed of cells of $\bar{N}$ and of B respectively, would then be a non-vacuous n-cycle composed of cells of D. That is impossible, since D does not cover S. Therefore $c_d^{n-1} = 0$, and the proof of Theorem 6 is complete.

THEOREM 7. *Let* M_i *denote the sum of the ith type numbers of all the connected critical sets of* f *in* R. *Let* R_i, H_i, L_i *denote the ith Betti numbers of* R, *of the sum of all loci of maximum, and of the sum of all loci of minimum, respectively. Then the following relations* (8.5), $\cdots$, (8.10) *are valid.*

$$M_i \geqq R_i + H_{n-i} + H_{n-i-1}, \quad i = 0, 1, \cdots, n-1; \tag{8.5}$$

$$M_i \geqq L_i + L_{i-1} - R_{i-1}, \quad i = 0, 1, \cdots, n; \tag{8.6}$$

$$M_i \geqq L_i + H_{n-i}, \quad i = 0, 1, \cdots, n; \tag{8.7}$$

$$(-1)^k \sum_{i=0}^{k} (-1)^i M_i \geqq (-1)^k \sum_{i=0}^{k} (-1)^i R_i + H_{n-k-1}, \quad k = 0, 1, \cdots, n-1; \tag{8.8}$$

$$(-1)^k \sum_{i=0}^{k} (-1)^i M_i \geqq (-1)^k \sum_{i=0}^{k-1} (-1)^i R_i + L_k, \quad k = 0, 1, \cdots, n-1; \tag{8.9}$$

$$\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i R_i. \tag{8.10}$$

Proof. Let us consider the changes in Betti numbers occuring when we pass from a locus $f \leqq a$ to a locus $f \leqq b$, where $a < b$ and neither is a critical value. According to Lemma 4 no changes occur unless there is a critical value between a and b. Therefore the ith Betti number of R is the sum of the ΔR_i appearing in Theorem 6, for all the critical values of f. Consequently relations (8.5), $\cdots$, (8.10) can be obtained by adding the corresponding relations with every symbol preceded by Δ (Theorem 6). Thus Theorem 7 is proved.

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Proof. According to Theorem 8 it will be sufficient to show that a critical set K which does not contain a locus of minimum can contribute nothing to M_0. We may assume that K is connected. According to

(6. 1.), $M_0(K) = a^0$, the number of 0-cycles on $\bar{N}$ independent of those on C. Since K is not a locus of minimum, C is not vacuous, and is connected to K. Hence any point of $\bar{N}$ can be connected with C by a 1-cell on $\bar{N}$. Therefore $a^0 = 0$, and the theorem is proved.

THEOREM 10. *If K is a locus of maximum, $M_i(K) = R_{n-i}(K)$, $i = 0, 1, \cdots, n$.*

Proof. In this case C is the complete point-set boundary of $\bar{N}$ in S, since N is a neighborhood of K in S. It follows that N and $S-D$ are not connected, so that $R_i(S-B) = R_i(N+S-D) = R_i(N) + R_i(S-D)$. By the Alexander duality theorem we have

$$\begin{aligned} R_i(D) &= R_{n-i-1}(S-D) + \delta_0^i - \delta_{n-1}^i, \\ R_i(B) &= R_{n-i-1}(S-D) + R_{n-i-1}(N) + \delta_0^i - \delta_{n-1}^i, \end{aligned} \qquad i = 0, 1, \cdots, n-1;$$

where the relations are seen to hold for $i = n$, under the convention that $R_{-1} = 0$ for any configuration. In the proof of Theorem 6 we showed that $\Delta R_i = R_i(D) - R_i(B)$. From the relations above we infer that $\Delta R_i = -R_{n-i-1}(N)$. By Lemma 3, N and K have the same Betti numbers. Therefore

$$\Delta R_i = -R_{n-i-1}(K), \qquad i = 0, 1, \cdots, n. \tag{9.1}$$

We shall now prove that $a^i = c_d^i = 0$, $i = 0, 1, \cdots, n$. That $a^i = 0$ follows easily from the facts that C is the complete boundary of $\bar{N}$ and that n-space (equals S minus point at infinity) contains no non-bounding cycles other than a single 0-cell. Also $c_d^i = 0$, for if an i-cycle, say C_3^i, composed of cells of C, bounds both on $\bar{N}$ and on B, the difference of two corresponding bounded chains composed of cells of D must form an $(i+1)$-cycle, which in turn bounds an $(i+2)$-chain composed of cells of S. The boundary of the part of the latter in $\bar{N}$, less the $(i+1)$-chain of $\bar{N}$ bounded by C_3^i, is then an $(i+1)$-chain of C bounded by $(-C_3^i)$. Thus C_3^i bounds on C. From the definition of c_d^i it follows that $c_d^i = 0$.

Since $a^i = c_d^i = 0$, from (6.1) and (8.2) we obtain the following relations.

$$M_i(K) = c_a^{i-1}; \tag{9.2}$$

$$\Delta R_i = -c_a^i. \tag{9.3}$$

(The proof of (8.2) is valid for any critical set K.) We can now verify Theorem 10 very easily by use of (9.1), (9.2) and (9.3).

THEOREM 11. *The type number M_n equals the number of parts in the sum of all loci of maximum.*

Proof. According to Theorem 10 it will be sufficient to show that if K is a critical set not containing a locus of maximum, then $M_n(K) = 0$. We shall use (6.1), with $i = n$:

$$M_n(K) = a^n + c_d^{n-1} + c_a^{n-1}. \tag{9.4}$$

Now a^n is zero, since $\bar{N}$ does not cover all of S. It remains to prove that $c_d^{n-1} = c_a^{n-1} = 0$.

Let C_4^{n-1} be any $(n-1)$-cycle on C composed of cells of D'' (§ 4). If C_4^{n-1} bounded on $\bar{N}$, it would bound an n-chain, say A_1^n, composed of cells of D'' in $\bar{N}$. I say that if A_1^n were not vacuous, the point-set boundary of the locus of A_1^n in S would be entirely on C. For if two n-cells of D'' are given having a common vertex on K, they can be made end terms of a sequence of n-cells and $(n-1)$-cells alternating in order and all incident on the vertex in question. Since none of the $(n-1)$-cells could figure in the chain boundary of A_1^n, it follows that if one of the n-cells were in A_1^n then the other would also be in A_1^n. Therefore the point-set boundary of A_1^n could have no points on N; and since it must be on $\bar{N}$ we conclude that it would be entirely on C, as was stated. Therefore there would be a locus of maximum in the interior part of the point-set locus of A_1^n; for A_1^n would contain some cells of K, $f = c$ on K, and $f < c$ on C. The locus of maximum would be on K, since K contains all the critical points in N. Since this would contradict our assumption that K contains no locus of maximum, we infer that A_1^n must be vacuous. Hence C_4^{n-1} cannot bound on $\bar{N}$ if it is not vacuous. Consequently $c_d^{n-1} = c_a^{n-1} = 0$, as was to be proved.

10. **Remarks on the definition of type number.** Let a critical set K be given, and a corresponding locus D, as defined in § 4. According to Theorem 5 $M_i(K)$ is the sum of the number of dropped $(i-1)$-cycles and the number of new i-cycles, in passing from $D-K$ to D. It can be shown that the definitions of type number in the papers Morse I and B I lead to this same result. Consequently the definition of the present paper, when applied to the cases treated in those earlier papers, gives the same values to the type numbers. The same remark can be made regarding the paper B II. For in that paper the definitions of M_0 and M_2 have the forms of Theorems 9 and 11, respectively, of this paper. From the relation

$$M_0 - M_1 + M_2 = R_0 - R_1 + R_2,$$

which holds in both cases, we infer that the definitions of M_1 are also equivalent.

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ON COMPACT SPACES.*

By Solomon Lefschetz.

The chief purpose of the present paper is to give certain applications of the sequences of complexes approximating compact metric spaces introduced by Alexandroff.[1] From an important lemma regarding the complexes we derive anew certain deformation theorems of Alexandroff's, then prove Menger's immersion theorem and his surmise regarding a certain type of universal n-space.[2] The paper ends with the study of an interesting class of spaces defined by approximating sequences analogous to, but in general different from the Alexandroff sequences of complexes.

The unavoidable link between dimensionality and combinatorial topology is the Lebesgue-Urysohn-Menger theorem on the order of covering sets. This theorem is all that we borrow from that theory. Aside from that, up to and including the universal space, we need only a few elementary properties of complexes. These are completely established in our Colloquium Lectures *Topology,* whose notations and terminology are used wherever practicable.[3] Indeed had we obtained our basic lemma early enough the present paper would have been incorporated in Ch. VII of the book after § 4.

1. Let L be a compact metric space, $\Sigma = \{F^\alpha\}$ a covering of L by a finite number of closed sets F^α. With the covering there is associated a certain complex Φ, the *skeleton* of Σ, introduced by Alexandroff,[4] as follows: There is a vertex A^α of Φ for each F^α, and a k-simplex $\sigma_k = A^{\alpha_0} A^{\alpha_1} \cdots A^{\alpha_k}$ of Φ for every intersection $F^{\alpha_0} F^{\alpha_1} \cdots F^{\alpha_k} \neq 0$.

* Received January 31, 1931.

[1] These Annals vol. 30 (1928–29), pp. 101–187.

[2] See Menger: Dimensionstheorie. As the present manuscript was practically finished there appeared a paper by Nöbiling, Math. Ann., vol. 104 (1930), pp. 71–80, in which there is a different proof of the immersion theorem, and also a proof of the existence of another type of universal n-space, likewise surmised by Menger. We show (No. 29) that this space contains one similar to our own, hence Nöbiling's result follows from ours. For $n = 0$ all these results coalesce and were established by Sierpiński; for $n = 1$ our results go back to Menger; he also sketched a proof of the immersion theorem for $n > 1$ but did not carry it through.

[3] We shall have occasion to consider only finite, simplicial complexes, no two cells of any complex K having the same vertices. The successive derived of K are denoted by $K', K'', \cdots$ the new vertices of $K^{(i)}$ being the centroids of the carrying cells of $K^{(i-1)}$. The chief purpose of this last condition is to insure that mesh $K^{(i)} \to 0$ with $1/i$. When we state that $K \subset S_r$ we mean that its cells are simplexes of S_r.

[4] The results of the present number are found in his paper already quoted. He uses the term "nerve" for Φ but "skeleton" seems more descriptive.

Φ is the sum of the σ's and it is closed. The dimension n of Φ is the *order* of Σ. It is characterized by the property that there are groups of $n+1$ sets F, but no more, whose intersection $\neq 0$.

Let $F^{\alpha_0} \cdots F^{\alpha_k} = 0$, and let P be any point of L. Since L is compact the average of the distances $d(F^{\alpha_i}, P)$ has a lower bound $\eta > 0$. Since the total number of sets F is finite there is only a finite number of η's, —let $\xi (> 0)$, be the least of them. ξ has the following important property: If any subset A of L whose diameter $< \xi$ meets a group of F's, their intersection $\neq 0$. For the average distance from any point of A to one of the F's is $< \xi$.

Let $\Sigma^* = \{F^{*\beta}\}$ be an ε-covering $(d(F^{*\beta}) < \varepsilon)$ of L, $\varepsilon < \frac{1}{2}\xi$, and let Φ^* be the skeleton of Σ^*, with $A^{*\beta}$ as its vertices. We associate with each $A^{*\beta}$ one of the vertices A^α of Φ belonging to an F which meets $F^{*\beta}$. To the vertices of any σ of Φ^* correspond those of a σ of Φ. From this follows immediately that by barycentric extension to the points of the simplexes, the correspondence defines a continuous single-valued simplicial transformation T of Φ^* into Φ. In the sequel we shall merely say "simplicial transformation", it being understood that it is barycentric and single-valued.

2. Lemma. *Given any ε-covering $\sum^*$, ε sufficiently small, there exists a simplicial transformation U of its skeleton Φ^* into the derived Φ' of the skeleton Φ of $\sum$, such that if A^* is any vertex of Φ^*, either $T \cdot A^*$ coincides with $U \cdot A^*$, or else they are joined by an edge of Φ'.*

To establish the first part it is only necessary to show that there exists a U transforming the vertices in the correct way.

Consider one of the sets $E_k = F^{\alpha_0} \cdots F^{\alpha_k} - \sum F^{\beta_0} \cdots F^{\beta_{k+1}} \neq 0$, where the sum is extended to all the combinations of upper indices taken $k+2$ at a time. E_k represents the part of $F^{\alpha_0} \cdots F^{\alpha_k}$ which belongs to no similar set of more than $k+1$ intersecting E's. The total number of E's is finite and we designate them by E_k^i. The following properties are obvious:

I. $E_h^i \cdot E_h^j = 0, \quad i \neq j$.

II. If $\bar{E}_{h-1} \cdot \bar{E}_h \neq 0$, the F's intersecting in E_h include the same for E_{h-1}.

3. From I, II plus an elementary induction we conclude that there may be associated with each E_h^i a closed set G_h^i whose points are not farther than $\xi/4$ from E_h^i and such that

III. $E_h^i \subset G_h^i + \sum G_{h+1}^l$, where, if G_{h+1}^l is any term of the sum, $\bar{E}_h^i \cdot \bar{E}_{h+1}^l \neq 0$.

IV. G_h^i meets no G_h^j, $i \neq j$, and the only sets G_{h+1} which it meets are those present in the sum in III.

V. $G_h^i \cdot E_k = 0$ for $k \geqq h$ unless $E_k = E_h^i$.

From II, III, IV and from V follow respectively

VI. If $G_h^i \cdot G_k^j \neq 0$, $k > h$, the F's intersecting in E_k^j include the same for E_h^i.

VII. The only F's which G_h^i meets are those intersecting in E_h^i.

4. We will now prove the lemma for $\epsilon < \xi/4$ and also $< 1/2$ the distance from any G to a G or to an F which it does not meet. With ϵ thus restricted we have:

VIII. If $F^{*\beta} \cdot F^{*\gamma} \neq 0$, $F^{*\beta} \cdot G_h^i \neq 0$, $F^{*\gamma} \cdot G_k^j \neq 0$ then

$h \neq k$, say $h < k$, implies that the F's intersecting in E_h^i are included among the same for E_k^j;

$h = k$ implies $i = h$ (if two F^*'s intersect they cannot intersect two different G's with the same subscript h).

In particular a given F^* can only meet one G_h with given h and if it meets G_h^i, G_k^j, $h < k$, then the two G's are related as above.

IX. If $F^{*\beta}$ meets G_h^i, the only sets F^α which $F^{*\beta}$ can meet are those intersecting in E_h^i.

Now for each E_h^i we have a definite group $F^{\alpha_0}, \cdots, F^{\alpha_h}$ intersecting in it. To their intersection there corresponds $\sigma_h = A^{\alpha_0} \cdots A^{\alpha_h}$ of Φ, and hence a unique vertex B^β of Φ', on that simplex.

Let now $A^{*\gamma}$ be any vertex of Φ^* with its associated set $F^{*\gamma}$. Among the sets G_h which $F^{*\gamma}$ meets there is one with the highest subscript h and it is unique by VIII; let it be G_h^δ. We will define as the transform of $A^{*\gamma}$ the vertex B^β of Φ' corresponding to E_h^δ in the manner just described. Let T be the transformation of vertices thus defined. In order to be able to extend T barycentrically over Φ^*, and to have a single-valued transformation such as announced, all that is necessary is to show that if $\sigma_k = A^{*\gamma_0} \ldots A^{*\gamma_k}$ is any simplex of Φ^*, its vertices have for transforms the vertices of a simplex of Φ' (*Topology* p. 85).

Since σ_k is a simplex of Φ^*, we have $F^{*\gamma_0} \ldots F^{*\gamma_k} \neq 0$. Let then $G_{h_i}^{\delta_i}$ correspond to $F^{*\gamma_i}$ in the same manner as above. According to VIII if the h's are in increasing order, we have that the F's intersecting in $E_{h_i}^{\delta_i}$ are included among the same for $i+1$. The first set of F's determines a σ_{h_i} of Φ, the second a $\sigma_{h_{i+1}}$, and since the second set includes the first, σ_{h_i} is a face of $\sigma_{h_{i+1}}$, or else coincides with it. As a consequence the simplexes σ_{h_i} constitute a mutually incident set and therefore the corresponding B's are vertices of a simplex of Φ' (*Topology,* p. 117). But these B's are precisely the vertices $T \cdot A^{*\gamma_i}$, so that the latter are actually the vertices of a simplex of Φ'. The barycentric extension of T to the whole of Φ^* is the required transformation. We shall also call it T.

5. We pass to the last part of the lemma. Let $A = U \cdot A^*$. If F, F^* are the sets of Σ and Σ^* associated especially with A and A^*, we know that F^* meets F. Therefore, by IX, F is one of the sets of Σ intersecting in E_h^δ. This implies that $B = T \cdot A^*$ is a vertex of Φ' on

a simplex of Φ which has A for vertex. Owing to the structure of Φ', B is then on the closed star of A as to Φ' and either $B = A$ or else AB is an edge of Φ'. This completes the proof of the lemma.

6. We will now consider a sequence of coverings $\{\Sigma^i\} = \{\{F^{i\alpha}\}\}$, where the mesh ε_i of $\Sigma^i \to 0$ with $1/i$ monotonely and so rapidely that any two Σ^i, Σ^{i+1} are related like the two coverings of the lemma, with T^i, U^i as the two associated transformations. We shall further restrict the ε's in a moment.

Consider the sets $E_h^{i\alpha}$ such as in No. 3, corresponding to Σ^i. The set $E_h^{i\alpha}$ is the intersection of a certain number of sets $F^{i\beta}$, and these in turn are met by certain sets $F^{i\gamma}$ (among which we include the sets $F^{i\beta}$ themselves). Let $V^{i\alpha}$ be the open *core* of $\Sigma F^{i\gamma}$:

$$V^{i\alpha} = L - \overline{(L - \Sigma F^{i\gamma})}.$$

Evidently $V^{i\alpha} \supset \bar{E}^{i\alpha}$ and hence $d(E^{i\alpha}, L - V^{i\alpha}) > 0$. For a given i these distances are in finite number; if η_i is the least of them, we shall take $\varepsilon_{i+1} < \frac{1}{2}\eta_i$. This is the other restriction on the ε's mentioned above.

It is to be observed that from any $\{\Sigma^i\}$ whatever, whose ε's merely $\to 0$, we can extract a subsequence whose ε's are restricted in the present manner. For all practical purposes the subsequence can take everywhere the place of the initial sequence.

We will now define a new space whose points are those of L and whose determining neighborhoods are the V's. A neighborhood of a point P is a $V^{i\alpha}$ whose $E^{i\alpha} \supset P$. There is one and only one for each i. Since the F^i's whose sum is $\bar{V}^{i\alpha}$ are not farther than ε_i from $E^{i\alpha}$, $d(V^{i\alpha}) \leqq 4\varepsilon_i$. From this follows readily that the space just defined is identical with L.

7. Let $V^{i\alpha}$, $V^{i+1,\beta}$ be two neighborhoods of P. Since $E^{i+1,\beta}$ intersects $E^{i\alpha}$ and since $\varepsilon_{i+1} < \frac{1}{2}\eta_i$, the F^{i+1}'s whose sum is $\bar{V}^{i+1,\beta}$ are on $V^{i\alpha}$ and hence $\bar{V}^{i+1,\beta} \subset V^{i\alpha}$. We have then $P = \Pi V^{i\alpha}$.

Let $\sigma^{i\alpha}$ be the simplex of Φ^i corresponding to $E^{i\alpha}$, that is whose vertices correspond to the sets $F^i \supset E^{i\alpha}$ and let $N(\bar{\sigma}^{i\alpha})$ be the Φ^i-neighborhood of $\bar{\sigma}^{i\alpha}$, (= the sum of the simplexes of Φ^i having a vertex in common with $\sigma^{i\alpha}$). The simplexes of $\bar{N}(\bar{\sigma}^{i\alpha})$ correspond to the sets $F^{i\gamma}$ and their intersections. Since the sets F^{i+1} whose sum is $V^{i+1,\beta}$ meet only the sets F^i whose sum is $\bar{V}^{i\alpha}$, $T^i \cdot \bar{N}(\bar{\sigma}^{i+1,\beta}) \subset \bar{N}(\sigma^{i\alpha})$.

8. Given any point P of L there is a unique set E^i, say $E^{i\alpha}$, carrying P and $P = \Pi E^{i\alpha}$. Conversely if we have a sequence $\{E^{i\alpha}\}$ such that $\Pi E^{i\alpha} \neq 0$, since $d(E^i) \to 0$ with $1/i$, $\Pi E^{i\alpha} = P$, a single point of L. If $\sigma^{i\alpha}$ corresponds to $E^{i\alpha}$ in Φ^i, each $\bar{N}$ of $\{\bar{N}(\bar{\sigma}^{i\alpha})\}$ is projected onto its predecessor by the proper T. A sequence of such $\bar{N}$'s associated with a sequence of E's whose intersection $\neq 0$, is called a *projection sequence*

(= p. s.). The set of all p. s. and the points of L are in $(1-1)$ correspondence.

Let $P \subset V^{i\alpha}$. The sets F^i through P are all included among the sets $F^{i\gamma}$, and hence, if $\{\bar{N}(\sigma^{i\beta})\}$ is the p. s. for P, $\sigma^{i\beta} \subset \bar{N}(\sigma^{i\alpha})$. Conversely if the latter condition is satisfied $P \subset V^{i\alpha}$. Thus the points of $V^{i\alpha}$ are in $(1-1)$ correspondence with the set of all p. s. $\{\bar{N}(\sigma^{i\beta})\} = Q$ such that $\sigma^{i\beta}$ is a simplex of $\bar{N}(\sigma^{i\alpha})$. Let $W^{i\alpha}$ designate the set of all p. s. $\{\bar{N}(\sigma^{i\beta})\}$ such as just considered. Let us consider the Q's as points of a new space $\mathfrak{L}$, with the W's as the determining neighborhoods, where $W^{i\alpha}$ is a neighborhood of any $Q' = \{\bar{N}(\sigma^i)\}$ whose $\sigma^i = \sigma^{i\alpha}$. Between the points of L and $\mathfrak{L}$ there is a $(1-1)$ correspondence wherein $V^{i\alpha}$ and $W^{i\alpha}$ correspond to one another. Therefore $\mathfrak{L}$ is homeomorphic to L.[5]

9. So far nothing has been said concerning dim L. *We now assume that L is an n-space.* As a consequence, we can choose an infinite sequence $\{\Sigma^i\}$ whose coverings are all of order $n+1$ (Lebesgue-Urysohn-Menger). Assume then that such a choice has been made. One of the first and simplest applications of our lemma is the proof of certain mapping theorems brought out by Alexandroff. The first is expressed in terms of *ε-mapping.* We understand by ε-mapping μ of L onto a metric space L' a continuous single-valued transformation of L into L', such that whatever the point Q of L', $d(\mu^{-1}Q) < \varepsilon$. In speaking of ε-deformation we shall understand that L and the Φ's are on the Hilbert parallelotope $\mathfrak{H}$, the Φ's being constructed as in *Topology* p. 326. Then

THEOREM. *Necessary and sufficient conditions in order that L be an n-space, are that n be the least integer such that for every $\varepsilon > 0$,*

(a) *L is ε-mappable on an n-complex whathever ε, or else,*

(b) *L is ε-deformable onto an n-complex over $\mathfrak{H}$.*

10. We begin with a preliminary observation. Let K, K^* be two complexes, K', $K^{*\prime}$, their first derived, and suppose that $K = T \cdot K^*$, T simplicial. Then $K' = T' \cdot K^{*\prime}$, where T' is simplicial and transforms the cells of K^* like T itself. Take indeed the T' transforming the vertices of K^* like T, and the vertex of $K^{*\prime}$, K'^* on σ^* of K^* into that of K' on $\sigma = T \cdot \sigma^*$, then extend T' barycentrically over $K^{*\prime}$. The transformation thus obtained behaves manifestly as required.

We need also the following property: Referring to the lemma and in the notations used in its proof, $U \cdot A^*$ is on the star of $T \cdot A^*$ relatively to Φ. Hence if σ is any simplex of Φ^*, $U \cdot \bar{\sigma}$ is on the Φ-neighborhood

[5] A space analogous to $\mathfrak{L}$ has been considered by Alexandroff, loc. cit., but in his treatment the elements of the p. s. are simplexes, which requires that $\{\Sigma^i\}$ be a so-called *subdivision* sequence. The p. s. here introduced do not demand that and hence enable us to proceed more speedily.

of $T \cdot \bar{\sigma}$, and therefore the closed Φ'-neighborhood of $U \cdot \bar{\sigma}$ is on the closed Φ-neighborhood of $T \cdot \bar{\sigma}$.

11. When L is an n-space we can assume that the Φ's are n-complexes. Let us write for convenience Φ^{ik} for $(\Phi^i)^{(k)}$, the kth derived of Φ^i. By repeated application of the lemma together with the second property in No. 10, we find that Φ^{i+k+1} is simplically transformable into Φ^{ik} in such a manner that if $\{\bar{N}(\bar{\sigma}^j)\}$ is any p. s., $\bar{N}(\bar{\sigma}^{i+1})$, $\bar{N}(\bar{\sigma}^{i+2})$, $\cdots$ are transformed into mutually inclusive subcomplexes of Φ^{i1}, Φ^{i2}, $\cdots$. Owing to the structure of the N's the diameter of the transform of $\bar{N}(\bar{\sigma}^{i+j}) < 3$ mesh Φ^{ij}, and hence $\rightarrow 0$ with $1/j$. Therefore the subcomplexes converge on a certain point Q of Φ^i. Let τ_i be the transformation of L into Φ^i carrying into Q the point P of L corresponding to the p. s. Through τ_i two points, P, P' corresponding to $\{\bar{N}(\bar{\sigma}^j)\}$, $\{\bar{N}(\bar{\sigma}'^j)\}$ go into points on a closed cell of Φ^{ik} provided that $\sigma^j \subset \bar{N}(\bar{\sigma}'^j)$ for $j \leqq i+k+1$. Since mesh $\Phi^{ik} \rightarrow 0$ with $1/k$, τ_i is continuous. Two points P, P' of L cannot have the same transform unless they belong to the same neighborhood $V^{i\alpha}$ of No. 6. Hence if ξ_i is the maximum diameter of $V^{i\alpha}$ for i fixed, τ_i is a ξ_i-mapping of L onto the n-complex Φ^i. Since $\xi_i \rightarrow 0$ with $1/i$, we see that the condition of part (a) of the theorem is necessary.

The sufficiency of (a), is proved as follows (Alexandroff loc. cit.): By what precedes since L cannot be mapped on a complex of dimension $< n$, $\dim L \geqq n$. Let μ be an ϵ-mapping of L onto K_n. There exists an $\eta > 0$ such that if Q, Q' are points on K_n not farther than η apart, $\mu^{-1}(Q+Q')$ is of diameter $< 2\epsilon + \eta$. Replace K_n by a simplicial subdivision, still to be called K_n, and so chosen that its first derived K'_n is of mesh $< \eta/2$. Then the system $\{s^\alpha\}$, of the closed stars of the vertices of K_n relatively to K'_n, is an η-covering of K_n whose order is n. Now the system $\{\mu^{-1} s^\alpha\}$ is a 3ϵ-covering of K_n whose order is likewise n. Since such a covering exists for every ϵ, $\dim L \leqq n$, hence $\dim L = n$. This proves part (a) of the theorem.

Regarding part (b) it is readily seen that if P is any point of L, $d(P, \mu P) \rightarrow 0$ with $1/i$, and hence, for i large enough, μ is an ϵ-deformation. The converse of (b) follows from the fact that when L is ϵ-deformable onto K_n it is η-mappable onto it, where $\eta \rightarrow 0$ with ϵ.

12. Remark. The transformation $\tau_{ij} = U_i \cdots U_{i+j-2}\, U_{i+j-1}$ represents a simplicial transformation of Φ^{i+j} into Φ^{ij} such that $\tau_i = \tau_{ij}\, \tau_{i+j}$. If P, Q are points of L or Φ^j, the points $\tau_i P$, $\tau_{ij} Q$ of Φ^i are called their *projections* on Φ^i.

13. We shall need the following elementary properties:

(a) Let K_n be an n-complex and $A^1, \cdots, A^q$ its vertices. Given q arbitrary points $B^1, \cdots, B^q$ of S_{2n+1} we can construct a K^*_n on S_{2n+1} having the

same structure as K_n and such that if A^{*i} is the vertex of K_n^* corresponding to A^i, every A^{*i} is in a prescribed neighborhood of B^i.

Proof. Choose q arbitrary points A^{*i} in S_{2n+1} and take all simplexes $A^{*i} \cdots A^{*j}$ such that $A^i \cdots A^j$ is a simplex of K_{2n}. Their totality will fail to add up to a complex having the structure of K_n when and only when the A^*'s satisfy certain algebraic equations in finite number. Hence in any vicinity of q points of S_{2n+1} there are groups of q points not restricted like the special groups.

(b) If $K_n \subset S_r$ has for mesh η, its derived K_n' has its mesh $\leqq \eta\left(1-\dfrac{1}{n+1}\right)$.

Proof. It is sufficient to take for K_n a $\bar{\sigma}_n$. Now the diameter of a σ is equal to the length of its longest edge. For in the first place, given a segment on a convex polyhedral region of S_n there is one at least as long, parallel to it and with an end point at a vertex. Hence the longest segment will be issued from a vertex. But in a triangle ABC the length of any segment issued from A is included between AB and AC. Hence in a σ with A for vertex the longest segment issued from A will be on the boundary of σ, hence on a σ of lower dimension. Therefore by induction it is proved that $d(\sigma) =$ the longest edge of σ.

Now if G is the centroid of σ and AB the segment through G joining the vertex A to the opposite face, G divides AB in the ratio 1 to n. Therefore $AG,\ BG < d(\sigma)\cdot\left(1-\dfrac{1}{n+1}\right)$. This implies property (b) for $\bar{\sigma}$ and hence for K.

14. The Immersion Theorem. *Every compact metric n-space can be mapped topologically on an* S_{2n+1} (Menger).

The proof rests on the construction of a sequence $\{\Psi^i\}$ where Ψ^i is a complex on a fixed S_{2n+1} whose structure is that of Φ^i, and with the following properties: Let σ^i, $N(\sigma^i)$, T^i, etc., apply now to the Ψ's as before to the Φ's. There exists on S_{2n+1} a neighborhood $\mathfrak{N}(\sigma^i)$ of $\bar{N}(\bar{\sigma}^i)$ whose points are not farther than η_i from $\bar{N}(\bar{\sigma}^i)$ and such that:

I. $\eta_i \leqq \varepsilon_i \leqq \left(1-\dfrac{1}{n+1}\right)^{i-1} \varepsilon_1$, where ε_i is the mesh of Ψ^i;

II. if $\bar{N}(\bar{\sigma}^i)\cdot\bar{N}(\bar{\sigma}'^i) = 0$, likewise $\mathfrak{N}(\sigma^i)\cdot\mathfrak{N}(\sigma'^i) = 0$;

III. if $\sigma^i = T^i\cdot\sigma^{i+1}$, $\mathfrak{N}(\sigma^i) \supset \mathfrak{N}(\sigma^{i+1})$.

From the construction of the Φ's in terms of the coverings follows that no two cells of Φ^i have the same vertices. We can therefore construct Ψ^1 as required. Suppose now that we have already constructed every Ψ up to the index i. Let $\Psi^{i\prime}$ designate the derived of Ψ^i and let T'^i, U'^i denote the simplicial transformation of Φ^{i+1} into Ψ^i and $\Psi^{i\prime}$ whereby each vertex A^{i+1} of Φ^{i+1} goes into the images of $T^i A^{i+1}$ and $U^i A^{i+1}$ on Ψ^i and $\Psi^{i\prime}$.

By No. 13a, we can construct Ψ^{i+1} with the structure of Φ^{i+1} and such that every B^{i+1} is not farther than ζ_i from the corresponding $U'^i A^{i+1}$. Then if σ'^{i+1} is any cell of Φ^{i+1}, for ζ_i small enough, its image σ^{i+1} will be as near as we please to $U'^i \sigma'^{i+1}$, and hence, referring to No. 10, $\bar{N}(\bar{\sigma}^{i+1})$ will be as near as we please to $\bar{N}(T'^i \cdot \bar{\sigma}'^{i+1})$ where the two N's are respectively Ψ^i- and Ψ^{i+1}-neighborhoods.

It follows that for suitably small ζ_i, $\bar{N}(\bar{\sigma}^{i+1}) \subset \mathfrak{N}(T'^i \cdot \sigma'^{i+1})$, and ε_{i+1} will differ as little as we please from $\left(1 - \dfrac{1}{n+1}\right)\varepsilon_i$.

Therefore we can fulfill condition I as regards ε_{i+1}, then choose $\eta_{i+1} < \varepsilon_{i+1}$ and at all events so small that $\mathfrak{N}(\sigma^{i+1}) \subset \mathfrak{N}(T'^i \cdot \sigma'^{i+1})$, thus complying with III, and also so small that II holds. The required construction of the Ψ's is thus completed.

We now apply the term p. s. to the image of a p. s. made up of subcomplexes of the Ψ's. If $\{\bar{N}(\bar{\sigma}^i)\}$ is a p. s., according to I, $d(\bar{N}(\bar{\sigma}^i))$ and hence also $d(\mathfrak{N}(\sigma^i)) \to 0$ with $(1/i)$. Therefore, by III, $\mathfrak{N}(\sigma^i)$ converges on a point P^* of S_{2n+1}. We designate by L^* the set of all points P^*.

Consider L^* as a space whose defining neighborhoods are $L^* \cdot \mathfrak{N}(\sigma^i)$, the latter being a neighborhood for every P^* associated with it in the preceding manner ($\mathfrak{N}$ is member of a sequence $\to P^*$). The association of P^* defined by $\{\mathfrak{N}(\sigma^{i\alpha})\}$ with $Q = \{\bar{N}(\bar{\sigma}^{i\alpha})\}$ is a $(1-1)$ correspondence between L^* and the space L of No. 8, in which as a consequence of II, $W^{i\alpha}$ and $L^* \cdot \mathfrak{N}(\sigma^{i\alpha})$ correspond to one another point for point. Therefore L and L^* are homeomorphic, and so are L and L^*. Since $L^* \subset S_{2n+1}$, *this proves Menger's immersion theorem.*

15. We shall now transfer to the relations between the Ψ's and L^* the terminology of the Remark No. 12, In particular the τ's are now projections of L^* and Ψ^j onto Ψ^i. Now it is apparent that the preceding construction of L^* could be modified as follows: Replace Ψ^1 by $(\Psi^1)^{(j+1)}$ and Ψ^2 by Ψ^j. Then the neighborhoods $\mathfrak{N}(\sigma)$ being chosen with their points never farther from σ than, say, three times its diameter, we would obtain an L^* such that no point is farther than three times the mesh of $(\Psi^1)^{(j)}$ from its projection on Ψ^1. Since that mesh $\to 0$ with $1/j$, and since Ψ' can be replaced by any Ψ^i we have:

THEOREM. *There exists a topological image L^* of L on S_{2n+1} such that no point is farther than an assigned $\varepsilon > 0$ from its projection on Ψ^i.*

16. **Universal n-space.** The term "universal space" is now in general use in the following sense: Given a class $\{\mathfrak{R}\}$ of metric sets or spaces, if there exists a member $\mathfrak{R}_0$ of the class such that every $\mathfrak{R}$ can be topologically mapped on $\mathfrak{R}_0$, the latter is said to be a *universal* set or space

for the whole class. The class to be considered here is that of all compact metric n-spaces and we propose to construct a *universal n-space* for that class.

17. Let G be a point-set coincident with a subcomplex of K_n. The sum N of all cells of K with a vertex on G is an open subcomplex of K, the *K-neighborhood* of G. We have already considered such an N in No. 7, when G consists of a single closed cell. If G has the property that every cell of K with all its vertices on G is a cell of G, N is said to be a *normal* neighborhood. An equivalent property is that every cell of $N-G$ has some vertex $\not\subset G$. When N is normal through every point P of $N-G$ there passes a unique segment QR, on the same simplex σ of K as the point, with Q on G and R on the boundary of N. Q is on the face of σ on G, R on the opposite face; both points vary continuously when P so varies on $N-G$. The segment QR is the *projecting segment of P relatively to N*, Q and R the *projections* of P on G and on the boundary of N. As regards the boundary of N it is the sum of the faces of the simplexes of N without vertices on G.

In the sequel the neighborhoods to be considered are K'-neighborhoods of a subcomplex of K and these are always normal (*Topology* p. 91). The preceding situation will then hold automatically.

18. We introduce an infinite sequence $\{K^i_{2n+1}\}$ defined as follows:— (a) K^0_{2n+1} is a $\bar{\sigma}_{2n+1}$; (b) if K^i_n is the sum of the closed n-cells of K^i_{2n+1} then K^{i+1}_{2n+1} is the closure of the $(K^i_{2n+1})''$-neighborhood of K^i_n. Setting

$$\Delta = \Pi K^i_{2n+1} = \overline{\Sigma K^i_n},$$

we shall show that *Δ is a universal compact n-space.* This space, obviously compact, differs only from the one announced by Menger in that in his construction our simplexes are replaced by parallelotopes. As a matter of fact, with insignificant modifications, our discussion would hold if $\{K^i_{2n+1}\}$ were any sequence of convex complexes, each a subdivision of its predecessor, their mesh $\to 0$ with $1/i$. However the sequence here adopted is decidedly the most convenient.

19. We will establish the universal space property of Δ independently of the considerations of this number. Granting that result it is a simple matter to show that Δ is an n-space. For if L, L' are compact metric spaces and $L' \subset L$, we have $\dim L' \leqq \dim L$. This is indeed an immediate consequence of the Urysohn-Menger definition plus an elementary recurrence. Since Δ contains compact metric n-spaces, $\dim \Delta \geqq n$. On the other hand, if ε_i is the mesh of K^i_{2n+1}, K^{i+1}_{2n+1} can be ε_i-deformed into K^i_n by sliding its points along the projecting segments of the $(K^i_{2n+1})'$-neighborhood

of K_n^i. Since $\varDelta \subset K_{2n+1}^{i+1}$, $\varDelta$ is thus ε_i-deformable onto a K_n, and as $\varepsilon_i \to 0$ with $1/i$, $\dim \varDelta \leqq n$, and hence $= n$.

20. Any simplex of K_{2n+1}^i not on K_n^{i-1} is of type $\sigma_p\,\sigma_q$ where $\sigma_p \subset K_n^{i-1}$, and $\bar{\sigma}_q$ does not meet K_n^{i-1}. The sum of the simplexes σ_q will be designated by $\mathfrak{B}_{2n}^i$ and the sum of the (open) simplexes $\sigma_p\,\sigma_q$ by $\mathfrak{K}_{2n+1}^i$. The first will also be called the *boundary* of K_{2n+1}^i.

21. Let now L be the same n-space as before. We propose to construct a sequence of homeomorphs $\{L^i\}$ of L, $L^i \subset \mathfrak{K}_{2n+1}^i$, and show that $\lim L^i = L^*$, a space homeomorphic to L. Since $L^* \subset \varDelta$, this will prove that $\varDelta$ is a universal n-space. At the same time as L^i we shall consider topological images on $\mathfrak{K}_{2n+1}^i$ of the sequence set $\{\Psi^j\}$ associated with $\{\Sigma^j\}$, or of points and sets on L. These will all be designated by an additional superscript i, as Ψ^{ij}, P^i, $\cdots$.

In the construction we need a special sequence of neighborhood-pairs on L. If $\{V^j\}$ is an enumerable determining set of neighborhoods for L, the pairs in question (V^j, V^k) are those such that $\bar{V}^j \cdot \bar{V}^k = 0$. They constitute likewise an enumerable set which, ranged in some order, shall be designated by $\{(V^j, W^j)\}$. Given any two distinct points P, Q of L, there exists at least one pair (V^j, W^j) such that $P \subset V^j$, $Q \subset W^j$. This is the essential property of the sequence for our purpose.

22. To obtain L^1 we first construct $\Psi^{11} = \Psi^1$ as in No. 14, except that its vertices are taken on $\mathfrak{K}^1$. Since the latter is a neighborhood of the sum of the n-faces of $K_{2n+1}^0 = \bar{\sigma}_{2n+1}$, if the vertices are taken near enough to those of the simplex, $\Psi^{11} \subset \mathfrak{K}^1$. We then construct the other Ψ's as in No. 14, taking care that all are on $\mathfrak{K}^1$. As a consequence there results the topological image L^1, of L, on K_{2n+1}^1. Suppose that for each index $\leqq i$, and large enough j in each case, we have constructed L^i, Ψ^{ij} and the nested neighborhoods $\mathfrak{N}(\sigma^{ij})$ of the construction in No. 14, all on $\mathfrak{K}^i$. We shall show how to obtain the same for $i+1$.

23. Let A^i, B^i designate the closures of the $(K_{2n+1}^i)''$-neighborhoods of K_n^{i-1} and $\mathfrak{B}_{2n}^i$. I say that $A^i \cdot B^i = 0$. It is only necessary to show that their cells on any simplex $\sigma_p\,\sigma_q$ of $\mathfrak{K}^1$, $\sigma_p \subset K_n^{i-1}$, $\sigma_q \subset \mathfrak{B}^i$, are distinct. This amounts to proving that two cells of $(\overline{\sigma_p\,\sigma_q})''$ having respectively a vertex on σ_p and σ_q can have no common vertices. But the cells of $(\overline{\sigma_p\,\sigma_q})'$ so related have only one vertex in common—the centroid of $\sigma_p\,\sigma_q$. Since the centroid is not a vertex of any cell of $(\overline{\sigma_p\,\sigma_q})''$ with a vertex on σ_p or σ_q our assertion follows.

If we set $C^i = K_{2n+1}^i - A^i - B^i$, C^i will be an interior subset of K_{2n+1}^i, i. e. on $\mathfrak{K}^i$. By sliding the elements, L^i, Ψ^{ij}, $\cdots$, along the projecting lines of $\mathfrak{K}^i$ we can bring them onto C^i. We shall assume this done and

continue to designate them as before. We observe that *the deformation of every point is over a closed cell of* K^i_{2n+1}.

24. Consider the pair of neighborhoods V^i, W^i and their images V^{ii}, W^{ii}. Now, on the one hand the closures of the latter do not meet, on the other hand ψ^{ij} approximates indefinitely to L^i with increasing j, in such manner that the distance from any point of L^i to its projection on ψ^{ij} tends uniformly to zero with $1/j$—a direct consequence of the construction of the ψ's and L^i. Therefore, for j above a certain value, the distances of the projections of V^{ii} and W^{ii} on ψ^{ij} will differ from $d(V^{ii}, W^{ii})$ by an arbitrary small quantity. Since mesh $\psi^{ij}\to 0$ with $1/j$, for j sufficiently large, it will be possible to decompose ψ^{ij} into a sum of three (closed) complexes θ, θ', θ'' such that: (a) θ' and θ'' have no common cells; (b) every cell of θ is of the form $\sigma'\sigma''$ where σ' is a cell of θ' and σ'' a cell of θ''; (c) the projection of V^{ii} on ψ^{ij} is on θ' and that of W^{ii} is on θ''.

25. We recall that if K is any finite complex there exists an $\eta>0$ such that every complex (singular or not) of mesh $<\eta$ on K is barycentrically deformable over K into a subcomplex of K in such a manner that no point leaves the closure of its carrying cell throughout the deformation. (Alexander-Veblen, see *Topology* p. 86.)

There exists in particular an η such as just considered for $\mathfrak{B}^i_{2n}+(K^{i-1}_n)''$ (the latter term represents the subcomplex of K^i_{2n+1} in coincidence with K^{i-1}_n). For reasons of continuity we can find an $\eta'>0$ such that if we slide onto K^{i-1}_n or $\mathfrak{B}^i_{2n}$ along the projecting segments of K^i_{2n+1}, a subset of C^i whose diameter $<\eta'$, the resulting set has its diameter $<\eta$. We choose j such that mesh $\psi^{ij}<\eta'$, which is clearly possible since the two previous conditions imposed upon ψ^{ij} merely demand that j be high enough.

26. Let us then first slide θ' and θ'' along the projecting segments until they come respectively onto K^{i-1}_n and $\mathfrak{B}^i_{2n}$. There result two singular complexes whose mesh $<\eta$. We can then deform them in the manner indicated in No. 25 so as to reduce them to subcomplexes θ'^*, θ''^* of $(K^{i-1}_n)''$ and $\mathfrak{B}^i_{2n}$. Under these circumstances the vertices of a generic simplex of θ are deformed onto vertices of θ'^* and θ''^* which belong to a closed cell of K^i_{2n+1}. It follows that, associated with the barycentric deformations of θ', θ'', there is one of θ into a subcomplex θ^* of K^i_{2n+1} of such a nature that $\psi^{ij}=\theta+\theta'+\theta''$ has been barycentrically deformed into the subcomplex $\theta^*+\theta'^*+\theta''^*$ of K^i_n, in such manner that no point leaves the closure of its cell on K^i_{2n+1}.

If P is any point of ψ^{ij} the corresponding point Q on the proper θ^* is on the closure of the cell that carries P, and hence the segment PQ is on that closed cell. If Q is on θ'^* or θ''^* there is an intervall QQ'

of the segment QP on A^i or B^i respectively. Since P remains on C^i we can find on each PQ a Q', $Q' \not\subset C^i$, whose locus Ω is a homeomorph of Ψ^{ij} and such that

$$Q' \subset A^i - K_n^i \text{ when } P \subset \theta',$$

$$Q' \subset B^i - \mathfrak{B}_{2n}^i \text{ when } P \subset \theta'',$$

$$Q' \subset \mathfrak{R}_{2n+1}^{i+1} \text{ when } P \subset \theta.$$

The third condition can always be taken care of since Q' can be chosen as near as we please to K_n^i and hence on the $(K_{2n+1}^i)''$-neighborhood of the latter. Under the circumstances $\Omega \subset \mathfrak{R}_{2n+1}^{i+1}$.

27. By the "projection" of a point of Ψ^{ih} or of L^i on Ω we shall now understand the image of its projection on Ψ^{ij} under the homeomorphism between Ω and Ψ^{ij}.

Since $\mathfrak{R}^{i+1}$ is a region of S_{2n+1} and Ω is a closed set on the region, there exists a positive ζ such that if a σ_k, $k \leqq n$, is not farther than ζ from Ω and its diameter also $<\zeta$, then $\sigma_k \subset \mathfrak{R}^{i+1}$. Now by (a) of No. 13, we can construct a complex $\Psi^{i+1,h}$, $h > j$, (which implies that it has the structure of Ψ^{h}), whose vertices are as near as we please to the projections of the corresponding vertices of Ψ^{ih} on Ω. But the mesh of the projection of Ψ^{ih} on $\Psi^{ij} \to 0$ with $1/h$, and hence this holds likewise for the mesh of the projection on Ω. It follows that for h large enough: (a) $\Psi^{i+1,h} \subset \mathfrak{R}^{i+1}$; (b) if P', P'' are the projections of any point P of L^i on Ω and on $\Psi^{i+1,h}$ (i. e. P'' is the image of the projection of P on Ψ^{ih}), then $d(P', P'') < \varepsilon$ arbitrarily assigned positive number. We now construct the complexes $\Psi^{i+1,k}$, $k > h$, and L^{i+1} as before and we can again so construct them that no point of L^{i+1} or of any Ψ^{i+1} is farther than ε from its projection on a given Ψ^{i+1}, in particular from its projection on $\Psi^{i+1,h}$. But for a proper choice of ε the various projections of the points of V^{ii}, W^{ii} will remain as near as we please to those of θ', θ''. Therefore we can so dispose of the situation that the projections of V^i, W^i on $\Psi^{i+1,h}$ be respectively on A^i and B^i without points on C^i. Since we can also manage to bring every point of L^{i+1} arbitrarily near to its projection on $\Psi^{i+1,h}$, we can so construct L^{i+1} that $V^{i,i+1} \subset A^i$, $W^{i,i+1} \subset B^i$. Finally as the deformation from Ψ^{ij} to Ω is such that no point leaves the closure of its carrying cell on K_{2n+1}^i, and so the points of L^{i+1} are arbitrarily close to their projections on Ω, we can also assume that we have:

I. If P is any point of L, then P^{i+1} is on the $(K_{2n+1}^i)''$-neighborhood N^i of the closure of the cell of K_{2n+1}^i that carries P^i.

In addition, according to the above:

II. If $P \subset V^i$, $Q \subset W^i$ are two points of L then $P^{i+1} \subset A^i$, $Q^{i+1} \subset B^i$.

Since no cell of $(K^{i+1}_{2n+1})''$ can have vertices on both A^i and B^i, we have from II:

III. The situation being as in II, N^{i+1} does not meet its analogue N'^{i+1} for Q.

Now there exists an $\eta > 0$ such that when $d(P^i, Q^i) < \eta$, P^i and Q^i are on the same star of a cell of K^i_{2n+1} and therefore P^{i+1}, Q^{i+1} will be on the $(K^i_{2n+1})''$-neighborhood of the closed star. Since L^i is a homeomorph of L we have:

IV. There exists a $\xi_i \to 0$ with $1/i$ such that if $d(P, Q) < \xi_i$ P^{i+1} and Q^{i+1} are on neighborhoods N^i, N'^i which intersect.

We have thus constructed L^{i+1} from L^i so as to satisfy the preceding properties in addition to the condition $L^{i+1} \subset \mathfrak{K}^{i+1}$. In particular we have thus an inductive construction for $\{L^i\}$.

28. Let $\mathfrak{N}^i$ denote the K^i_{2n+1}-neighborhood of the closed cell of K^i_{2n+1} that carries P^i. From I and the fact that K^{i+1}_{2n+1} is a subcomplex of $(K^i_{2n+1})''$, we conclude that $\mathfrak{N}^i \supset \mathfrak{N}^{i+1}$. Furthermore $d(\mathfrak{N}^i) < 3$ mesh K^i, and hence $\to 0$ with $1/i$. Therefore the $\mathfrak{N}$'s converge on a single point $P^* \subset \Lambda$, since $P^* \subset K^i_{2n+1}$ whatever i. The set L^* of all points P^* is then a subset of Λ. I say that it is homeomorphic to L.

We have already that every point P of L has a single image P^* on L^*. Let Q be a point of L other than P. There exists then a pair V^i, W^i such that $V^i \supset P$, $W^i \supset Q$. It follows from III that P^{i+1} and Q^{i+1} are on certain non-intersecting neighborhoods of cells of $(K^{i+1}_{2n+1})''$. But these neighborhoods include the neighborhoods $\mathfrak{N}^{i+2}$ and its analogue $\mathfrak{N}'^{i+2}$ for Q. Therefore $\mathfrak{N}^{i+2} \cdot \mathfrak{N}'^{i+2} = 0$ and as the first carries P^* and the second Q^*, $P^* \neq Q^*$. Therefore the correspondence between L and L^* is $(1-1)$.

Finally let $d(P, Q) < \xi_i$. From IV and $N^i \subset \mathfrak{N}^i$, $N'^i \subset \mathfrak{N}'^i$, follows $\mathfrak{N}^i \cdot \mathfrak{N}'^i \neq 0$. Since $P^* \subset \mathfrak{N}^i$, $Q^* \subset \mathfrak{N}'^i$, $d(P^*, Q^*) < 6$ mesh K^i_{2n+1}, and hence $\to 0$ with $1/i$, i.e. with ξ_i. Therefore the correspondence from L to L^* is continuous and as L is compact it is a homeomorphism. Thus L has been mapped topologically on a subset L^* of Λ. *Therefore Λ is a universal n-space.*

29. We have already alluded (footnote 2) to a universal space whose existence has been established by Nöbiling. That space, which we shall call Λ^*, is the set of all points of S_{2n+1} having at most n rational coördinates. It is a simple matter to derive Nöbiling's result from ours, as we shall now show.

In the first place it is clear that, if in the construction of Λ in No. 18, we replace $(K^i_{2n+1})''$ by $(K^i_{2n+1})^{(h_i)}$, $h_i \geqq 2$, the ulterior treatment goes through as before, and therefore the resulting space, say Λ', is still a

universal n-space. We will take advantage of this new degree of freedom in a moment.

Suppose now that we have a certain number s of positive transcendental numbers $\alpha_1, \cdots, \alpha_s$, between which there exists no relation $f(\alpha_1, \cdots, \alpha_s) = 0$, where f is a polynomial with integer coefficients. The set of all positive numbers α_{s+1} satisfying an equation $f_1(\alpha_1, \cdots, \alpha_{s+1}) = 0$, where f_1 is of the same type as f, is enumerable. Hence we can always increase the number of α's having the property of the initial set by one and therefore find $(2n+1)^2$ such numbers. It follows that we can find a σ_{2n+1} such that the coördinates of its vertices are transcendental numbers that satisfy no polynomial equation with integral coefficients.

Under the circumstances if we take any derived $(\bar{\sigma}_{2n+1})^{(h)}$ its n-simplexes will have no point with more than n rational coördinates. It follows that $K_n^i \subset \varDelta^*$ whatever i, where K_n^i is as in No. 18 but for the modified construction corresponding to $\varDelta'$.

Now the set of points having more than n rational coördinates is the sum of all S_n of S_{2n+1} of the form $x_{k_j} =$ a rational number, $j = 1, 2, \cdots, n+1$. These spaces constitute an enumerable aggregate $\{S_n^j\}$, and we have (Nöbiling): $\varDelta^* = S_{2n+1} - \Sigma S_n^j$. According to the above K_n^{i-1} does not meet any space S_n^j, so that the distance δ_i from K_n^{i-1} to S_n^i is positive. We will choose h_{i-1} such that mesh $(\bar{\sigma})^{(h_{i-1})} \leqq \frac{1}{2}\delta_i$. Since K_{2n+1}^i is the closure of the $(K_{2n+1}^{i-1})^{(h_{i-1})}$-neighborhood of K_n^{i-1} it will not meet S_n^i, and hence S_n^i will not meet $\varDelta' = \Pi K_{2n+1}^i$. This implies that $\varDelta' \subset S'_{2n+1} - \Sigma S_n^i = \varDelta^*$, and since $\varDelta'$ is a universal n-space this is likewise true for the space $\varDelta^*$ of Nöbiling.

30. **Separable spaces.** There is in existence an incomplete proof of the following proposition: *Every separable metric n-space can be mapped topologically on a compact metric n-space*[6]. Granting this result, it follows from what we have proved that a separable metric n-space can be mapped topologically on the universal space $\varDelta$, and, in particular, can be topologically imbedded in an S_{2n+1}.

It is possible to obtain the preceding result for an important class of separable metric spaces, the locally compact metric spaces, independently of the theorem of Hurewicz. These spaces can in fact be approximated by sequences $\{\Phi^i\}$ analogous to those of Alexandroff except that the Φ's are infinite. With relatively unimportant modifications in the treatment we can prove in particular the imbedding theorem. In constructing the initial complex Ψ^1 of No. 14, we choose its vertices $A^1, A^2, \cdots \to \infty$ with

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[6] Hurewicz, Amsterdam Proc., vol. 30 (1927), pp. 425. See the observation in Menger's book at the end of p. 284.

the order, but the rest of the construction is substantially as before, distance conditions being replaced wherever need be, by suitable neighborhood restrictions.

31. **A certain class of metric spaces.** Let $\{\Omega^i\}$ be a sequence of compact metric spaces, such that for every i there exists a continuous single-valued transformation τ_i of Ω^{i+1} into the *whole* of Ω^i, i. e. such that $\Omega^i = \tau_i \Omega^{i+1}$. If P is any point of Ω^{i+1}, the points $\tau_i P$, $\tau_{i-1} \tau_i P$, $\cdots$ are called the *projections* of P on Ω^i, Ω^{i-1}, etc. By a *projection sequence* (= p. s.) we now understand a sequence $\{P^i\}$, where P^i is a point of Ω^i and $\tau_i P^{i+1} = P^i$ for every i.

We will now consider the set of all p. s. as points of a new space L. The result of No. 11 implies that every compact metric space is of this type. We shall show that the converse also holds: every space such as L is compact and metric. What we have then is a construction of compact metric spaces in terms of (generally simpler) spaces of same type. By means of it we may hope to control to some slight extent the properties of L.

We may think of the consecutive points of a p. s. as joined by arcs which vary continuously with their extremities. The points of L correspond then to the curves of a certain continuous family, in some respects similar to a family of dynamical trajectories.

32. Through immersing the Ω's in the Hilbert parallelotope (*Topology* p. 323) we can choose for their totality a metric such that the distances between the points of any Ω^i are less than a certain constant A independent of i. We will now define $d(Q, Q')$, $Q = \{P^i\}$, $Q' = \{P'^i\}$ by the expression

$$d(Q, Q') = \sum_{i=1}^{+\infty} \frac{d(P^i, P'^i)}{i!}.$$

The verification of the two basic distance axioms (*Topology* p. 5) is immediate, hence L is a metric space.

We will now show that L is compact. For that purpose it is sufficient to prove that it possesses the following two properties[7]: (a) it is *totally bounded,* i. e. it can be ε-covered by a finite number of closed sets whatever ε; (b) it is *complete,* i. e. every Cauchy-sequence of points on L has an actual limit.

Now given ε we can choose i such that

$$A \sum_{j=i+1}^{+\infty} \frac{1}{j!} < \frac{1}{2}\varepsilon.$$

[7] Hausdorff, Grundzüge der Mengenlehre, 1st ed., p. 314.

Since the Ω's are compact, we can cover every Ω^h, $h \leqq i$, with a finite number of closed sets $F^{\alpha h}$ whose diameter $< \frac{\varepsilon h!}{2i}$. Consider now the set G of all points Q of L such that $P^h \subset F^{\alpha_h h}$, where the α's are given. We will have $G \neq 0$ for certain sets of α's and then G is closed. Furthermore every point of L belongs to a G. Therefore the G's $\neq 0$ constitute a finite covering of L. But if Q, Q' belong to the same G, we find immediately $d(Q, Q') < \varepsilon$. Therefore L is totally bounded.

Now let $\{Q^\alpha\} = \{\{P^{\alpha i}\}\}$ be a Cauchy-sequence. From the expression of the distance-function follows that $\{P^{\alpha i}\}$ is a Cauchy-sequence for Ω^i. Since Ω^i is compact the sequence has a limit P^i. Owing to the continuity of τ_i, $P^i = \tau_i P^{i+1}$, so that $Q = \{P^i\}$ is a point of L. Now in order that $d(Q, Q^\alpha) < \varepsilon$ it is sufficient that $d(P^i, P^{i\alpha})$ be less than a certain η for i less than a certain h. Since $P^{\alpha i} \to P^i$ we can choose an α_0 such that these conditions be fulfilled for every $\alpha > \alpha_0$ and $i < h$. Hence $d(Q, Q^\alpha) < \varepsilon$ for $\alpha > \alpha_0$, and $Q^\alpha \to Q$. Thus L is complete, and hence it is compact.

33. A basic property of L is that *it is ε_i-mappable on Ω^i, where $\varepsilon_i \to 0$ with $1/i$.* Let indeed T^i be the mapping of L on Ω^i whereby if $Q = \{P^i\}$, $T^i \cdot Q = P^i$. Then if $Q' = \{P'^j\}$ is also mapped on P^i by T^i, we have $P^j = P'^j$ for $j \leqq i$, hence $d(Q, Q') \leqq A \sum_{j=i+1}^{+\infty} \frac{1}{j!} = \varepsilon_i \to 0$ with $1/i$.

An immediate consequence is that *if dim $\Omega^i \leqq n$, whatever i, likewise dim $L \leqq n$.* For Ω^i is ε-mappable on an n-complex whatever ε, and hence this holds likewise for L.

34. It appears probable that when dim $\Omega^i = n$ for i above a certain value likewise dim $L = n$, but we have not been able to establish the fact. We shall then endeavor to restrict the Ω's in such manner as to have dim $L \geqq n$, and hence dim $L = n$. This will be done by choosing for the Ω's appropriate complexes.

Let indeed every Ω^i be an n-complex and suppose that there exist for every i an n-cycle Γ_n^i of Ω^i such that we have homologies:

$$\alpha_i \tau_i \Gamma_n^{i+1} \sim \beta_i \Gamma_n^i \text{ on } \Omega^i. \tag{1}$$

Under the circumstances *L is an n-space.* For each Ω^i has a subdivision which is the skeleton of some covering Σ^i of L whose mesh is $3\varepsilon_i$, where ε_i is as in No. 33, (Alexandroff, loc. cit. p. 18). Replacing if need be, every Ω^i by the subdivision in question, still called Ω^i, and $\{\Omega^i\}$ by a subsequence, we can treat the new $\{\Omega^i\}$ as the sequence of *Topology* Ch. VII § 4. In our treatment indeed we assumed that we had a sub-

division sequence. Now the only place where that mattered at all was in the proof of the deformation theorem, p. 328. It is however easily verified that this proof goes through provided $\varepsilon_i \to 0$ sufficiently fast, as we may well assume.

Under the circumstances we conclude from (1) to the existence of a Vietoris sequence $\{\Delta_n^i\}$, where Δ_n^i is a rational cycle of Ω^i and we have an ε_i-homology

$$\Delta_n^{i+1} \approx \Delta_n^i \text{ on } L,$$

in the sense of Vietoris (see *Topology* p. 330). It follows that $\{\Delta_n^i\} = \gamma_n$ is an n-cycle on L. Referring to the discussion loc. cit., it will be seen that $\gamma_n \not\approx 0$, on L, since n is the common dimension of all the Ω's. Therefore the Betti-number $R_n(L) \geqq 1$, and consequently dim $L \geqq n$, (*Topology* p. 335). Hence finally dim $L = n$ as asserted.

It is thus seen that we have obtained the inequality dim $L \geqq n$ as a consequence of the fact that some homology character of L for the dimension n is $\neq 0$. As it happens it was the Betti-number, but any other character would do equally well.

35. A very simple method to construct effectively an n-space L of the preceding type is as follows: Let K_{n-1} be an absolute $(n-1)$-circuit. We will take $\Omega^i = K_{n-1}^i \times \gamma^i$ where K_{n-1}^i is a copy of K_{n-1} and γ^i a circumference. Let u_i be an angular parameter for γ^i, $A^{i+1} \times B^{i+1}$ any point of Ω^{i+1} where A^{i+1} is the image of a point A of K and B^{i+1} corresponds to the value u_{i+1} of the parameter. We define τ_i by $\tau_i A^{i+1} \times B^{i+1} = A^i \times B^i$, where A^i is the image of A and B^i corresponds to $u_i = k_i u_{i+1}$, k_i an arbitrary integer. The number k_i is the Brouwer degree of τ_i. If Γ_{n-1} is the fundamental $(n-1)$-cycle on K_{n-1} and Γ_{n-1}^i its image on K^i, the fundamental n-cycle on Ω^i is $\Gamma_{n-1}^i \times \gamma^i$ and we have

$$\tau_i \Gamma_{n-1}^{i+1} \times \gamma^{i+1} \sim k_i \Gamma_{n-1}^i \times \gamma^i \quad \text{on} \quad \Omega^i, \tag{2}$$

which is (1) for the present system. Therefore L is an n-space.

36. The space just constructed depends upon the arbitrary sequence of integers $\{k_i\}$. For the infinite K_{n+1} of *Topology* Ch. VII, § 4, associated with $\{\Omega^i\}$, we have finite chains

$$C_{n+1}^{i+1} \to \Gamma_n^{i+1} - k_i \Gamma_n^i \tag{3}$$

where in the notation loc. cit., $C^i \subset N^i - N^{i+1}$. Furthermore there are no other boundary relations except (3) between the Γ's. It follows that

$$C_{n+1} = \sum h_i C_{n+1}^i \to -k_1 \Gamma_n^1, \quad \frac{1}{h_i} = k_1 k_2 \cdots k_i.$$

Thus C_{n+1} is a rational infinite chain with finite boundary. This chain defines an n-cycle Γ_n on L, and from what preceeds we conclude that every other n-cycle on L depends on Γ_n. Therefore the absolute Betti-number $R_n(L) = 1$. It is not difficult to show that the same holds for the numbers $R_n(L; m)$ whatever m. However, unless all the numbers k_i for i above a certain value, are ± 1, there is no integral n-cycle and the Betti-number for the integral cycles (Vietoris-cycles proper) is zero.

By taking products $L \times K$, where L is of the type just constructed and K a complex, we can obtain spaces with non-zero Betti-numbers for dimensions $<$ the dimension of the spaces themselves, and also having a numerical value >1.

37. Another interesting modification is in a different direction: Let Ω^i have a subcomplex ω^i, such that $\{\omega^i\}$ constitutes a sequence analogous to $\{\Omega^i\}$ attached to the same sequence of transformations $\{\tau_i\}$. Then, Γ_n^i designating now a cycle of $\Omega^i \bmod \omega^i$, let (1) hold $\bmod \omega^i$. Under the circumstances $\{\omega^i\}$ defines a closed subset l of L and we have $R_n(L; l) = 1$, hence again $\dim L = n$. The extension of the construction of No. 35 and the considerations of No. 36 hold here also with boundary relations, etc., taken $\bmod l$, instead of absolute.

Suppose that we merely know that the Ω's are all n-complexes and the τ's simplicial, it being always understood that $\tau_i \, \Omega^{i+1} = \Omega^i$. Then I say that we have a space of the type just considered. Let indeed σ_n^1 be any n-simplex of Ω^1. Then there exists a simplex σ^2 of Ω^2 of which it is the projection. But the projection of a σ_p is a simplex of dimension $\leqq p$. Hence σ^2 is of dimension $\geqq n$ and since it is a simplex of an n-complex its dimension can only be n, so that its proper designation is σ_n^2. Similarly there exists a σ_n^3 of Ω^3 whose projection on Ω^2 is σ_n^2, etc. We have thus a sequence of simplexes $\{\sigma_n^i\}$, each the projection of its successor. If we set $\omega^i = \Omega^i - \sigma_n^i$, the sequence $\{\omega^i\}$ has the same properties as above except that now we have as the basic boundary relation

$$\sigma_n^i \to 0 \quad \bmod (\omega^i, 2) \quad \text{on} \quad \Omega^i.$$

Therefore here $R_n(L; l, 2) = 1$ and L is again an n-space.

PRINCETON, N. J.

THE POINCARÉ DUALITY THEOREM FOR TOPOLOGICAL MANIFOLDS.

By William W. Flexner.[1]

1. This paper is a sequel to "On Topological Manifolds"[2] and carries out the program described first in a note to the Proceedings of the National Academy by Lefschetz and Flexner[3] and also in the introduction to "On Topological Manifolds". In that paper the invariance of the Betti numbers of an orientable manifold M_n as there defined was proved and the definition of the Kronecker index of two cycles on M_n specified. It was also shown that if any multiple of a p-cycle γ_p on M_n is homologous to zero, its Kronecker index with any $(n-p)$-cycle is zero. For the proof of the duality theorem it is sufficient to demonstrate the proposition inverse to this: "if every $(n-p)$-cycle intersects γ_p with a Kronecker index zero, some multiple of γ_p must bound", or, what is equivalent, that every non-bounding p-cycle is intersected by some $(n-p)$-cycle with a Kronecker index one.

This is one of the conclusions of Theorem 5 of the present paper and its proof depends on the Alexander duality theorem[4] generalized to compact spaces by Alexandroff[5] and Lefschetz,[6] which says that to every non-bounding cycle γ_p on M_n there is an $(r-p)$-chain, C_{r-p} in H_r, a Euclidean r-sphere in which M_n is immersed, such that the boundary of C_{r-p} does not meet M_n and such that $(C_{r-p} \cdot \gamma_p) = 1$. Theorems 3 and 4 of this paper show that the intersection of C_{r-p} and M_n defines an $(n-p)$-cycle in H_r near M_n. Theorem 5 contains the demonstration that this cycle can be deformed into a cycle Γ_{n-p} on M_n in such a way that $(\Gamma_{n-p} \cdot \gamma_p) = 1$.

From this result the Veblen Kronecker index theorem[7] and the Poincaré duality theorem[8] follow immediately. The invariance of the degree (Abbildungsgrad)[9,10] of a single-valued transformation of M_n into itself,

[1] Received March 10, 1931. Presented to the American Mathematical Society, December 31, 1930.

[2] Flexner, W. W., Annals of Math., (2) 32 (1931): refered to in the sequel as "F. M.".

[3] Lefschetz, S. and Flexner, W. W., Proc. Nat. Acad. Sci. 16 (1930), pp. 530–533.

[4] Alexander, J. W., Trans. Am. M. S. 23 (1922), pp. 333–349.

[5] Alexandroff, P., Annals of Math. (2) 30 (1929), pp. 101–187.

[6] Lefschetz, S., "Colloquium Lectures on Topology", Am. Math. Soc. Colloquium Publications, vol. XII (1930): refered to in the sequel as "L. T.".

[7] Veblen, O., Trans. Am. Math. Soc. 25 (1923), pp. 540–550.

[8] Poincaré, H., Journal de l'Ec. Polyt. (2) 1 (1895), pp. 1–123.

[9] Brouwer, L. E. J., Math. Ann. 71 (1912), pp. 97–115.

[10] Wilson, W., Math. Ann. 100 (1928), pp. 552–578.

originally proved by Hopf[11] for topological manifolds whose defining neighborhoods are n-simplexes, is, for the M_n here considered, a consequence of the duality theorem. Open manifolds which Hopf considers are not treated here, but are reserved for a latter occasion. Though all theorems here are proved for orientable manifolds, they also hold modulo 2 as in F. M., and so cover the non-orientable case.

I want to express my thanks to Professors Alexander and Lefschetz for their help in this work, particularly to Professor Lefschetz with whom, as our joint Proceedings note shows, this paper was planned, and who gave many valuable suggestions during the writing.

2. Throughout this paper notations and technical terms are used in the sense of L. T.[6] and F. M.[2]. The first result shows that any chain on M_n can be deformed into one made up of simplicial pieces connected by singular parts in the neighborhood of a complex of less dimension.

LEMMA. *Any chain C_p on M_n is ε-deformable over M_n into a chain $C'_p = \sum_i D^i_p + d_p$ such that:*

1. *D^i_p is simplicial in an n-cell E^i_n of the covering set $\{E^i_n\}$ of M_n, and the D's do not meet one another.*
2. *There exists an arbitrarily small neighborhood $\mathfrak{N}$ of W, the complex containing the sum over i of the complexes $F(D^i_p)$, such that $\mathfrak{N} \supset d_p$ and that every p-cycle on $\mathfrak{N}$ is ~ 0 on M_n.*

A chain such as C'_p will be called *semi-simplicial.* The reduction to C'_p satisfying the lemma is much the same as the reduction of the p-cycle in Theorem 9 of F. M., but as the details are different, a proof of this lemma will now be outlined.

On $\bar{E}^i_n$ take a complex K^i of mesh so small that if J^i is the sum of the closed cells of K^i with no points on $F(E^i_n)$, $\{J^i\}$ covers M_n. Let $G^{k-1} = J^1 + \cdots + J^{k-1}$. It is assumed that C_p is τ-deformable, $\tau > 0$ arbitrarily small, into a chain

$$C^{k-1}_p = \sum_{j=1}^{k-1} D^j_p + d^{k-1}_p + \delta^{k-1}_p$$

such that:

a. $\sum_{j=1}^{k-1} D^j_p + d^{k-1}_p$ is the sum of the closed cells of C^{k-1}_p meeting G^{k-1}.

b. D^j_p is simplicial on E^j_n and the D's do not meet one another.

c. d^{k-1}_p is on an arbitrarily small neighborhood $\mathfrak{N}^{k-1}$ of W^{k-1}, the sum of the complexes containing $\sum_{j=1}^{k-1} F(D^j_p)$, such that every p-cycle on $\mathfrak{N}^{k-1}$ is ~ 0 on M_n.

[11] Hopf, H., Math. Ann. 100 (1928), pp. 579–608 and 102 (1929), pp. 562–623.

Let $G^k = J^1 + \cdots + J^k$. If a, b and c can be satisfied for the index k, a finite number of deformations, each reducing the part of C_p on one n-cell E_n^i will suffice to reduce C_p to C_p'. Hence at each step the deformations can be taken so small that each member of the set $\sum_{j=1}^{k-1} D_p^j$, $\sum_{j=1}^{k} D_p^j, \cdots$ is not affected by any deformation successive to the one giving rise to it.

Let g_p be the sum of the cells of a subdivision of $d_p^{k-1} + \delta_p^{k-1}$ that are at a distance ζ from $\sum_{j=1}^{k-1} D_p^j$. ζ is to be so chosen that $F(g_p) \subset \mathfrak{N}^{k-1}$ or at a distance greater than 2ζ from W^{k-1}. Now with the general principle just stated about deformations in mind, a suitable subdivision of C_p^{k-1} can be made so that when the closed cells of g_p on E_n^k are deformed onto a subdivision K'^k of K^k no deformation cells impinge on $\sum_{j=1}^{k-1} D_p^j$. Call D_p this sum of closed cells after deformation and call C_p^k the deform of C_p^{k-1}. Let D_p^k be the sum of the cells of a subdivision of mesh α of D_p whose closure meets J^k and which have no points in common with the deform of d_p^{k-1} plus the deformation chain of its boundary. If the mesh of K'^k is small enough, D_p^k contains only cells of K'^k.

Now construct a neighborhood M of $F(D_p^k)$ such that any cycle on M can be deformed over E_n^k onto $F(D_p^k)$. Since $\mathfrak{N}^{k-1}$ is arbitrarily small it can have been taken so small that the deformation chain obtained by deforming any chain on $\mathfrak{N}^{k-1} \cdot M$ onto $F(D_p^k)$ lies in a neighborhood $\mathfrak{N}'^{k-1}$ of W^{k-1} chosen before hand so small that any chain on $\mathfrak{N}'^{k-1}$ is deformable over M_n onto W^{k-1}. All that need be required of M is that it shall be far enough inside of E_n^k and, near d_p^{k-1}, inside $\mathfrak{N}^{k-1}$ so that spheres with centers in $\mathfrak{N}^{k-1} \cdot M$ and of radius only slightly greater than 2α will lie in E_n^k and $\mathfrak{N}^{k-1}$ respectively. But as α decreases, the distance of $F(D_p^k)$ from $F(E_n^k)$ and $F(\mathfrak{N}'^{k-1})$ increases.

Call d_p the sum of the closed cells of C_p^k not in D_p^k or d_p^{k-1} which are contained in M. Then let $\mathfrak{N}^k = \mathfrak{N}^{k-1} + M$ and $d_p^k = d_p^{k-1} + d_p$. Conditions a, b and c are now fulfilled for the index k. Condition c is the only one that needs verification. Any p-cycle Γ_p on $\mathfrak{N}^k$ can be written $\Gamma_p = A_p + A_p'$ where $A_p \subset \mathfrak{N}^{k-1}$, $A_p' \subset M$ and $F(A_p) = -F(A_p') \subset \mathfrak{N}^{k-1} \cdot M$. By assumption A_p' can be deformed onto $F(D_p^k)$ which is of dimension less than p so $A_p' \equiv 0$. Now $\Gamma_p = A_p + A_p''$ where A_p'' is the deformation chain of $F(A_p')$, and $A_p + A_p''$ lies in $\mathfrak{N}'^{k-1}$ so is ~ 0.

This completes the step from $k-1$ to k. The induction starts for at the zeroth step

$$D_p^0 = d_p^0 = W^0 = \mathfrak{N}^0 = 0, \quad C_p = \delta_p^0.$$

If s is the number of n-cells E_n^i covering M_n, $\delta_p^s = 0$ and the proof of the lemma is complete.

THEOREM 1. *Given h cycles, Γ_p^q, a reduction similar to that of the lemma may be applied to them, replacing them by a new set*

$$\Gamma_p^{\prime q} = \sum_i D_p^{qi} + d_p^q,$$

where the D's are as before, no two D^{qi}, D^{qj}, $i \neq j$, intersect, and condition 2 of the lemma holds as to the neighborhoods containing all the d_p^q.

The argument is as before, d_p^q being now in an arbitrarily small neighborhood of the complex containing the sum of the chains $F(D_p^{qi})$.

Corollary. It is a consequence of this theorem that no non-zero cycles of dimension higher than n exist on M_n. For $p > n$, the D's, being degenerate p-chains, are $\equiv 0$, so the $F(D)$'s are $\equiv 0$ and hence, finally, the neighborhood $\mathfrak{N} \equiv 0$.

3. A very particular set $\Delta_p^1, \Delta_p^2, \cdots, \Delta_p^h$ of semi-simplicial p-cycles on M_n is now chosen as the set defining the pth Betti number of M_n.

THEOREM 2. *Given a set of h independent semi-simplicial p-cycles, Δ_p^1, $\Delta_p^2, \cdots, \Delta_p^h$ on M_n there can be constructed semi-simplicial cycles homologous to these such that if L'_p is the point-set that covers them, its pth Betti number is reduced to zero by the removal of h simplicial n-cells no two of which are adjacent.*

If L is the point set carrying the given set of cycles, call the simplicial p-cells of L $E_p^1, E_p^2, \cdots$. Suppose $E_p^{b_1}, E_p^{b_2}, \cdots, E_p^{b_q} (b_i < b_{i+j})$ is the maximal subset of these, each member of which is the cell with least upper index of some cycle on L. It can be assumed that no two of the cells $E_p^{b_i}$ are adjacent, for if two were adjacent a subdivision and renumbering of the cells would bring about the desired condition. Select some p-cycle of L with $E_p^{b_i}$ as its first cell, choosing a cycle of type Δ^*, that is a cycle some multiple of which bounds on M_n, if such a Δ^* is to be found. Dividing the cycle by the coefficient of $E_p^{b_i}$ in it gives a rational semi-simplicial cycle

$$\Gamma_p^i = E_p^{b_i} + t_1 E_p^{b_i+1} + \cdots + d_p^i.$$

By L. T. pp. 302–303 and because every cycle on $\mathfrak{N}$ (Theorem 1) bounds, the subset

$$A_p^j = E_p^{a_j} + t'_1 E_p^{a_j+1} + \cdots + d_p^j \qquad (j = 1, \cdots, h)$$

comprehending the cycles of the set Γ_p^i which are not Δ^*'s is a maximal independent set of non-bounding cycles on L and on M_n. By construction $\{A_p^j\}$ lies on L. Let L'_p be the subset of L carrying $\{A_p^j\}$.

Now suppose the removal of $E_p^{a_1}, E_p^{a_2}, \cdots, E_p^{a_h}$ does not reduce the pth Betti number of L'_p to zero. Then there is a non-bounding cycle γ_p in L'_p containing none of the cells $E_p^{a_j}$. Moreover $\gamma_p - \sum_j \alpha_j \Lambda_p^j \sim 0$ on M_n where not every α_j is zero. Since γ_p is on L'_p the cell of $\gamma_p - \sum_j \alpha_j \Lambda_p^j$ with least upper index must be one of the set $E_p^{a_i}$. Suppose, $E_p^{a_b}$ is that cell. But $E_p^{a_b}$ cannot by construction be the first cell of a bounding cycle on M_n, which is a contradiction. Hence the removal of the non-adjacent p-cells $E_p^{a_1}, E_p^{a_2}, \cdots, E_p^{a_h}$ reduces the pth Betti number L'_p to zero.

4. In F. M. section 3 it was proved that M_n can, if r is large enough, be immersed in a Euclidean r-sphere H_r. Assume $r \geqq 2n+1$. H_r is itself a topological manifold and so can be covered by an elemental complex $\mathfrak{K}_r$ (F. M., sect. 1.). For the present purposes a $\mathfrak{K}_r$ is needed of the type about to be described.

THEOREM 3. *If M_n is immersed in H_r, H_r can be covered by an elemental complex $\mathfrak{K}_r$ of arbitrarily small mesh whose elements are chains of a simplicial complex on H_r and such that no s-element, $s < r - n$, of $\mathfrak{K}_r$ meets M_n.*

The construction of $\mathfrak{K}_r$ is exactly as in F. M. except that, being in H_r, the elements can be chosen at each step to be simplicial chains, and that the s-elements $s < r - n$, can be chosen not intersecting M_n as will now be proved. For $s = 0$ there is no problem since r being greater than n, M_n is nowhere dense in H_r and so the zero-elements can be chosen not on M_n. These points can also be chosen as the vertices of a simplicial complex on H_r. For $s > 0$, proceeding as in the inductive construction used in F. M., the theorem is assumed for $s = k - 1$. The induction will be complete if it can be shown that a $(k-1)$-cycle in an r-cell E_r of U^{r-k}, must bound a k-chain in E_r not meeting M_n, for this means that the k-element defined by $k+1$ vertices in E_r can be chosen not intersecting M_n.

Since H_r is Euclidean the r-cells, E_r, determining $\mathfrak{K}_r$ can be taken to be spherical r-cells. Also they can be so small that $M_n \cdot E_r$ is covered, with a wide margin, by an n-cell E_n of M_n. Assume that γ_{k-1} fails to bound in $E_r - \overline{M_n \cdot E_r}$. Let $L = H_r - E_r + M_n$. Then $E_r - \overline{M_n \cdot E_r} = H_r - L$ and to each non-bounding $(k-1)$-cycle on $H_r - L$ there corresponds a non-bounding Vietoris cycle $\{\Gamma^i_{r-k}\}$ on L an vice versa (L. T. p. 341). It will now be shown that every cycle $\{\Gamma^i_{r-k}\}$ must bound in the sense of Vietoris on L from which the theorem follows.

5. A Vietoris cycle $\{\Gamma^i_{r-k}\}$ on L is a sequence of cycles Γ^i_{r-k} $(i = 1, 2, \cdots)$ on H_r and approaching more and more nearly to L as i increases. Since M_n is locally connected the cells of Γ^i_{r-k} that are near M_n, when i is large enough to bring Γ^i_{r-k} near enough to L, can be deformed onto M_n

and then into a subchain of a complex on E_n which, because $r-k>n$, is identically zero. The rest of Γ^i_{r-k} was very near $F(E_r)$ to start with. The displacement just applied to Γ^i_{r-k} approaches zero as i increases, so for i large enough, the deformed Γ^i_{r-k} is arbitrarily near $F(E_r)$. Therefore it can be deformed into the interior of $H_r - E_r$ by sliding its cells along rays from the center of E_r. But $H_r - E_r$ is an r-cell so any cycle on it is homologous to zero. Therefore $\Gamma^i_{r-k} \sim 0$ and this is true for every i greater than a certain i_0. This means that $\{\Gamma^i_{r-k}\} \sim 0$ in the sense of Vietoris.

6. Having eliminated the cells of dimension less than $r-n$ from the problem, it is now possible to study the intersection of $\mathfrak{K}_r$ and M_n. Let $\widehat{M_n \cdot B}$ represent the point set common to M_n and a configuration B of H_r intersecting M_n.

THEOREM 4. *If* $t = r - n$ *it is possible to define for each* $(t+k)$*-element,* a^i_{t+k}*, of the elemental complex* $\mathfrak{K}_r$ *a* k*-chain,* A^i_k*, in* H_r *within* ζ *of* $\widehat{M_n \cdot a^i_k}$ *where* ζ *approaches zero with the mesh* ϵ *of* $\mathfrak{K}_r$ *and such that if*

$$a^i_{t+k} \to \sum_j \mu^i_j a^j_{t+k-1} \quad \textit{then} \quad A^i_k \to \sum_j \mu^i_j A^j_{k-1}$$

and $A^i_k \equiv 0$ *when* a^i_{t+k} *does not meet* M_n.

A^i_k will be called the "intersection" of M_n and a^i_{t+k}. In Theorem 3, ϵ was chosen so small that if a^i_s is an s-element of $\mathfrak{K}_r$, $\widehat{M_n \cdot a^i_s}$ for every i and s is covered by some n-cell E^i_n of the covering of M_n. The $(r-n)$-elements a^i_t are those of least dimension which meet M_n. By the Lefschetz intersection theory, E^i_n and a^i_t have as intersection a family of simplicial chains $b^{i1}_0, b^{i2}_0, \cdots$ approaching $\widehat{M_n \cdot a^i_t}$ more and more closely and each chain homologous in a small neighborhood of $\widehat{M_n \cdot a^i_t}$ to the succeeding zero-chains (L. T., ch. IV). For A^i_0 choose a particular one of these b^i_0's. If $a^i_{t+1} \to \sum_j \mu^i_j a^j_t$ and E^i_n covers $\widehat{M_n \cdot a^i_{t+1}}$, E^i_n and a^i_{t+1} have for intersection a homologous family of chains, $b^{i1}_1, b^{i2}_1, \cdots$. But because the boundary of each such one-chain is by construction a member of an homologous family defining the intersection of $\sum_j \mu^i_j a^j_t$ and M_n, it must be homologous to $\sum_j \mu^i_j A^j_0$. Hence $\sum_j \mu^i_j A^j_0$ bounds. Call the simplicial complex it bounds A^i_{t+1} and form A^i_{t+1} for every i.

The rest of the proof is by induction. Assume that to every $(t+s)$-element, a^i_{t+s}, for which $s \leq k-1$ an s-chain A^i_s is defined such that $a^i_{t+s} \to \sum_j \mu^i_j a^j_{t+s-1}$ implies $A^i_s \to \sum_j \mu^i_j A^j_{s-1}$ and make the construction for $s = k$ as follows. By the condition of the induction $\sum_j \mu^i_j A^j_{k-1}$ is a cycle.

A choice of A_0^i near enough $\widehat{M_n \cdot a_t^i}$ makes it possible to have A_1^i arbitrarily close to $\widehat{M_n \cdot a_{t+1}^i}$, so it could have been assumed that $\sum_j \mu_j^i A_{k-1}^j$ was arbitrarily close to $\widehat{M_n \cdot a_{t+k}^i}$. Therefore, as the diameter of a_{t+k}^i is arbitrarily small, $\sum_j \mu_j^i A_{k-1}^j$ lies in an arbitrarily small r-cell σ_r of H_r which contains $\widehat{M_n \cdot a_{t+k}^i}$. Take as A_k^i a simplicial k-chain in σ_r bounded by $\sum_j \mu_j^i A_{k-1}^j$. This chain satisfies the required conditions, which completes the induction.

Corollary. If C_k is a subchain of $\mathfrak{K}_r$, the boundary of its intersection with M_n is the intersection with M_n of its boundary.

7. The next is the central theorem towards whose proof the previous ones have been directed. It gives, when combined with Theorem 2 and the theorem that two homologous p-cycles have the same Kronecker index with an $(n-p)$-cycle (F. M., Th. 13), the result described in the first section: a sufficient condition that a multiple of γ_p shall bound is that $(\Gamma_{n-p} \cdot \gamma_p) = 0$ for every Γ_{n-p}.

THEOREM 5. *Given h independent cycles Δ_p^i on M_n there exist associated rational cycles Γ_{n-p}^i such that $(\Gamma_{n-p}^i \cdot \Delta_p^i) = \delta^{ij}$ on M_n.*

By Theorem 2 it may be assumed that the Δ's are the simplicial cycles there obtained. In the notations used in the proof of Theorem 2, it is true that there can be found h chains C_{r-p}^i whose boundaries do not meet M_n and such that first, on H_r $(C_{r-p}^i \cdot \Delta_p^j) = \delta^{ij}$, and next, the C's meet the Δ's only in the h p-cells $E_p^{a_i}$. For $p = 0$, C_{r-p}^i is the fundamental cycle on the sphere H_r and Δ_0^j is a point of M_n and $(C_r \cdot \Delta_0) = 1$ so that for $p = 0$ the theorem holds. Let now $p > 0$. That, when $p > 0$, some C's exist of the type required except as regards the intersecting in the cells $E_p^{a_i}$, is implicit in a result in L. T., p. 342 (generalized Alexander Duality Theorem). When the cells $E_p^{a_i}$ are removed from L'_p, the set carrying the Δ's, the Betti number of the set L'' remaining is zero. Therefore, by the result of L. T. (p. 339) on $H_r - L''$, $C_{r-p}^i \approx C_{r-p}^{\prime i}$, where $C_{r-p}^{\prime i}$ does not meet L''. It follows that $(C_{r-p}^i \cdot \Delta_p^j) = (C_{r-p}^{\prime i} \cdot \Delta_p^j)$. But $C_{r-p}^{\prime i} \subset H_r - L''$, and hence it can only meet L'_p on $L'_p - L''$, that is on the cells $E_p^{a_i}$. Since $C_{r-p}^{\prime i}$ otherwise behaves like C_{r-p}^i it can take its place. This shows that C's can be found behaving in every way as required. It is possible to go a step further. If $\mathfrak{K}_r$ is an elemental complex whose mesh is suitably small, applying the deformation theorem of F. M. will reduce the C's to subchains of $\mathfrak{K}_r$ without destroying the properties already imposed upon them.

Consider now the intersections of M_n and C_{r-p}^i. They are by Theorem 4 cycles whose maximum distance from M_n approaches zero with the mesh

of $\mathfrak{R}_r$. Therefore since M_n is locally connected, when the mesh is taken small enough, the cycles are ε-deformable into cycles Γ^i_{n-p} on M_n, where ε is arbitrarily small. By taking ε small enough the Γ's will be made to intersect the Δ's arbitrarily near to their intersections with the C's. Take ε so small that the intersections continue to be on the cells E^a_p.

To take care of straightness and the like it is convenient to surround M_n with a large r-simplex σ_r and operate wholly within it. It is no restriction to assume that $\mathfrak{R}_r$ covers $\bar{\sigma}_r$.

Returning to the theorem, as the situation is now it is merely necessary to show that

$$(C^i_{r-p} \cdot \Delta^j_p)_{\bar{\sigma}_r} = (\Gamma^i_{r-p} \cdot \Delta^j_p)_{M_n}.$$

(The subscripts $\bar{\sigma}_r$ and M_n refer to the configurations with respect to which the intersections are taken). The C's and the Γ's intersect the Δ's only in the D chains and for a given Δ no two of the D's intersect (Th. 2). Let E_n be one of the n-cells covering M_n. D_p will be a simplicial subchain of the complex K_n covering E_n, and D_p will be met by C_{r-p} and Γ_{n-p} (an arbitrary one of the pairs C^i_{r-p}, Γ^i_{n-p} above) on the cell E^a_n of K_n which can be taken small enough not to meet $F(D_p)$. Now it is sufficient to prove that,

$$(C_{r-p} \cdot D_p)_{\bar{\sigma}_r} = (\Gamma_{n-p} D_p)_{M_n}.$$

Since after all only the part of Γ on E^a_n now matters replace Γ by a chain G'_{n-p} which represents as large a part of Γ on E_n as is desirable. Then it suffices to prove

$$(C_{r-p} \cdot D_p)_{\bar{\sigma}_r} = (G_{r-p} \cdot D_p)_{\bar{E}_n}.$$

Here it is best to make use of the convex intersection,[12] which is permissible because (L. T., pp. 210—216) these intersections give Kronecker indexes identical with those used heretofore, and because the intersection cycles of the two types are homologous.

8. The first thing that must be done is to approximate $\bar{E}^a_n$ itself by a polyhedral cell in H_r. To effect this let a^i_0, $i = 1, 2, \cdots, s$ be the vertices of K_n, the complex of which E^a_n is a subcomplex. Within any prescribed vicinity of each of these points take new points b^i_0 and for each simplex of K_n, say $a^{i_0}_0, \cdots, a^{i_q}_0$, construct the corresponding simplex $b^{i_0}_0, \cdots, b^{i_q}_0$. There results a rectilinear complex $\bar{E}'^a_n$ whose structure for suitable chosen b's is the same as that of K_n. Since $r > 2n$, if two simplexes of $\bar{E}'^a_n$ intersect which do not have identical vertices, the b points that are their vertices satisfy certain linear relations and hence constitute

[12] Lefschetz, S., Trans. Am. M. S. 28 (1926), pp. 1–50.

an algebraicly restricted system of points. As a consequence in any prescribed vicinity of the a's there is a group of b's which is not a system thus restricted, that is, whose simplexes will only intersect when their vertices are identical. $\bar{E}'^a_n$ is then an n-cell and it approximates K_n cell for cell arbitrarily close, the distance depending solely upon the mesh of the complex K_n which is arbitrarily small.

9. There is now a well defined barycentric transformation T of K_n into $\bar{E}'^a_n$ whereby each a^i_0 goes into the corresponding b^i_0, and T is a homeomorphism. The image of any configuration of E_n on $\bar{E}'^a_n$ will be designated by accenting its symbol. It then remains to prove

$$(C_{r-p}\cdot D_p)_{\bar{\sigma}_r} = (G'_{n-p}\cdot D'_p)_{\bar{E}'^a_n}.$$

But if the deformation T is small enough (and its smallness can be controlled by the mesh of K_n which can be chosen freely) the Lefschetz intersection theory says that at the left D can likewise be replaced by D'. Now it remains to prove that

$$(C_{r-p}\cdot D'_p)_{\bar{\sigma}_r} = (G'_{n-p}\cdot D'_p)_{\bar{E}'^a_n}.$$

But in the neighborhood of the cell E^a_n where C_{r-p} meets D_p it is possible to choose as the elements of the intersection of M_n and C_{r-p} the very intersections of the elemental cells of C_{r-p} with $\bar{E}'^a_n$. This merely demands that C_{r-p} and $\bar{E}'^a_n$ be in general position as regards their cells, and may always be achieved by subjecting one of them to a slight translation. Finally in the deformation from the intersection cycle in H_r to Γ_{n-p} on M_n choose for the particular deformation of the cells near E_n the deformation imposed by T^{-1} itself. Under the circumstances $G'_{n-p} = \bar{E}'_n\cdot C_{r-p}$ and therefore it is merely necessary to prove

$$(C_{r-p}\cdot D'_p)_{\bar{\sigma}_r} = (\bar{E}'^a_n\cdot C_{r-p}\cdot D'_p)_{\bar{E}'^a_n}.$$

But this is implicit in a proposition due to Lefschetz.[13] The proof of Theorem 5 is therefore complete.

10. THEOREM 6. *If $\gamma^1_p, \gamma^2_p, \cdots, \gamma^h_p$ is a maximal set of independent non-bounding p-cycles on M_n there exists a maximal set of non-bounding $(n-p)$-cycles $\Gamma^1_{n-p}, \Gamma^2_{n-p}, \cdots, \Gamma^h_{n-p}$ on M_n such that $(\Gamma^i_{n-p}\cdot\gamma^j_p) = \delta^{ij}$.*

By Theorems 1 and 2 there is for each γ^j_p a semi-simplicial cycle Δ^j_p satisfying the conditions of Theorem 5 and such that $\gamma^j_p \approx \Delta^j_p$ and consequently $(\Gamma^i_{n-p}\cdot\gamma^j_p) = (\Gamma^i_{n-p}\cdot\Delta^j_p)$ (F. M., Th. 13). If s is the number of independent non-bounding $(n-p)$-cycles on M_n and $\Gamma^1_{n-p}, \Gamma^2_{n-p}, \cdots, \Gamma^h_{n-p}$

[13] Lefschetz, S., Trans. Am. M. S. 29 (1927), pp. 429-462.

is the set of $(n-p)$-cycles obtained by Theorem 5, it follows by the argument of L. T. (p. 179–180) that $s = h$. This also proves the next theorem.

THEOREM 7. *Poincaré duality theorem. If P_i is the ith Betti number of M_n as defined in F. M., then $P_k = P_{n-k}$.*

11. Brouwer[9] has defined a number, the "degree" (Abbildungsgrad) connected with a single-valued transformation T of M_n into itself and giving the excess of the number of positive coverings of M_n by TM_n over the number of negative coverings. For simplicial manifolds Brouwer showed that the degree remains unchanged in passing from T to any transformation that can be derived from T by continuous deformation. Wilson[10] defined the degree for topological manifolds whose fundamental neighborhoods are simplexes, whereupon Hopf[11] extended Brouwer's invariance proof to such manifolds. Such an extension follows for the M_n here treated from Theorem 7.

THEOREM 8. *Brouwer's degree remains unchanged on passing by a continuous deformation from a transformation T of M_n into itself, to a transformation T'.*

Since M_n is connected, P_0 for M_n is 1 (L. T., p. 16). Hence by Theorem 7, $P_n = 1$ which means there is only one independent non-bounding n-cycle Γ_n on M_n to a multiple of which every n-cycle on M_n is homologous. This cycle, Γ_n, covers every point P on M_n, for if Q is the one independent zero-cycle on M_n which has a Kronecker index of 1 with Γ_n, then $P \sim Q$ and $(\Gamma_n \cdot P) = 1$ (F. M., Th. 13). Hence Γ_n must cover P.

Let $\Delta_n = T\Gamma_n$. Since Δ_n is a cycle on M_n, $\Delta_n \sim g\Gamma_n$. The number g is Brouwer's degree. Now if $\Delta'_n = T'\Gamma_n$, using Lefschetz's conventions (L. T., pp. 73–74) $\Delta'_n \sim \Delta_n$ on M_n and therefore $\Delta'_n \sim g\Gamma_n$ which proves the degree invariant.

12. THEOREM 9. *The Kronecker index theorem (Th. 6), the Poincaré duality theorem (Th. 7) and the invariance of the degree (Th. 8) hold modulo 2.*

The theorems just mentioned are proved modulo 2 just as in the oriented case, except that Kronecker indexes, boundaries of chains, the degree, and so on, are all computed modulo 2. The modulo 2 treatment is used when M_n is non-orientable.

CLOSED EXTREMALS.*

(FIRST PAPER.)

By Marston Morse.[1]

1. **Introduction.** The problem is one of the characterization and existence of closed extremals. It belongs both to the Calculus of Variations and to Differential Topology.

In problems of this sort it is important to distinguish between the general case and the special case. A closed extremal g is called general or "*non-degenerate*" if the corresponding equations of variation possess no periodic solutions except the null solution. To a non-degenerate extremal g the author attaches an *index* which characterizes g in the present theory in the same way that the number of conjugate points on an extremal segment characterizes the extremal segment in the theory of extremals joining two fixed points.[2] (See Trans. III and IV.) This index is shown to be not only a geometric invariant in the ordinary sense but also a semi-topological invariant. It is a geometric invariant in that it is independent of the parametric system or choice of coordinates. It is a semi-topological invariant in that it is definable by means of deformations of closed extremals of a restricted class. Cf. § 35 Trans. II.

This index of g equals the index of an associated quadratic form and depends upon an associated linear boundary value problem.[3] It is determined explicitly for the "principal ellipses" on an r-dimensional ellipsoid E_r whose semi-axes are unequal and have lengths near unity. It is shown that the closed geodesics covering these principal ellipses are the only closed geodesics on E_r with lengths less than a prescribed constant N, provided the semi-axes of E_r have lengths sufficiently near unity.

In the second paper we shall show that on a regular, analytic image S of an r-sphere there will always exist closed extremals with indices (suitably counted) such as the principal ellipses on E_r possess. If all the closed geodesics on S with lengths less than a sufficiently large constant N

* Received March 13, 1931.

[1] Morse, Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 856–859. The principal results of the present paper were outlined here.

[2] Morse, Transactions of the American Mathematical Society I, vol. 27 (1925), pp. 345–396; II, vol. 30 (1928), pp. 213–274; III, vol. 31 (1929), pp. 379–404; IV, vol. 32 (1930), pp 599–631; V, vol. 33 (1931), pp. 72–92.

These papers will be referred to as Trans. I, II, etc.

[3] Morse, American Journal of Mathematics, vol. 53 (1931).

are non-degenerate, the preceding statement is true without qualifications as to the count of indices. The final theorem of the preceding paragraph now assumes a peculiar importance because it shows that the result of the present paragraph cannot be made any stronger in the non-degenerate case. For E_r furnishes an example in which no more closed extremals exist than are affirmed to exist on S in general.

In the second paper the characterization of closed extremals will be made to depend on the author's theory of critical sets of functions regardless of whether the extremal is degenerate or not. (See Trans. I and V.) It is found that in the special case, where two closed extremals on S which are affirmed to exist and possess different indices, give the same value to the integral, there will appear a p-dimensional family of closed extremals, where p is the difference between the two indices.

The characterization of closed extremals is the authors. Except in the 2-dimensional reversible case where special methods[4] have been used, and except for the existence of at least one closed geodesic on S, as proved by Birkhoff,[5] the author's results on the existence of closed extremals are new. In particular the general theorem on the existence of extremals in the non-degenerate case is new without exception. In this case the author proves the existence of at least as many closed extremals on S as there are combinations of $r+1$ objects taken 2 at a time.

The results of the present paper will also be used in determining certain topological invariants to be called *circular connectivities* in a theory in which the closed curve replaces the point.

2. The r-spread S and the integral. Let

$$(w) = (w_1, \cdots, w_m) \qquad m > 2$$

be rectangular coördinates in an Euclidean m-space. Let $(v) = (v_1, \cdots, v_r)$ be rectangular coördinates in an auxiliary r-space, $(1 < r < m)$. By an *r-element* in the space (w) will be meant a set of points (w) homeomorphic with a set of points (v) within an $(r-1)$-sphere in the space (v). An r-element will be said to be of class $C^{(n)}$ if the w's are functions of the v's of class $C^{(n)}$, that is possess continuous nth order partial derivatives. An r-element will be called *regular* if it is of at least class C', and on it at least one of the jacobians of r of the w's with respect to the r v's is not zero.

We shall be concerned with a regular r-spread S of the following nature.

The r-spread S shall consist of a bounded, connected, closed set of

[4] M. M. Lusternik et Schnirelmann, Topological Methods in Variational Problems. Research Institute of Mathematics and Mechanics. Gosizdat Moscow (1930).

[5] Birkhoff, Dynamical Systems, American Mathematical Society Colloquium Publications, vol. 9.

points (w) the neighborhood of every point of which consists of a regular element of class C'''. We shall *admit* no representation of neighborhoods on S other than by regular elements of class C'''.

For each point (w) on S and for each set of direction numbers (σ) of a direction, tangent at (w) to S, let there be given a single-valued function,

$$F(w_1, \cdots, w_m, \sigma_1, \cdots, \sigma_m) = F(w, \sigma),$$

positive and *continuous* in (w) and (σ), for (w) on S and $(\sigma) \neq (0)$. Suppose further that

$$F(w, k\sigma) \equiv kF(w, \sigma)$$

for any positive constant k.

We shall suppose S *orientable*. For our purposes the following definition is convenient.

Let $D(p)$ be an ordered set of r mutually orthogonal directions each tangent to S at a variable point p of S. Let p_0 be any point on S. Suppose p traverses a closed curve h on S, starting and ending at p_0, and suppose that $D(p)$ varies continuously with p on h. If at the end of any such variation the final set D can be continuously turned about p_0 in its own r-plane so as to coincide with the original set D, then S will be termed *orientable.*

The hypothesis of orientability could readily be dispensed with. It is made for the sake of simplicity.

Corresponding to any admissible element $w_i = \overline{w_i}(v)$ of S we set

$$F\left[\overline{w}_1, \cdots, \overline{w}_m, \frac{\partial \overline{w}_1}{\partial v_j}\varrho_j, \cdots, \frac{\partial \overline{w}_m}{\partial v_j}\varrho_j\right] \equiv G(v, \varrho) \qquad (j = 1, \cdots, r)$$

for (v) on the element and for $(\varrho) \neq (0)$. Here j is to be summed following the usual conventions of tensor analysis to which we adhere.

We assume that the functions $G(v, \varrho)$ are of class C''' for (v) on the corresponding element and for $(\varrho) \neq (0)$.

The curves $w_i = w_i(t)$ to be admitted on S shall be of the "ordinary" type[6] with continuously turning tangents except at most at a finite number of corners.

Along any such curve g our integrand is to have the form

$$J = \int_{t^1}^{t^2} F(w, \dot{w})\, dt \tag{2.1}$$

[6] See Bolza, Vorlesungen über Variationsrechnung (1909), p. 192.

where the dot indicates differentiation with respect to t. If g lies on an element of parameters (v) then g can be represented in the form $v_j = v_j(t)$ and along g our integral will take the form

$$J = \int_{t^1}^{t^2} G(v, \dot v)\, dt. \tag{2.2}$$

We assume that the integrand G is positively regular.

That is we assume that on each element the corresponding function G is such that

$$G_{\varrho_h \varrho_k}(v, \varrho)\, z_h z_k > 0 \qquad (h, k = 1, \cdots, r)$$

for $(\varrho) \neq (0)$, and (z) any set not (0) nor proportional to (ϱ).

An extremal shall be a regular curve of class C''' on S which satisfies the Euler equations set up for integrals (2.2). One proves readily that a curve which satisfies the Euler equations corresponding to any one admissible representation of an element will also satisfy the Euler equations arising from any other admissible representation of the whole or part of that element.

3. **A representation of the neighborhood of a closed curve.** Let g be a regular closed curve of class C''' and of length ω. Concerning g we shall prove the following lemma.

LEMMA. *The part of S near g can be admissibly represented in the form*

$$w_i = w_i(x, y_1, \cdots, y_n) \qquad (n = r - 1;\ i = 1, \cdots, m)$$

in such a fashion that g corresponds to the x axis in the space (x, y) and the functions $w_i(x, y)$ have a period ω in x.

The proof will be made to depend upon the following statements.

Let s be the arc length along g. For each value of s for which $0 \leqq s \leqq \omega$ there exists a set of m directions with direction cosines given by the columns of a matrix

$$\| a_{ij}(s) \| \qquad (i, j = 1, \cdots, m)$$

with the following properties.

A. The functions $a_{ij}(s)$ are continuous in s and $|a_{ij}(s)| > 0$. The first direction of the set is positively tangent to g at the point s, while the first r directions of the set are tangent to S at this point s. The sets $a_{ij}(0)$ and $a_{ij}(\omega)$ each consist of orthogonal directions.

B. The functions $a_{ij}(s)$ have a period ω in s.

To prove these statements we assume that there exists a set $\bar a_{ij}(s)$ which satisfies A but not necessarily B. The proof of this assumption can be given by a process of continuation along g and will be left to the reader.

We shall now show how to replace the set $\bar{a}_{ij}(s)$ by a set $a_{ij}(s)$ which satisfies B as well as A. That this can be done depends upon the orientability of S.

Let π_s be the r-plane tangent to S at the point s on g, and let L_s be the straight line tangent to g at the same point. As the time t varies from 0 to 1 let T represent a continuous rigid motion of our m-space which turns π_0 in itself, leaves the points of L_0 fixed, and which turns the directions of the set $\bar{a}_{ij}(0)$ into the corresponding directions of $\bar{a}_{ij}(\omega)$. Such a motion is possible because of our hypothesis of orientability of S.

We now return to the set $\bar{a}_{ij}(s)$ as originally defined.

Let e be a small positive constant. As the time t varies from 0 to 1 let each direction of the set $\bar{a}_{ij}(s)$ for which s lies between 0 and e be turned under T until t reaches that time t_s which divides the interval (0, 1) in the ratio inverse to that in which s divides the interval $(0, e)$. Of the directions into which the directions $\bar{a}_{ij}(s)$ are thereby turned we now replace the first direction by the direction of L_s and the 2nd to the rth by their orthogonal projections on π_s. The set $\bar{a}_{ij}(0)$ will thereby be replaced by the set $\bar{a}_{ij}(\omega)$, and if e be sufficiently small the final sets $a_{ij}(s)$ will satisfy both A and B.

To return to the proof of the lemma we now approximate the elements $a_{ij}(s)$ by elements $b_{ij}(s)$ which are analytic in s and have the period ω, except that we take the first column of $b_{ij}(s)$ as the first column of $a_{ij}(s)$. We make the approximations so close that $|b_{ij}(s)|$ is not zero.

Suppose that g is given in the form $w_i = w_i(s)$. Near g we shall make a transformation from the variables (w) to variables

$$(x, y_1, \cdots, y_{m-1}) \tag{3.1}$$

of the form

$$w_i = w_i(x) + b_{i\,p+1}(x) y_p \qquad (i = 1, \cdots, m;\ p = 1, \cdots, m-1) \tag{3.2}$$

where x is any number, and the variables y_p are near zero.

Under this transformation the x axis corresponds to g. The Jacobian of the right hand number of (3.2) with respect to the variables (3.1) evaluated for $y_p = 0$, is seen to be $|b_{ij}(x)| \neq 0$.

Suppose $w_i = \bar{w}_i(v)$ is an admissible representation of S near a point P on g. We shall see that the variables

$$(x, y_1, \cdots, y_n) \qquad n = r - 1 \tag{3.3}$$

can serve as parameters in an admissible representation of S near P.

To establish this fact we note that the equations

$$\bar{w}_i(v) = w_i(x) + b_{i\,p+1}(x) y_p \tag{3.4}$$

have an initial solution corresponding to the point P on g. In the neighborhood of this initial solution these equations can be solved for the parameters (v) and the variables

$$y_r, \cdots, y_{m-1}$$

in terms of the variables (3.3). In fact the pertinent Jacobian D is not zero if the elements $b_{ij}(x)$ are sufficiently good approximations of the elements $a_{ij}(x)$. For one notes that in D the columns of partial derivatives of $\bar{w}_i(v)$ define a set of r independent directions in π_s, while the remaining columns of D define $m-r$ additional independent directions if our approximations are sufficiently close.

Thus the variables (3.3) form a set of admissible parameters in a representation of S near g, and the lemma follows directly.

In terms of the parameters (x, y) of the preceding lemma our integral J will now be given the form (cf. § 5, Trans. III)

$$J = \int_a^b f(x, y, y')\, dx, \qquad y'_k = \frac{dy_k}{dx} \tag{3.5}$$

where we restrict ourselves to ordinary curves near the x axis composed of a finite succession of closed segments on each of which y_k is a function of x of class C'.

4. **The fundamental form Q of a closed extremal.** Let g be a closed extremal on S of length ω. With g we shall now associate a quadratic form Q which brings to light certain remarkable characteristics of g and enables one to define in a very simple way certain integral geometric invariants of g.

We represent S near g as in the preceding lemma so that g is mapped on the x axis. In particular g will correspond to a segment γ of the x axis from $x = 0$ to $x = \omega$.

We cut orthogonally across γ with $p + 1$ successive n-planes

$$t_0, \cdots, t_p \tag{4.1}$$

of which the first is $x = 0$ and the last $x = \omega$ and which are placed so near together that there are no pairs of conjugate points on the successive segments into which γ is divided. (See § 9, Trans. III). Let P_i be any point on t_i near γ arbitrarily placed on t_i except that P_0 and P_p shall have the same coördinates (y).

The points $P_0, \cdots, P_p$ can be successively joined by extremal segments neighboring γ. Let the resulting broken extremal be denoted by E. Let (u) be a set of $\mu = pn$ variables of which the first n are the coördinates (y) of P_1, the second n the coördinates (y) of P_2, and so on until

finally the last are the coördinates (y) of P_p. The value of the integral J taken along E will be denoted by $J(u)$.

The function $J(u)$ will have a critical point at $(u) = (0)$, that is a point at which all of its first partial derivatives are zero.

We come next to our *fundamental form*

$$Q(z) = J^0_{u_h u_k} z_h z_k \qquad (h, k = 1, \cdots, \mu)$$

where the superscript 0 calls for the evaluation of the partial derivatives at $(u) = (0)$.

By the *index* of a quadratic form is meant the number of negative terms that appear in the form after the form has been reduced by a real, non-singular, linear transformation to squared terms alone. By the *nullity* of the form is meant the order of its matrix minus its rank.

We shall classify our closed extremals according to the index and nullity of the form Q.

A closed extremal whose nullity is zero will be called *non-degenerate*.

We set $y'_i = p_i$ and further set

$$2\,\Omega(x, \eta, \eta') = f^0_{p_i p_j} \eta'_i \eta'_j + 2 f^0_{p_i y_j} \eta'_i \eta_j + f^0_{y_i y_j} \eta_i \eta_j$$

where $i, j = 1, \cdots, n$, and where the partial derivatives are evaluated for $(x, y, p) = (x, 0, 0)$ as indicated by the superscript zero. Curves in the space (x, η) which are represented by functions $\eta_i = \eta_i(x)$ of class C'' and which satisfy the Jacobi differential equations

$$\Omega_{\eta_i} - \frac{d}{dx}\,\Omega_{\eta'_i} = 0 \qquad (i = 1, \cdots, n) \tag{4.2}$$

will be called *secondary* extremals.

It will convenient to suppose the spaces (x, η) and (x, y) are identical. With this understood we see that the successive points $P_0, \cdots, P_p$ can be joined by secondary extremals. The set of these secondary extremals form a broken secondary extremal which we shall say is *determined* by and *determines* the set (u) previously defined.

With this understood we state the following lemma.

LEMMA. *The fundamental form Q can be represented as follows,*

$$Q(z) = J^0_{u_h u_k} z_h z_k = 2 \int_0^{\omega} \Omega(x, \eta, \eta')\, dx, \qquad (h, k = 1, \cdots, \mu)$$

where $\eta_j = \eta_j(x)$ represents the broken secondary extremal determined by (z).

The proof is essentially that in § 10, Trans. III.

We next consider the integral

$$2 \int_0^{\omega} [\Omega(x, \eta, \eta') - \sigma \eta_i \eta_i]\, dx \qquad (i = 1, \cdots, n) \tag{4.3}$$

in which the parameter σ may have any value independent of x.

Let the system of n Euler equations corresponding to the integral (4.3) be represented by

$$\Omega^{\sigma i} = 0 \qquad (i = 1, \cdots, n).$$

We state the following theorem.

THEOREM I. *The nullity of the fundamental form $Q(z)$ equals the number of linearly independent solutions of the equations $\Omega^{\sigma i} = 0$ which have the period ω for $\sigma = 0$.*

The index of the form $Q(z)$ equals the number of linearly independent solutions of the equations $\Omega^{\sigma i} = 0$ which have a period ω and for which $\sigma < 0$.

This theorem is proved in § 15, Amer. Journ. loc. cit.

5. **The non-degenerate case when $n = 1$.** In this case, in § 18, Trans. II we have further analyzed the preceding segments γ as follows.

If $x = 0$ is not conjugate to $x = \omega$ on γ, there exists a secondary extremal E on which $\eta(0) = \eta(\omega) = 1$. Let E' be a secondary extremal obtained from E by replacing each point (x, η) on E by the point $(x + \omega, \eta)$. We see that E and E$'$ intersect at $(\omega, 1)$. Moreover, E and E' will not be tangent at $(\omega, 1)$ unless E and E' are identical and hence periodic. This could happen only in case the nullity of $Q(z)$ is positive, that is in the so called "*degenerate*" case. We exclude the degenerate case in this section.

As x increases through ω, E crosses E' at $(\omega, 1)$. If E then enters the region between E' and the x axis, we say that γ is *relatively concave*, in the contrary case *relatively convex.*

We have the following theorem from § 21, Trans. II.

THEOREM 2. *In the case $n = 1$ the index of a non-degenerate periodic extremal g may be determined as follows.*

If on the x axis representing g, $x = 0$ is not conjugate to $x = \omega$, the index of g equals the number of conjugate points of $x = 0$ which precede $x = \omega$ plus 0 or 1 according as the x axis between 0 and ω is relatively convex or concave.

If a point $x = a$ is conjugate to $x = a + \omega$, the index of g is the number of conjugate points of $x = a$ between a and $a + \omega$ including $a + \omega$.

We shall now establish the following theorem.

THEOREM 3. ($n = 1$). *If to each point x on the x axis representing the closed extremal there corresponds k conjugate points ($k > 0$) between x and $x + \omega$, while the point x is never conjugate to $x + \omega$, then the index of the closed extremal is k if k is odd and $k + 1$ if k is even.*

By virtue of the preceding theorem we have merely to show that γ is relatively convex if k is odd and relatively concave if k is even. The proof is largely an application of the Sturm separation theorem.

Suppose $k = 2r - 1$ where $r > 0$.

I say that E cannot pass below E' at the point $(\omega, 1)$, that is E cannot there enter the region between E' and the x axis.

A comparison of E with a secondary extremal which vanishes at $x = 0$ shows that E must vanish $2r$ times on the interval $(0, \omega)$. Let $x = a$ be the first zero of E following $x = 0$.

The first conjugate point of $x = 0$ following $x = 0$ must follow $x = a$. Hence the first conjugate point of $x = \omega$ following $x = \omega$ must follow $x = a + \omega$. Accordingly E cannot intersect E' except at $x = \omega$ on the closed interval $(\omega, a + \omega)$.

Moreover, E cannot intersect the x axis on the interval $(\omega, a + \omega)$ since that would mean that the point $x = a$ had $2r$ conjugate points on the interval $(a, a + \omega)$ contrary to our hypothesis.

It follows topologically that E cannot pass below E' at the point $(\omega, 1)$.

Thus if k is odd, γ is convex, and the index is k.

One can treat the case where k is even in a similar fashion.

6. **Invariance of the index and degeneracy of a closed extremal.** Let us call a set of parameters (x, y) of the type whose existence is affirmed in the lemma of § 3 a *canonical* set of parameters neighboring g. In terms of these parameters let the fundamental form Q be set up as in § 4. If Q is degenerate g will be momentarily termed *degenerate relative to the parameters* (x, y). The index of Q will be momentarily called the *index* of g *relative to the parameters* (x, y).

We shall prove the following theorem.

THEOREM 4. *The property of degeneracy of a closed extremal g and value of the index of a non-degenerate closed extremal are independent of the particular system of canonical parameters with the aid of which the fundamental form is defined.*

Let (x, y) be a set of canonical parameters and $f(x, y, y')$ the corresponding integrand of § 3. Let k be the index of g relative to the parameters (x, y).

If g is non-degenerate its index k can be characterized in a way clearly independent of the canonical parameters used. To proceed we need a definition.

Let e be a small positive constant. An ordinary closed curve h on S representable in the space (x, y) in the form $y_i = y_i(x)$ will be said to lie in the *neighborhood* R_e of g if y_i and y'_i are in absolute value less than e along h. $0 \leqq x \leqq \omega$.

The case $k = 0$ can now be characterized as follows.

(a) *In case g is non-degenerate relative to the parameters (x, y) a necessary and sufficient condition that its index k be zero relative to (x, y), is that g afford a proper minimum to J relative to all ordinary closed curves which lie in a sufficiently small neighborhood R_e of g.*

In characterizing the case $k \neq 0$ we shall *admit* only those ordinary closed curves which give J a value less than J_g. This characterization is an obvious generalization of a part of Theorem 10 in Trans. II.

(b) *Suppose g is non-degenerate relative to (x, y) and has an index $k \neq 0$. Corresponding to any sufficiently small neighborhood R_e of g there exists an arbitrarily small neighborhood $R_{e'}$ with the following property. The index k of g relative to (x, y) is one more than the minimum order m of closed m-families of admissible curves on $R_{e'}$ which cannot be deformed among admissible curves on R_e into a single admissible curve.*

To return to the proof of Theorem 4 we consider the following integral:

$$I^\sigma = \int_0^\omega [f(x, y, y') - \sigma y_i y_i]\, dx, \qquad (i = 1, \cdots, n). \tag{6.1}$$

The integrand has a period ω in x, and the x axis is still an extremal. If the integral be represented in terms of the original parameters of S, g will still appear as a closed extremal.

The second variation of the above integral will take the form (4.3) and the corresponding Euler equations will be represented by the set of equations,

$$\Omega^{\sigma i} = 0 \qquad (i = 1, \cdots, n). \tag{6.2}$$

Values of σ for which (6.2) has solutions (not null) of period ω are here called characteristic roots.

(c) *Relative to the parameters (x, y), and for $\sigma = 0$ the extremal g is now assumed degenerate.*

According to Theorem 1, $\sigma = 0$ must then be a characteristic root. Let $k(\sigma)$ be the index of g relative to the parameters (x, y) and the integral I^σ. According to Theorem 1 $k(\sigma)$ will suffer an increase as σ increases through 0 equal to the number of linearly independent periodic solutions of (6.2) for $\sigma = 0$. For σ near 0 but not 0, g will be non-degenerate, and the preceding characterizations of k in (a) and (b) are at our disposal.

Let $(\bar{x}, \bar{y})$ be a second set of canonical parameters. Restricting ourselves to ordinary curves in the (x, y) space in a sufficiently small neighborhood R_e of the x axis we see that the integral I^σ can be represented by an integral $\bar{I}^\sigma$ in terms of the parameters $(\bar{x}, \bar{y})$ with $\bar{x}$ as the independent variable.

(d) *I say that under the assumption* (c) g *must be degenerate for* $\sigma = 0$, *relative to the parameters* $(\bar{x}, \bar{y})$.

For in the contrary case the index $\bar{k}(\sigma)$ of g relative to $(\bar{x}, \bar{y})$ and I^σ would be constant for σ sufficiently near zero. But from (a) and (b) we see that $k(\sigma) = \bar{k}(\sigma)$ for σ near 0 but not 0. Thus $k(\sigma)$ would not change as σ increases through $\sigma = 0$. From this contradiction we infer the truth of (d).

The part of the theorem concerning the index now follows from (a) and (b).

7. **The indices of the principal ellipses on an $(m-1)$-ellipsoid.** We shall consider an $(m-1)$-ellipsoid E_{m-1} of the form

$$a_1^2 w_1^2 + \cdots + a_m^2 w_m^2 = 1, \quad a_1 > a_2 > \cdots > a_m > 0 \tag{7.1}$$

for which the constants a_i are near 1.

By *principal ellipse* g_{ij} of E_{m-1} $(i \neq j)$ will be meant the ellipse on E_{m-1} which lies on E_{m-1} and in the 2-plane of the w_i and w_j axes. The number of principal ellipses is $m(m-1)/2$. The principal ellipses are closed geodesics.

The determination of the indices of the principal ellipses on E_{m-1} can be reduced to a determination of the indices of the principal ellipses on E_2. We now proceed with the latter determination.

8. **The ellipsoid E_2.** First recall that if s is the arc length along any closed geodesic g on a 2-dimensional spread S, the conjugate points of the point $s = s_0$ on g are the zeros other than $s = s_0$ of a non-null solution of the differential equation

$$\frac{d^2 u}{d s^2} + K(s) u = 0 \tag{8.1}$$

where $K(s)$ is the total curvature of S at the point s on g, and u vanishes at $s = s_0$. (See Bolza, p. 231).

As in Trans. II we shall call g doubly-degenerate, simply-degenerate, or non-degenerate according as (8.1) possesses 2, 1, or 0 linearly independent, non-null solutions with a period ω.

We shall use the following lemma. (See § 19, Trans. II).

LEMMA 8.1. *If g is a simply-degenerate closed extremal and $u(s)$ is a non-null solution of* (8.1) *with a period ω, the only points s on g for which s is conjugate to $s + \omega$ are the points at which $u(s) = 0$.*

We shall now prove the following lemma.

LEMMA 8.2. *If the semi-axes of the ellipsoid E_2 of § 7 have lengths sufficiently near unity the principal ellipses of E_2 have the following properties.*

(a) *To each point s on g_{12} there corresponds just one conjugate point prior to $s+\omega$, while s is never conjugate to $s+\omega$.*

(b) *To each point s on g_{23} there correspond just two conjugate points prior to $s+\omega$ while s is never conjugate to $s+\omega$.*

(c) *On g_{13} opposite umbilical points are conjugate to each other and to no other points. The geodesic g_{13} is non-degenerate.*

Let the ellipsoid

$$b_1^2 x_1^2 + b_2^2 x_2^2 + b_3^2 x_3^2 = 1, \qquad b_i > 0$$

be denoted by $E(b_1, b_2, b_3)$.

We need the fact that the total curvature $K(s)$ of $E(b_1, b_2, b_3)$ along its principal ellipse g_{12} will be increased if b_3 is replaced by a larger positive constant. In fact $K(s)$ is the product of the curvature k_1 of g_{12} at the point s and the curvature k_2 of the ellipse π in which a plane orthogonal to g_{12} at the point s, cuts $E(b_1, b_2, b_3)$. An increase of b_3 will not alter k_1 but it will diminish the axis of the ellipse orthogonal to the plane g_{12}. It will accordingly increase k_2 and hence $K(s)$.

We now return to the constants $a_1 > a_2 > a_3$ of § 7, and consider the spheroid $E(a_1, a_2, a_2)$. On the ellipse g_{12} of this spheroid we shall measure s from the point $(a_1, 0, 0)$. We shall prove the following statement.

(A) *On the ellipse g_{12} of $E(a_1, a_2, a_2)$ the distance Δs from a point s to its first following conjugate point exceeds $\omega/2$ for all points except the points conjugate to $s = 0$, for which $\Delta s = \omega/2$.*

First note that $s = 0$ is conjugate to $\omega/2$ and ω, and that the corresponding solution of (8.1) has the period ω.

On the other hand on the ellipse g_{12} of $E(a_1, a_2, a_2)$ $K(s)$ is less than the total curvature at the same point on $E(a_1, a_2, a_1)$. But on $E(a_1, a_2, a_1)$ the point $s = \omega/4$ is conjugate to the opposite point on g_{12}. An application of the Sturm comparison theorem to (8.1) now shows that the distance from $s = \omega/4$ to its first conjugate point on g_{12} exceeds $\omega/2$ on $E(a_1, a_2, a_2)$. It follows from Lemma 8.1 that the same is true for all points of g_{12} on $E(a_1, a_2, a_2)$, except the points conjugate to $s = 0$.

Thus the statement (A) is proved.

We can now prove statement (a) of Lemma 8.2.

The curvature $K(s)$ on the ellipse g_{12} of $E(a_1, a_2, a_3)$ is less than that at the same points on $E(a_1, a_2, a_2)$. It follows from (A) and the Sturm comparison theorem that to each point s on the ellipse g_{12} of $E(a_1, a_2, a_3)$ there corresponds at most one conjugate point prior to or including $s+\omega$. If we bear in mind what happens on the unit sphere, we see that if the constants a_i are sufficiently near unity there will be exactly one conjugate point of the point s prior to $s+\omega$, and $s+\omega$ will not be conjugate to s.

Thus (a) is proved.

We can prove (b) similarly, first proving the following.

(B) *On the ellipse* g_{23} *of* $E(a_2, a_2, a_3)$ *the distance* Δs *from a point* s *to the first following conjugate point is less than* $\omega/2$ *for all points except the points conjugate to the point* $(w) = (0, 0, a_3)$, *for which* $\Delta s = \omega/2$.

To prove (B) we compare g_{23} on $E(a_2, a_2, a_3)$ with g_{23} on $E(a_3, a_2, a_3)$, and then use Lemma 8.1 as in the proof of (A).

To prove (b) we make use of (B) comparing g_{23} on $E(a_1, a_2, a_3)$ with g_{23} on $E(a_2, a_2, a_3)$.

To prove (c) we recall that the geodesics through an umbilical point pass through the opposite umbilical point but form a field otherwise. Hence, each umbilical point on g_{13} is conjugate to the opposite umbilical point.

Further, g_{13} cannot be doubly-degenerate. For if so $s + \omega$ would be the second conjugate point, not only of the umbilical points s, but also of all points s. After a slight increase of a_2 no point $s + \omega$ on g_{13} would be a conjugate point of s, contrary to the fact that there would still be umbilical points on g_{13}. Thus g_{13} cannot be doubly-degenerate.

Finally g_{13} cannot be simply-degenerate because it would then follow from Lemma 8.1 that the four umbilical points would be mutually conjugate, which is false.

Thus (c) is proved.

From Theorems 2 and 3 and the preceding lemma we now have the following theorem.

THEOREM 5. *If the semi-axes of the ellipsoid*

$$a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2 = 1 \qquad a_1 > a_2 > a_3 \tag{8.2}$$

are sufficiently near unity the indices of the principal ellipses g_{12}, g_{13}, *and* g_{23} *are* 1, 2, *and* 3, *respectively.*

9. **The index formula on E_{m-1}.** We shall now determine the indices of the principal ellipses g_{ij} on the $(m-1)$-ellipsoid E_{m-1} of (7.1).

By the *principal ellipsoids* of E_{m-1} we shall mean those ellipsoids E_2 which are obtained from E_{m-1} by setting all of the coördinates (w) equal to zero excepting three.

We note that the geodesics on any principal ellipsoid are geodesics on E_{m-1}. This is seen most simply by putting the equations of the geodesics in the form

$$\frac{d^2w_i}{ds^2} = \lambda w_i a_i^2 \qquad (i = 1, \cdots, m;\ i \text{ not summed})$$

where (w) lies on E_{m-1}, and λ is a function of s.

We shall now fix our attention on a particular principal ellipse, $g_{m-1,m}$, *which we denote by* $\bar{g}$.

We can represent E_{m-1} near $\bar{g}$ in terms of parameters (x, y) as in the Lemma of § 3. In fact if we set

$$r^2(y) = 1 - a_h^2 y_h^2 \qquad (h = 1, \cdots, n = m-2)$$

we can represent E_{m-1} near $\bar{g}$ in the form

$$\begin{aligned} w_h &= y_h && (h = 1, \cdots, n), \\ a_{m-1} w_{m-1} &= r(y) \cos x && r > 0, \\ a_m w_m &= r(y) \sin x. && \end{aligned} \tag{9.1}$$

The integral of arc length on E_{m-1} near $\bar{g}$ becomes the integral (3.5). We continue with the notation of sections 3, 4, and 5.

The principal ellipsoids E_2^k. The principal ellipsoid which lies in the 3-space of the w_k, w_{m-1}, w_m axes $(k < m-1)$ will be denoted by E_2^k.

If we set all of the parameters (y) except y_k equal to zero, (9.1) will give a representation of the neighborhood of $\bar{g}$ on E_2^k. The geodesics of E_{m-1} near $\bar{g}$ on E_2^k will be represented by extremals in the 2-plane of x and y_k.

One readily verifies the fact that the present integral of arc length in the form (3.5) will be unchanged if we replace y_h by $-y_h$ and y'_h by $-y'_h$ for any particular integer h. It follows that under this substitution each extremal arc is replaced by an extremal arc.

Now consider the fundamental form $Q(z)$ set up for $\bar{g}$ as in § 4.

Recall that (z) is composed of the successive sets of coördinates (y) of the points $P_1, \cdots, P_p$ of § 4. Let the coördinates y_h of P_k now be denoted by

$$y_h^k \qquad (k = 1, \cdots, p;\ h = 1, \cdots, n).$$

It follows from the preceding symmetry considerations that $Q(z)$ is unchanged if for any h we replace

$$(y_h^1, \cdots, y_h^p) \quad \text{by} \quad (-y_h^1, \cdots, -y_h^p).$$

As a matter of quadratic forms it then follows that we can write

$$Q(z) = Q_1(y_1^1, \cdots, y_1^p) + \cdots + Q_n(y_n^1, \cdots, y_n^p)$$

where Q_k is a quadratic form in its arguments alone.

Hence the index of $Q(z)$ will be the sum of the indices of the separate forms Q_k.

But if we set all of the variables (z) equal to zero except those in Q_k we see that Q_k is the fundamental form (except for the symbols for its arguments) which would be associated with $\bar{g}$ if $\bar{g}$ were regarded as a closed geodesic on the principal ellipsoid E_2^k.

We have made the preceding analysis for $\bar{g} = g_{m-1,m}$. It is not essentially different for g_{ij} in general. We therefore have the following theorem.

THEOREM 6. *The index of the principal ellipse g_{ij} on the ellipsoid E_{m-1} in m-space is the sum of the indices of g_{ij} regarded as an ellipse on each of the $m-2$ principal 2-dimensional ellipsoids on which g_{ij} lies.*

When the semi-axes of E_{m-1} have lengths sufficiently near unity we can evaluate the index k of g_{ij} by using Theorems 5 and 6.

Suppose $i<j$. Then the index k is three times the number of integers between i and 0, plus two times the number of integers between i and j, plus one times the number of integers between j and $m+1$. Thus

$$k = 3(i-1)+2(j-i-1)+(m-j) = m+i+j-5.$$

We accordingly have the following theorem.

THEOREM 7. *If the semi-axes of the $(m-1)$-ellipsoid* (8.1) *have lengths sufficiently near unity, the index of the principal ellipse g_{ij} is $m+i+j-5$.*

The number of distinct indices of principal ellipses is $2m-3$.

The preceding theorem can be immediately extended as follows.

THEOREM 8. *If the semi-axes of the $(m-1)$-ellipsoid* (7.1) *have lengths sufficiently near unity, the index k of the closed geodesic which covers a principal ellipse g_{ij} $r+1$ times is given by the formula:*

$$k = m+i+j-5+2r(m-2).$$

To prove this extension we review the proof of Theorem 7 and successively verify the following statements.

If the semi-axes of the ellipsoid E_2 have lengths sufficiently near unity the indices of the closed geodesics which cover the principal ellipses g_{12}, g_{13}, and g_{23}, $r+1$ times are $2r+1$, $2r+2$, and $2r+3$ respectively.

The index of a closed geodesic g on E_{m-1} which covers a principal ellipse is the sum of the indices of g regarded as a closed geodesic on each of the $m-2$ principal 2-dimensional ellipsoids on which it lies.

The index k of a closed geodesic g which covers g_{ij} on E_{m-1} $r+1$ times $(i<j)$ is accordingly

$$\begin{aligned} k &= (2r+3)(i-1)+(2r+2)(j-i-1)+(2r+1)(m-j) \\ &= m+i+j-5+2r(m-2) \end{aligned}$$

and the theorem is proved.

10. **The exclusiveness of the principal ellipses as closed geodesics.** In this section we shall prove the following theorem.

THEOREM 9. *Let N be an arbitrarily large positive constant. Upon any $(m-1)$-ellipsoid whose semi-axes are unequal and sufficiently near unity in*

length there are no closed geodesics with lengths less than N other than multiples of the principal ellipses.

We state the following lemma without proof. See § 19, Trans. II and Lemma 8.1.

LEMMA. *Let there be given a differential equation*

$$w'' + q(s)\, w = 0 \tag{10.1}$$

in which $q(s)$ is continuous and has a period ω. If $u(s)$ is a solution ($\not\equiv 0$) of period ω upon which all periodic solutions of (10.1) *are dependent, the only solutions whose zeros have the period ω are dependent on $u(s)$.*

Let the $(m-1)$-ellipsoid have the form (7.1). The equations of the geodesics then take the form

$$w_j'' + \lambda w_j a_j^2 = 0 \qquad (j \text{ not summed}), \tag{10.2}$$

$$a_i^2 w_i^2 = 1 \qquad (i, j = 1, \cdots, m) \tag{10.3}$$

where the independent variable is the arc length s, and where λ is an analytic function of s which we now determine. If we differentiate (10.3) twice with respect to s and use (10.2) we find that

$$\lambda = \frac{a_i^2 w_i' w_i'}{a_i^4 w_i w_i} \qquad w_i' w_i' = 1. \tag{10.4}$$

When the constants (a) of $(10.3) = (1)$, $\lambda \equiv 1$. Accordingly for (a) sufficiently near (1), λ will be uniformly near 1 for any point (w) on (10.3) and $w_i' w_i' = 1$.

Let g be any geodesic on E_{m-1} and $\lambda(s)$ the corresponding function λ. Let $w_j(s, c)$ be a solution of the jth equation (10.2) such that

$$w_j(s, c) = 0, \quad w_j'(s, c) = 1.$$

We shall now restrict the constants (a) to a neighborhood R of the set $(a) = (1)$ so small that any variation of (a) in R will cause the successive zeros of $w_j(s, c)$ on the interval

$$c \leqq s \leqq c + N$$

to vary by less than $\pi/4$ regardless of the choice of g, of c, or of j.

Suppose now that g is a closed geodesic $w_i = u_i(s)$, with a length $\omega < N$, and that $\lambda(s)$ is the corresponding function λ. Suppose that the theorem is false. For definiteness assume that

$$u_k(s) \not\equiv 0 \qquad (k = 1, 2, 3). \tag{10.5}$$

We shall make repeated use of the Sturm comparison theorem which we denote by (S).

An obvious use of (S) shows that $u_k(s)$ must vanish at least once, say at s_k. Because of our restrictions on the constants (a) it follows that the zeros of $u_k(s)$ on the interval

$$s_k \leqq s \leqq s_k + \omega$$

lie respectively within $\pi/4$ of the points

$$s_k,\ s_k + \pi,\ s_k + 2\pi,\ \cdots,\ s_k + 2r\pi$$

where r is a positive integer dependent on ω.

Moreover the solution $u_k(s)$ can have no zero, say $s = a$, in common with

$$u_h(s) \qquad (h \neq k;\ h, k = 1, 2, 3).$$

For the conditions

$$u_k(a) = u_h(a) = u_k(a + \omega) = 0$$

taken with (S), imply that $u_h(a + \omega) \neq 0$ contrary to the periodicity of $u_h(s)$.

The equation (k not summed)

$$(k) \qquad w_k'' + \lambda(s)\, a_k^2\, w_k = 0 \qquad (k = 1, 2, 3)$$

possesses no solution ($\neq 0$) of period ω, independent of $u_k(s)$. For if all solutions of (k) had the period ω a use of (S) would show that equations (h), $h \neq k$, could have no solutions of period ω not identically zero.

Let $D_k(s)$ be the distance along the s axis from a zero of a solution of (k) which is not identically zero and which vanishes at s, to the $2r$th following zero. It follows from the lemma that the only points at which $D_k(s) = \omega$ are the zeros of $u_k(s)$.

A use of (S) shows that $D_2(s) > \omega$ at the zeros of $u_1(s)$. But there is at least one zero of $u_1(s)$ between each two consecutive zeros of $u_2(s)$. Hence, $D_2(s) \geqq \omega$ without exception. This fact and a use of (S) now show that $D_3(s) > \omega$ at each point s.

From this contradiction we infer that (10.5) is impossible.

The theorem follows readily.

11. **The set of all closed extremals in the analytic case.** In the paper which is to follow we shall restrict ourselves to the analytic case, that is to the case where the functions defining the elements of S are analytic as well as the integrands $G(v, \varrho)$. In the analytic case certain very definite statements can be made about the set of all closed extremals neighboring a given closed extremal g. It is convenient to make these statements in this place.

In the lemma of § 3 we showed how to map the neighborhood of a closed curve g on S onto the neighborhood of the x axis in a space of

variables $(x, y_1, \cdots, y_n)$. A review of the proof of this lemma shows that in the analytic case and in case g is analytic the mapping functions $w_i = w_i(x, y)$ there defined are also analytic.

We represent S neighboring g in terms of these parameters (x, y). The extremals neighboring g can then be represented by giving their coördinates y_i as functions $\varphi_i(x, \alpha)$ of x and $2n$ parameters (α) giving the initial values of (y) and (y') when $x = 0$. The functions φ_i will be analytic in their arguments for (α) near the set (0). The conditions that one of these extremals have a period ω are that

$$\varphi_i(0, \alpha) = \varphi_i(\omega, \alpha), \quad \varphi_{ix}(0, \alpha) = \varphi_{ix}(\omega, \alpha) \quad (i = 1, \cdots, n). \tag{11.1}$$

These equations may be satisfied for real sets (α) only when $(\alpha) = (0)$. Apart from this case the real solutions (α) of (11.1) will be representable as functions "in general" analytic on one or more suitably chosen "Gebilde" G[7] of r independent variables with $0 < r \leqq 2n$, each G including the point $(\alpha) = (0)$. To each such set (α) corresponds a periodic extremal.

These periodic extremals neighboring g all give the same value to J.

To see this we consider any regular curve h on one of the above "Gebilde" G, a curve h along which the parameters (α) are analytic on G. We evaluate J along the corresponding extremals from $x = 0$ to $x = \omega$. Upon differentiating J with respect to the arc length along h and making the usual reduction of the first variation we readily obtain the result zero. It follows that J is constant on G, and in fact takes on the value afforded by g.

We shall regard a continuous family of closed curves as *connected* if any closed curve of the family can be continuously deformed into any other closed curve of the family through the mediation of curves of the family.

From the preceding analysis "in the small" we now infer the following result "in the large".

In the analytic case the set of all closed extremals on S on which J is less than a prescribed positive constant make up at most a finite set of connected and continuous families of closed extremals on each of which J is constant, and which are either 0*-dimensional or else analytically representable neighboring a given closed extremal in terms of a variable x and parameters (α) on one or more of the above "Gebilde".*

[7] See Osgood, Funktionentheorie II. To Osgood's "Gebilde" we add the space of the variables (α) as a special "Gebilde".

SUFFICIENT CONDITIONS IN THE PROBLEM OF LAGRANGE WITH FIXED END POINTS.[1]

By Marston Morse.

1. **Introduction.**[2] Essentially the following theorem is stated by Bliss. Its terms will be defined later.

Theorem A. *In order that the extremal E afford a proper, strong, relative minimum to J it is sufficient that E be identically normal on its interval (a, b), that the Clebsch and Weierstrass sufficient conditions hold, and that there be no conjugate point of $x = a$ on (a, b).*

The first object of the present paper is to prove this theorem. It does not appear to have been previously proved, although several proofs exist if E be assumed identically normal on an interval containing (a, b) in its interior.

The difficulties arise in connection with trying to set up a Mayer field containing E. The conventional method of taking a family of extremals through a point P on E's extension just prior to E fails because examples will show that such a family need not form a field near P.

These difficulties are associated with the so called oscillation theorems of which the author gives two, one new, and one old, both proved by new methods. Starting with the excessively complicated methods of Von Escherich this part of the theory has gradually been stripped of its cumbersome algebra. The methods of the author go further in this direction. They are suggested by methods which he has found useful in treating oscillation and separation problems in general.

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[1] Received October 17, 1930.

[2] A list of references follow.

Bliss, The problem of Lagrange in the calculus of variations. Lectures by G. A. Bliss. The University of Chicago, Summer Quarter 1925. These lectures have proved particularly useful to the author. See also American Journal of Mathematics, vol. 52 (1930).

Bolza, Vorlesungen über Variationsrechnung.

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Carathéodory, Die Methode der geodätischen Äquidistanten und das Problem von Lagrange. Acta Mathematica, vol. 47, 1926, pp. 199–236.

Morse and Myers, The problems of Lagrange and Mayer with variable end points. Proceedings of the American Academy of Arto and Sciences, vol. 66 (1931).

J. Radon, Über die Oszillationstheoreme der konjugierten Punkte beim Problem von Lagrange. Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayrischen Akademie der Wissenschaften zu München, (1927), p. 243.

The author's contributions to the theory begin with section four. However the form in which the material is presented in sections two and three varies somewhat from previous work.

2. **The problem and its extremals.** In the space of the variables x and $(y) = (y_1, \cdots, y_n)$ let there be given a curve E,

$$y_i = \bar{y}_i(x), \qquad a \leqq x \leqq b \qquad (i = 1, \cdots, n) \tag{2.1}$$

of class C'' for x on (a, b).

We consider other curves $y_i(x)$ neighboring E of class C' on (a, b). Such curves are called *differentially admissible* if they satisfy m differential equations of the form

$$\varphi_\beta(x, y, y') = 0 \qquad (\beta = 1, \cdots, m;\ m < n). \tag{2.2}$$

We suppose E differentially admissible, and that along E the functional matrix of the functions (2.2) with respect to the variables y'_i is of rank m. A set (x, y, p) will be called *differentially admissible* if it satisfies $\varphi_\beta(x, y, p) = 0$.

A curve of class C' is called *admissible* if it is differentially admissible and joins the ends of E.

We seek conditions sufficient for E to afford a minimum to the integral

$$J = \int_a^b f(x, y, y')\, dx \tag{2.3}$$

among admissible curves.

The functions f and φ_β are to be of class C''' for any admissible set (x, y, y') for which (x, y) is near E.

A constant λ_0 and a set of m functions $\lambda_\beta(x)$ are termed an *admissible set of multipliers* if they are not all zero at one point at least of (a, b), and if the functions $\lambda_\beta(x)$ are of class C' for x on (a, b).

If E affords a minimum to J it is necessary that there exist a set of admissible multipliers such that along E

$$\frac{d}{dx} F_{y'_i} \equiv F_{y_i} \tag{2.4}$$

where[3]

$$F = \lambda_0 f + \lambda_\beta \varphi_\beta \qquad (\beta = 1, \cdots, m). \tag{2.5}$$

An *extremal* is a differentially admissible curve of class C'' which with admissible multipliers satisfies (2.4). It is *normal*[4] on (a, b) if it possesses

[3] The summation convention of tensor analysis is used throughout.

[4] If we were dealing with the problem under the more general end conditions of Morse and Myers, loc. cit., it would be better to replace the term normal by "normal relative to the conditions (2.4)."

no set of multipliers of which $\lambda_0 = 0$. It is *identically normal* on (a, b) if it is normal on every subinterval of (a, b). We suppose throughout that E is normal on (a, b), and take $\lambda_0 = 1$.

The differential equations of the extremals can be put into compact form as follows. Set

$$v_i = F_{y_i'}, \quad 0 = \varphi_\beta, \quad \lambda_0 = 1. \tag{2.6}$$

The jacobian R (see Bolza p. 589) of the right hand members of these equations with respect to the variables $y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_m$ is assumed not zero along E. We can accordingly solve these equations neighboring E for (y') and (λ) in terms of (v) and (y) and put the equations (2.4) and (2.2) in the well known form

$$\frac{dy_i}{dx} = Y_i(x, y, v), \quad \frac{dv_i}{dx} = V_i(x, y, v) \tag{2.7}$$

so that we see that extremals neighboring E are determined by the initial values of (x, y, v).

We now define the function $\Omega(\eta, \eta', \mu)$ as does Bolza p. 621. The second variation then becomes

$$J_2 = 2\int_a^b \Omega(\eta, \eta', \mu)\, dx.$$

Secondary extremals are defined as the extremals of the problem of minimizing the second variation subject to the differential conditions

$$\Phi_\beta(\eta) \equiv \varphi_{\beta y_i'} \eta_i' + \varphi_{\beta y_i} = 0 \qquad (i = 1, \cdots, n;\ \beta = 1, \cdots, m). \tag{2.8}$$

With admissible multipliers $\mu_0, \mu_1(x), \cdots, \mu_m(x)$ they must satisfy (2.8) and

$$\frac{d}{dx}\Omega_{\eta_i'} \equiv \Omega_{\eta_i}. \tag{2.9}$$

If E is normal on (a, b) we see that every secondary extremal is normal on (a, b). For from the definition of normalcy it is necessary and sufficient for the normalcy of E on (a, b) that the equations

$$\frac{d}{dx}(\lambda_\beta\, \varphi_{\beta y_i'}) - \lambda_\beta\, \varphi_{\beta y_i} \equiv 0, \qquad a \leqq x \leqq b \tag{2.10}$$

be satisfied only if $(\lambda) \equiv (0)$. By virtue of (2.9) an identical test for normalcy of a secondary extremal on (a, b) is obtained upon setting $\mu_0 = 0$. Thus each secondary extremal is normal on (a, b). We set $\mu_0 = 1$ hereafter.

As in the case of extremals the differential equations of the secondary extremals can be put into a simplified form

$$\text{(2.11)} \qquad \frac{d\eta_i}{dx} = \eta_i(x, \eta, \zeta), \qquad \frac{d\zeta_i}{dx} = \zeta_i(x, \eta, \zeta)$$

upon setting

$$\text{(2.12)} \qquad \zeta_i = \Omega_{\eta_i'}, \qquad \Phi_\beta(\eta) = 0.$$

The relations (2.12) are to be used to solve for the variables (η') and (μ) in terms of the variables (η), (ζ) and x. The jacobian involved is exactly the jacobian R used previously for the case of extremals.

We recall that the equations (2.11) are linear and homogeneous.

3. Certain sufficient conditions. By the *Clebsch sufficient condition* we mean the condition that

$$\text{(3.1)} \qquad F_{y_i' y_j'} z_i z_j > 0$$

for all sets $(z) \neq (0)$ for which

$$\text{(3.2)} \qquad \varphi_{\beta y_i'} z_i = 0$$

and for (x, y, y', λ) on E.

By the *Weierstrass sufficient condition* we mean the condition

$$\text{(3.3)} \quad F(x, y, q, \lambda) - F(x, y, p, \lambda) - (q_i - p_i) F_{y_i'}(x, y, p, \lambda) = E(x, y, p, q, \lambda) > 0$$

for distinct admissible sets (x, y, p) and (x, y, q) for which (x, y, p, λ) is near the set (x, y, y', λ) on E.

We come now to Mayer fields. (See Bliss p. 64). A family of differentially admissible curves satisfying differential equations of the form

$$\text{(3.4)} \qquad \frac{dy_i}{dx} = p_i(x, y) \qquad (i = 1, \cdots, n)$$

at each point of an open region S in which $p_i(x, y)$ is of class C' will define a so called *field* in S.

If with the slopes $p_i(x, y)$ there can be associated multipliers $\lambda_\beta(x, y)$ also of class C' in S, with $\lambda_0 = 1$, such that the Hilbert integral

$$\text{(3.5)} \qquad I^* = \int (F - y_i' F_{y_i'})\, dx + F_{y_i'}\, dy_i$$

with y_i' and λ_β replaced by $p_i(x, y)$ and $\lambda_\beta(x, y)$ respectively, is independent of the path in S, then the field is called a *Mayer field.*

We shall take up an important test for a Mayer field.

Let there be given a family of extremals of the form

$$\text{(3.6)} \qquad y_i = y_i(x, \alpha_1, \cdots, \alpha_n), \qquad v_i = v_i(x, \alpha_1, \cdots, \alpha_n),$$

containing E for $(\alpha) = (0)$, with $y_i(x, \alpha)$, $y_{ix}(x, \alpha)$, and $v_i(x, \alpha)$ of class C' for x on an interval containing (a, b) in its interior and (α) near (0). If we have

$$\frac{D(y_1, \cdots, y_n)}{D(\alpha_1, \cdots, \alpha_n)} \neq 0, \qquad (\alpha) = (0), \qquad a \leqq x \leqq b, \tag{3.7}$$

then at each point (x, y) near E we can solve the equations $y_i = y_i(x, \alpha)$ for the variables α_i obtaining thereby functions $\alpha_i(x, y)$. We can obtain slope functions $p_i(x, y)$ and multipliers $\lambda_\beta(x, y)$ at each point near E by solving the equations

$$v_i(x, \alpha(x, y)) = F_{y'_i}(x, y, p, \lambda), \qquad \varphi_\beta(x, y, p) = 0$$

for p_i and λ_β as functions of x and (y). Moreover these solutions will be of class C' when (x, y) is near E.

We combine these results with the result stated by Bliss on page 68 to obtain the following lemma.

LEMMA 1. *A family of extremals neighboring E, of the form* (3.6), *with a non-vanishing jacobian* (3.7), *will define a Mayer field neighboring E if on the intersection of the field with an n-plane $x = c$ $(a \leqq c \leqq b)$ the Hilbert integral of the field*

$$I^* = \int v_i(c, \alpha(c, y))\, dy_i$$

is independent of the path.

We shall make use of the following well known theorem on sufficient conditions.

THEOREM 1. *In order that the extremal E afford a proper, strong, relative minimum among admissible curves, it is sufficient that the Weierstrass sufficient condition hold along E, and that there exist a Mayer field containing E as a member and covering a neighborhood of E.*

4. **The first oscillation theorem.** The question of the existence of Mayer fields containing E is connected with the theory of conjugate points which we now define.

Two points c and c' on the interval (a, b) are called *conjugate* if $c \neq c'$, and if there exists a secondary extremal vanishing at c and c' but not identically zero between c and c'.

We shall have occasion to consider families of solutions of the accessory equations (2.11) linearly dependent upon n such solutions given by the respective columns of a matrix

$$\left\| \begin{matrix} \eta_{ij}(x) \\ \zeta_{ij}(x) \end{matrix} \right\| \qquad (i, j = 1, \cdots, n), \qquad a \leqq x \leqq b. \tag{4.1}$$

The set of solutions (4.1) will be called a *base* of the family.

Consider now the base taking on the initial values

$$\eta_{ij}(c) = 0, \qquad \zeta_{ij}(c) = \delta_i^j, \tag{4.2}$$

where δ_i^j is the Kronecker delta. For this base set

$$D(x, c) = |\eta_{ij}(x)|. \tag{4.3}$$

We shall prove the following lemma.

LEMMA 2. *If x_1 is conjugate to c then $D(x_1, c) = 0$. Conversely if $D(x_1, c) = 0$, $x_1 \neq c$, and E is normal on (x_1, c), then x_1 is conjugate to c.*

Any accessory solution for which (η) vanishes at c will be linearly dependent on the columns of (4.1). It follows that $D(x_1, c) = 0$ if x_1 is conjugate to c.

Conversely if $D(x_1, c) = 0$ for $x_1 \neq c$, there must exist a solution of the accessory equations of the form

$$\eta_i = a_j \eta_{ij}(x), \qquad \zeta_i = a_j \zeta_{ij}(x), \tag{4.4}$$

where the a_i's are constants $(a) \neq (0)$, so chosen that (η) vanishes at x_1. If now (η) were identically zero on (x_1, c), it would follow from (2.9) that the corresponding multipliers $\mu_\beta(x)$ determined by (η, ζ) using (2.12) would satisfy

$$\frac{d}{dx}(\mu_\beta \varphi_{\beta y'_i}) - \mu_\beta \varphi_{\beta y_i} \equiv 0$$

on (x_1, c) and hence be identically zero, since E is normal on (x_1, c). The corresponding functions $\zeta_i(x) = \Omega_{\eta'_i}$ would then be zero on (x_1, c). But this is impossible since (4.4) and (4.2) show that $\zeta_i(c) = a_i$, and we have taken $(a) \neq (0)$.

Thus the functions (η) defined by (4.4) are not identically zero on (x_1, c), and x_1 is conjugate to c. Thus the lemma is proved.

We shall continue with the following lemma.

LEMMA 3. *If the Clebsch sufficient condition holds on (a, b) there exists a positive lower bound of the distances from a point $x = c$ on (a, b) to its conjugate points.*

Consider now a base (4.1) taking on the values

$$\eta_{ij}(c) = \delta_i^j, \qquad \zeta_{ij}(c) = 0. \tag{4.5}$$

The corresponding family of solutions may be represented in the form

$$\eta_i = u_j \eta_{ij}(x), \qquad \zeta_i = u_j \zeta_{ij}(x), \tag{4.6}$$

where (u) is any set of n constants. The jacobian

$$\frac{D(\eta_i, \cdots, \eta_n)}{D(u_1, \cdots, u_n)} = |\eta_{ij}(x)| = \Delta(x, c) \tag{4.7}$$

is unity for $x = c$, and hence does not vanish in some closed neighborhood $(c - e, c + e)$ of c. $(e > 0)$. Moreover e can be chosen independently of the position of c on (a, b), as follows from the uniform continuity of $\Delta(x, c)$.

We now prove a statement A.

A. *The family* (4.6) *will form a Mayer field of secondary extremals for x on $(c - e, c + e)$.*

This appears at once from Lemma 1, of § 3 if we note that the Hilbert integral on the n-plane $x = c$ here takes the form

$$I^* = 2\int \zeta_i(c)\, d\eta_i = 2\int 0\, d\eta_i = 0$$

and so is independent of the path.

Recall now that the Clebsch sufficient condition relative to E entails the Weierstrass sufficient condition relative to the second variation for the segment (a, b) of the x axis regarded as a secondary extremal. This follows from an obvious use of Taylor's formula.

We can now prove statement B.

B. *There are no conjugate points of the point $x = c$ on that part of the interval $(c - e, c + e)$ on (a, b).*

For if x_1 were conjugate to c on $(c - e, c + e)$ and (a, b), by our definition of a conjugate point there would exist a secondary extremal $(\bar{\eta})$ vanishing at x_1 and c, but not identically zero on (x_1, c).[5] If we now apply Theorem 1 to the second variation with the segment (x_1, c) of the x axis regarded as the secondary extremal, we see that $(\bar{\eta})$ must make the second variation positive. For we have a Mayer field covering (x_1, c), according to A, and the Weierstrass sufficient condition satisfied, and we can clearly suppose $(\bar{\eta})$ arbitrarily near (x_1, c).

On the other hand the usual process of integration of the second variation by parts will show that the second variation is zero along $(\bar{\eta})$. From this contradiction we infer the truth of B.

The lemma follows readily.

If we combine the two preceding lemmas we obtain our first oscillation theorem.

THEOREM 2. *If E is identically normal on (a, b), and if the Clebsch sufficient condition holds on (a, b), there exists a positive constant e such that*

[5] We treat the case where $x_1 < c$.

$$D(x, c) \neq 0, \quad x \neq c, \quad c-e \leqq x \leqq c+e,$$

for x and c on (a, b).

5. A second oscillation theorem. We shall now establish the following theorem previously unproved.

THEOREM 3. *Suppose E is identically normal on (a, b). If the Clebsch sufficient condition holds, and there is no conjugate point of $x=a$ on (a, b), there exists a Mayer field of secondary extremals containing the x axis as a member and covering a neighborhood of the segment (a, b).*

The family of secondary extremals vanishing at $x=a$ represented in the form (4.4), possesses a determinant

$$D(x, a) = |\eta_{ij}(x)| \neq 0, \quad a < x \leqq b, \tag{5.1}$$

as follows from Lemma 2 § 4. Accordingly we can take a new base for this family of the form, $\eta^0_{ij}(x)$, $\zeta^0_{ij}(x)$ requiring that

$$\eta^0_{ij}(a) = 0, \quad \eta^0_{ij}(b) = \delta^j_i. \tag{5.2}$$

The secondary extremal of this family which joins the point $(x, \eta) = (a, 0)$ to $(x, \eta) = (b, u)$ can then be represented in the form $(i, j = 1, \cdots, n)$

$$\eta^0_i = u_j \eta^0_{ij}(x), \quad \zeta^0_i = u_j \zeta^0_{ij}(x). \tag{5.3}$$

Let the value of the second variation taken along this family from $x=a$ to $x=b$ be denoted by $J^0_2(u)$. If we integrate the second variation by parts in the usual way we find that

$$J^0_2(u) = \eta^0_i(b)\,\zeta^0_i(b) \equiv u_i u_j \zeta^0_{ij}(b), \tag{5.4}$$

noting that $\eta^0_i(b) = u_i$. We note that $J^0_{2u_i} = 2\,\zeta^0_i(b) = 2\,u_j \zeta^0_{ij}(b)$.

We shall now define a new base $\bar{\eta}_{ij}(x)$, $\bar{\zeta}_{ij}(x)$ of which the corresponding family will define the desired Mayer field.

We require that

$$\bar{\eta}_{ij}(b) = \eta^0_{ij}(b), \quad \bar{\zeta}_{ij}(b) = \zeta^0_{ij}(b) - \delta^j_i. \tag{5.5}$$

The secondary extremal of the resulting family which passes through the point $(x, \eta) = (b, u)$ may be represented in the form

$$\bar{\eta}_i = u_j \bar{\eta}_{ij}(x), \quad \bar{\zeta}_i = u_j \bar{\zeta}_{ij}(x). \tag{5.6}$$

We shall now prove the following.

A. *A secondary extremal $(\bar{\eta})$ of the family* (5.6) *which is not identically zero cannot vanish on (a, b).*

Such a secondary extremal $(\bar{\eta})$ cannot vanish at $x = b$, in particular, without (u) being (0), and hence $(\bar{\eta})$ identically zero.

Suppose that $(\bar{\eta})$ vanished at c where $a \leqq c < b$. Let us evaluate the second variation along the x axis from $x = a$ to $x = c$, and then along $(\bar{\eta})$ to its intersection (b, u) with the n-plane $x = b$. Denote the resulting value of the second variation by $\bar{J}_2(u)$. The usual integration by parts shows that

$$\bar{J}_2(u) = \bar{\eta}_i(b)\,\bar{\zeta}_i(b).$$

Upon using (5.6), (5.5), and (5.2) we find that $\bar{\eta}_i(b) = u_i$ and that

$$\bar{J}_2(u) = u_i u_j(\zeta^0_{ij}(b) - \delta^j_i) \qquad (i, j = 1, \cdots, n).$$

With the aid of (5.4) we see then that for the *above* set (u)

$$\bar{J}_2(u) = J^0_2(u) - u_i u_i.$$

The conclusion that we have arrived at, namely that $\bar{J}_2(u) < J^0_2(u)$ for the above set (u), is impossible. For the family of secondary extremals vanishing at $x = a$ forms a Mayer field for $a \leqq x \leqq b$ except at the point $(a, 0)$. In spite of the singularity at $x = a$ one can see that the result of Theorem I still holds for this special case for curves in the field, so that we must have $\bar{J}_2(u) \geqq J^0_2(u)$ for our set (u).

From this contradiction we infer that $(\bar{\eta})$ cannot vanish on (a, b) and the statement A is proved.

It follows that

$$|\bar{\eta}_{ij}(x)| \neq 0, \qquad a \leqq x \leqq b. \tag{5.7}$$

For if the determinant (5.7) vanished at c there would exist a set of constants $(u) \neq (0)$ such that the secondary extremal $(\bar{\eta})$ given by (5.6) would vanish at c. But $(\bar{\eta})$ would not be identically zero since $\bar{\eta}_i(b) = u_i$, thus contradicting statement A.

To prove that the field defined by (5.6) is a Mayer field we evaluate the corresponding Hilbert integral on the plane $x = b$. We find that at the intersection of the field with $x = b$

$$I^* = 2\int \bar{\zeta}_i(b)\, d\bar{\eta}_i.$$

Using (5.6) and (5.5), noting that on $x = b$, $d\bar{\eta}_i = du_i$, we find that

$$I^* = 2\int u_j(\zeta^0_{ij}(b) - \delta^j_i)\, du_i.$$

Referring to (5.4) and integrating we obtain the result

$$I^* = J_2^0(u) - u_j u_j + \text{const.}$$

Thus I^* is independent of the path and (5.6) forms a Mayer field.

The theorem is thereby proved.

6. **Sufficient conditions for a minimum.** A Mayer field of secondary extremals linearly dependent on secondary extremals of a base $\eta_{ij}(x)$, $\zeta_{ij}(x)$ of secondary extremals will be called a *linear Mayer field.* It can be shown that all Mayer fields of secondary extremals are linear, but such a proof is not necessary here and will be omitted. We note that throughout a linear Mayer field $|\eta_{ij}(x)| \neq 0$, for otherwise the field would fail to cover the neighborhood of all its points.

We have affirmed the existence of a linear Mayer field of secondary extremals in Theorem 3. We now prove the following theorem.

THEOREM 4. *Let there be given a family of secondary extremals defining a linear Mayer field L covering the neighborhood of the segment (a, b) of the x axis. There will then exist a family of ordinary extremals forming a Mayer field containing E which may be so represented that the functional matrix of y_i and v_i with respect to the parameters of the family reduces on E to a base of the field L.*

Without loss of generality we can suppose the base of the given field L so chosen that

$$\eta_{ij}(a) = \delta_i^j. \tag{6.1}$$

The secondary extremal of the field through the point $(a, \eta) = (a, u)$ will be given by

$$\eta_i = u_j \eta_{ij}(x), \qquad \zeta_i = u_j \zeta_{ij}(x). \tag{6.2}$$

We suppose the extremal E given in the form

$$y_i = \bar{y}_i(x), \qquad v_i = \bar{v}_i(x), \qquad a \leqq x \leqq b. \tag{6.3}$$

We now define a family of ordinary extremals

$$y_i = y_i(x, u), \qquad v_i = v_i(x, u) \tag{6.4}$$

for (u) near (0), and x on a slight enlargement of the interval (a, b) by giving the conditions at $x = a$,

$$y_i(a, u) = \bar{y}_i(a) + u_j \eta_{ij}(a), \qquad v_i(a, u) = \bar{v}_i(a) + u_j \zeta_{ij}(a)$$

for each (u) near (0).

The functional matrix of the family (6.4) evaluated for $(u) = (0)$ is the given base of the family L. We see then that the family (6.4) defines a field of extremals covering a neighborhood of E.

To show that the field defined by (6.4) is a Mayer field we evaluate the Hilbert integral on the intersection of the field with the n-plane $x = a$. If we note that for $x = a$, $dy_i = du_i$, we see that this Hilbert integral on $x = a$ becomes

$$I^* = \int v_i(a, u)\, du_i = \int \overline{v}_i(a)\, du_i + \int u_j\, \zeta_{ij}(a)\, du_i.$$

That the last integral is independent of the path follows at once from the observation that it is one half the Hilbert integral on $x = a$ set up for the given Mayer field of secondary extremals. Thus the family (6.4) defines a Mayer field containing E.

Theorems 1, 3, and 4 now combine to give us Theorem A of the introduction.

Referring to (5.4) and integrating we obtain the result

$$I^* = J_2^0(u) - u_j u_j + \text{const.}$$

Thus I^* is independent of the path and (5.6) forms a Mayer field.

The theorem is thereby proved.

6. Sufficient conditions for a minimum. A Mayer field of secondary extremals linearly dependent on secondary extremals of a base $\eta_{ij}(x)$, $\zeta_{ij}(x)$ of secondary extremals will be called a *linear Mayer field*. It can be shown that all Mayer fields of secondary extremals are linear, but such a proof is not necessary here and will be omitted. We note that throughout a linear Mayer field $|\eta_{ij}(x)| \neq 0$, for otherwise the field would fail to cover the neighborhood of all its points.

We have affirmed the existence of a linear Mayer field of secondary extremals in Theorem 3. We now prove the following theorem.

THEOREM 4. *Let there be given a family of secondary extremals defining a linear Mayer field L covering the neighborhood of the segment (a, b) of the x axis. There will then exist a family of ordinary extremals forming a Mayer field containing E which may be so represented that the functional matrix of y_i and v_i with respect to the parameters of the family reduces on E to a base of the field L.*

Without loss of generality we can suppose the base of the given field L so chosen that

$$\eta_{ij}(a) = \delta_i^j. \tag{6.1}$$

The secondary extremal of the field through the point $(a, \eta) = (a, u)$ will be given by

$$\eta_i = u_j \eta_{ij}(x), \qquad \zeta_i = u_j \zeta_{ij}(x). \tag{6.2}$$

We suppose the extremal E given in the form

$$y_i = \bar{y}_i(x), \qquad v_i = \bar{v}_i(x), \qquad a \leqq x \leqq b. \tag{6.3}$$

We now define a family of ordinary extremals

$$y_i = y_i(x, u), \qquad v_i = v_i(x, u) \tag{6.4}$$

for (u) near (0), and x on a slight enlargement of the interval (a, b) by giving the conditions at $x = a$,

$$y_i(a, u) = \bar{y}_i(a) + u_j \eta_{ij}(a), \qquad v_i(a, u) = \bar{v}_i(a) + u_j \zeta_{ij}(a)$$

for each (u) near (0).

The functional matrix of the family (6.4) evaluated for $(u) = (0)$ is the given base of the family L. We see then that the family (6.4) defines a field of extremals covering a neighborhood of E.

To show that the field defined by (6.4) is a Mayer field we evaluate the Hilbert integral on the intersection of the field with the n-plane $x = a$. If we note that for $x = a$, $dy_i = du_i$, we see that this Hilbert integral on $x = a$ becomes

$$I^* = \int v_i(a, u)\, du_i = \int \bar{v}_i(a)\, du_i + \int u_j\, \zeta_{ij}(a)\, du_i.$$

That the last integral is independent of the path follows at once from the observation that it is one half the Hilbert integral on $x = a$ set up for the given Mayer field of secondary extremals. Thus the family (6.4) defines a Mayer field containing E.

Theorems 1, 3, and 4 now combine to give us Theorem A of the introduction.

ON THE NECESSARY CONDITION OF WEIERSTRASS IN THE MULTIPLE INTEGRAL PROBLEM OF THE CALCULUS OF VARIATIONS.[1]

By E. J. McShane.[2]

1. The proof of the necessity of the Weierstrass condition for the curves minimizing an integral $\int f(x, y, y')\,dx$ has been established by Tonelli under hypotheses which could hardly be weakened. The extension to curves in space has not been carried out, to the best of my knowledge; but it seems that it should be possible without great difficulty. On the other hand, the analogous condition for the multiple integral problem has by no means been established with the same generality. Levi[3] considered $\int\int f(x, y, z, p, q)\,dx\,dy$ and showed that $E(x, y, z, p, q, \bar{p}, \bar{q}) \geqq 0$ at all points near which the minimizing surface possesses continuous partial derivatives; this is, I believe, the only paper dealing with the Weierstrass condition for multiple integrals in non-parametric form.

The present paper develops partially the theory of the Weierstrass condition for the case of the n-tuple integral $\int_D f(x, z, z_x)\,dx$, where f depends on n independent variables x, p functions $z(x)$, and their first-order partial derivatives. From the rather fragmentary results on the general case we deduce as corollaries a theorem on curves in space which when further restricted to the plane is somewhat weaker than Tonelli's; and a theorem on n-spreads in $n+1$-space, which for $n = 2$ is decidedly stronger than Levi's.

In the last two sections Legendre's condition is established, and also a theorem on semi-continuity.

2. **Notations and definitions.** Except when statement is made to the contrary, we shall use matrix notation;

x will mean the n-tuple $(x^1, x^2, \cdots, x^n)$,

$z(x)$ will mean $(z^1(x^1, \cdots, x^n), \cdots, z^p(x^1, \cdots, x^n))$,

$z_x(x)$ will mean the matrix $\dfrac{\partial}{\partial x^i} z^j(x^1, \cdots, x^n)$ $(i = 1, \cdots, n;\ j = 1, \cdots, p)$,

$\zeta_x f_{z_x}$ will mean $\sum_{i=1}^{n} \sum_{j=1}^{p} \zeta^j_{x^i} f_{z^j_{x^i}}$, etc.

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[1] Received November 6, 1930.

[2] National Research Fellow.

[3] E. E. Levi, *Sulla necessita della condizione di Weierstrass per l'estremo degli integrali doppi,* Atti della R. Acc. dei Lincei, ser. 5ª, vol. 24 (1915), p. 353.

The use of the letter D (with or without subscript) to indicate a point set of n-dimensional space will connote that the point set consists of a connected open set plus its boundary, the boundary having measure zero. The boundary of D will be designated by $\bar{D}$. The symbol

$\|x\|$ will stand for $\sqrt{x^{1^2}+x^{2^2}+\cdots+x^{n^2}}$,

$\|z\|$ for $\sqrt{z^{1^2}+\cdots+z^{p^2}}$ and

$\|z_x\|$ for $\sqrt{(z^1_{x^1})^2+\cdots+(z^j_{x^k})^2+\cdots+(z^p_{x^n})^2}$.

Let there be defined on a point set D a system of p continuous functions $z^1(x), \cdots, z^p(x)$; the set S of points of $n+p$-dimensional space determined by $(x, z(x))$ (x on D) will be called a *continuous n-spread in $n+p$-space.* Our notation will always be so chosen that a spread S_k will be defined by $(x, z_k(x))$ (x on D_k). By appr $(\bar{D}_1, \bar{D}_2)$ we shall mean the greater of the two quantities: (1) the upper bound of the distance of x_2 from $\bar{D}_1$ as x_2 ranges over $\bar{D}_2$; (2) the upper bound of the distance of x_1 from $\bar{D}_2$ as x_1 ranges over $\bar{D}_1$. If $S_1 : (x, z_1(x))$ (x *on* D_1) and $S_2 : (x, z_2(x))$ (x on D_2) be two continuous n-spreads, the symbol appr(z_1, z_2) shall designate the greatest of the three quantities:

(1) the upper bound of $\|z_1(x)-z_2(x)\|$ as x ranges over $D_1 \cdot D_2$;

(2) the upper bound of $\|z_1(x_1)-z_2(x_2)\|$, where x_1 ranges over D_1-D_2, and for each value of x_1, x_2 ranges over the set of points of $\bar{D}_2$ whose distance from x_1 is least;

(3) the upper bound of $\|z_1(x_1)-z_2(x_2)\|$, where x_2 ranges over D_2-D_1, and for each value of x_2, x_1 ranges over the set of points of $\bar{D}_1$ whose distance from x_2 is least.

Given now a sequence of continuous n-spreads S_k: $(x, z_k(x))$ (x on D_k) and a continuous n-spread S_0: $(x, z_0(x))$ (x on D_0), we shall say that $\lim_{k\to\infty} S_k = S_0$ if the following three conditions are satisfied:

(1) Each point x_0 interior to D_0 belongs to all D_k after a certain $D_{k(x_0)}$, and each point x_0 exterior to D_0 belongs to no D_k after a certain $D_{k(x_0)}$;

(2) $\lim_{k\to\infty}$ appr $(\bar{D}_k, \bar{D}_0) = 0$;

(3) $\lim_{k\to\infty}$ appr $(z_k, z_0) = 0$.

Let us suppose that we are given a class $\mathfrak{K}$ of continuous n-spreads S, all within a closed point-set A of $n+p$-space, and a functional $F(S) \equiv \int_D f(x, z(x), z_x(x))\,dx$, where the function $f(x, z, z_x)$ and all its first and second partial derivatives are defined and continuous in all $n+p+np$ arguments, for all (x, z) in A and all z_x. The problem with

which we are concerned is that of finding a spread S of the class $\mathfrak{K}$ for which $F(S)$ assumes its minimum value.

An admissible n-spread is a continuous n-spread $(x, z(x))$ (x on D) such that (1) each point $(x, z(x))$ is in A; (2) each $z^j(x)$ is absolutely continuous in each x^k separately for almost all values of $x^1, \cdots, x^{k-1}, x^{k+1}, \cdots, x^n$; (3) each partial derivative $z^i_{x^j}$ is summable over D; (4) $\int_D f(x, z(x), z_x(x))\, dx$ exists. A spread S: $(x, z(x))$ (x on D) is said to satisfy a *Lipschitz condition of constant* K if $\| z(x_1) - z(x_2) \| \leqq K \| x_1 - x_2 \|$ for every x_1, x_2 in D. If S lies in A and satisfies some Lipschitz condition, it is clearly admissible.

The point $(x_0, z(x_0))$ of the admissible spread S: $(x, z(x))$ (x on D) will be called a *point of indifference of S with respect to $\mathfrak{K}$ and A* if there exists a positive number ϵ such that if Σ: $(x, \zeta(x))$ (x on D) be an admissible n-spread for which $\| z(x) - \zeta(x) \| < \epsilon$ for all x in an ϵ-neighborhood (x_0) of x_0 and $\| z(x) - \zeta(x) \| = 0$ for all other x of D, then Σ is also a spread of $\mathfrak{K}$. We define the Weierstrass E-function in the usual manner: $E(x, z, z_x, \bar{z}_x) \equiv f(x, z, \bar{z}_x) - f(x, z, z_x) - (\bar{z}_x - z_x) f_{z_x}(x, z, z_x)$.

Occasionally we shall neglect to mention exceptional sets of measure zero, when such neglect does not affect the accuracy of the conclusion and simplifies the language.

3. Lemma 1. *If the sequence of n-spreads $\{S_k\}$ and the n-spread S_0 all satisfy the same Lipschitz condition, and $\lim_{k\to\infty} S_k = S_0$, then*

$$\overline{\lim_{k\to\infty}} [F(S_n) - F(S_0)] = \overline{\lim_{k\to\infty}} \int_{D_0} E(x, z_0(x), z_{0x}(x), z_{nx}(x))\, dx,$$

$$\underline{\lim_{k\to\infty}} [F(S_n) - F(S_0)] = \underline{\lim_{k\to\infty}} \int_{D_0} E(x, z_0(x), z_{0x}(x), z_{nx}(x))\, dx,$$

$E(x, z_0(x), z_{0x}(x), z_{nx}(x))$ being set equal to zero wherever undefined.

It is clear that all the S_k lie in some bounded closed portion of $n+p$-space. The functions E and f being continuous on a bounded closed set are themselves bounded, say less than M in absolute value. Since $\lim_{k\to\infty} S_k = S_0$, for every $\delta > 0$ we can find an n_δ such that for every $n \geqq n_\delta$ D_n is contained in $D_0 + (\bar{D}_0)_\delta$ ($(\bar{D}_0)_\delta$ being the set of all points whose distance from the boundary $\bar{D}_0$ of D_0 is $\leqq \delta$), while D_n contains $D_0 - (\bar{D}_0)_\delta$; and the measure of $(\bar{D}_0)_\delta$ can be made as small as desired. Let ϵ be an arbitrary positive number; we choose δ small enough so that $m((\bar{D}_0)_\delta) < \epsilon/12M$.

Consider now a parallelopiped Q: $a^i \leqq x^i \leqq b^i$ ($i = 1, \cdots, n$), and use the symbol $F(S(Q))$ to denote $\int_Q f(x, z(x), z_x(x))\, dx$. We can then write

$$
\begin{aligned}
F(S_k(Q)) - F(S(Q)) &= \int_Q f(x, z_k(x), z_{kx}(dx))\,dx - \int_Q f(x, z_0(x), z_{0x}(x))\,dx \\
&= \int_Q [f(x, z_k(x), z_{kx}(x)) - f(x, z_0(x), z_{kx}(x))]\,dx \\
&\quad + \int_Q [f(x, z_0(x), z_{kx}(x)) - f(x, z_0(x), z_{0x}(x)) \\
&\qquad\qquad - (z_{kx} - z_{0x}) f_{z_x}(x, z_0(x), z_{0x}(x))]\,dx \\
&\quad + \int_Q [(z_{kx} - z_{0x}) f_{z_x}(x, z_0(x), z_{0x}(x))]\,dx.
\end{aligned}
$$

In the first integral the function f is continuous and its arguments bounded, and $z_k(x)$ approaches $z_0(x)$ uniformly, hence the first integral has limit zero. In the last integral,[4] let us consider any one term

$$\int_Q (z^i_{kx^j} - z^j_{0x^j}) f_{z^i_{x^j}}(x, z_0(x), z_{0x}(x))\,dx,$$

which for ease in printing we will designate as

$$\int_Q (\zeta_k - \zeta_0) f_\zeta(x, z_0(x), z_{0x}(x))\,dx.$$

For almost all fixed values of $x^1, \cdots, x^{j-1}, x^{j+1}, \cdots, x^n$, we have

$$
\begin{aligned}
\int_{a^j}^{x^j} (\zeta_k - \zeta_0)\,dx^j &= z^i_k(x^1, \cdots, x^j, \cdots, x^n) - z^i_k(x^i, \cdots, a^j, \cdots, x^n) \\
&\quad - z^i_0(x^1, \cdots, x^j, \cdots, x^n) + z^1_0(x^1, \cdots, a^j, \cdots, x^n),
\end{aligned}
$$

which tends to zero with $1/k$, uniformly for all x^j; hence[5]

$$\lim_{k\to\infty} \int_{a^j}^{b^j} (\zeta_k - \zeta_0) f_\zeta(x_0, z_0(x), z_{0x}(x))\,dx = 0;$$

and now integrating with respect to the remaining variables, we have

$$\lim_{k\to\infty} \int_Q (\zeta_k - \zeta_0) f_\zeta(x, z_0(x), z_{0x}(x))\,dx = 0.$$

The second integrand is $E(x, z_0(x), z_{0x}(x), z_{kx}(x))$, so that

$$\lim_{k\to\infty} \left\{ F(S_k(Q)) - F(S_0(Q)) - \int_Q E(x, z_0(x), z_{0x}(x), z_{nx}(x))\,dx \right\} = 0.$$

The set $D_0 - (\bar{D}_0)_\delta$ is an open set; hence by the definition of measure[6] we can find a finite set of closed parallelopipeds $Q_1, \cdots, Q_s$ contained in $D_0 - (\bar{D}_0)_\delta$ such that $m(D_0 - (\bar{D}_0)_\delta - Q_1 - \cdots - Q_s) < \varepsilon/12M$.

---

[4] To verify the permissibility of the operations used, see e. g. Carathéodory, Vorlesungen über Reelle Funktionen, §§ 557, 550.

[5] Hobson, Theory of Functions of a Real Variable, 2nd ed., vol. II, p. 422.

[6] Hobson, loc. cit., vol. I, p. 161.

Let us now write

$$F(S_k)-F(S_0)-\int_{D_0} E(x, z_0(x), z_{0x}(x), z_{nx}(x))\,dx$$

$$=\sum \int_{Q_j} + \int_{D_0-(\bar{D}_0)_\delta-\Sigma Q_i} + \int_{D_k-(D_0-(\bar{D}_0)_\delta)} [f(x, z_k(x), z_{kx}(x))]\,dx$$

$$-\sum \int_{Q_i} - \int_{D_0-(\bar{D}_0)_\delta-\Sigma Q_i} - \int_{D_0-(D_0-(\bar{D}_0)_\delta)} f(x, z_0(x), z_{0x}(x))\,dx$$

$$-\sum \int_{Q_i} - \int_{D_0-(\bar{D}_0)_\delta-\Sigma Q_i} - \int_{D_0-(D_0-(\bar{D}_0)_\delta)} E(x, z_0(x), z_{0x}(x), z_{kx}(x))\,dx.$$

Denote the nine terms of the right hand member by $i_1, i_2, \cdots, i_9$. Since the functions f and E are for all x less in absolute value than M, and the sets $D_0-(\bar{D}_0)_\delta-\Sigma Q_j$, $D_k-(D_0-(\bar{D}_0)_\delta)$ and $D_0-(D_0-(\bar{D}_0)_\delta)$ each have measure at most equal to $\varepsilon/12M$, we find that the sum of i_2, i_3, i_5, i_6, i_8, and i_9 has absolute value at most equal to $\varepsilon/2$. By (3), we can choose k_ε large enough so that $i_1+i_4+i_7$ is less in absolute value than $\varepsilon/2$ for all $k \geqq k_\varepsilon$; so that for every positive ε we can find a k_ε such that for every $k \geqq k_\varepsilon$

$$\left| F(S_k)-F(S_0)-\int_{D_0} E(x, z_0(x), z_{0x}(x), z_{kx}(x))\,dx \right| < \varepsilon$$

and our lemma is proved.

REMARK: We have in fact proved that under the hypotheses of the lemma

$$\lim_{k\to\infty}\left[F(S_k)-F(S_0)-\int_{D_0} E(x, z_0(x), z_{0x}(x), z_{nx}(x))\,dx\right]=0,$$

which is a somewhat stronger conclusion than that of the lemma.

4. LEMMA 2. *Let z_x be a $n\times p$ matrix of numbers of rank 1, and let ε be a positive number $<\frac{1}{2}$. There exists a system of continuous functions $\omega(x)=(\omega^1(x), \cdots, \omega^p(x))$ defined on a cube Q: $-b \leqq x^i \leqq b$, and having the following properties:*

(1) *$\omega(x)$ is identically zero on the boundary of Q;*

(2) *On a subset Π_ε of Q, $\omega_x(x)=z_x$;*

(3) *Π_ε has measure $=K_2\varepsilon m(Q)$, where K_2 is independent of ε;*

(4) *On $Q-\Pi_\varepsilon$, all partial derivatives $\omega^i_{x^j}$ are less in absolute value than $K_1\varepsilon$, where K_1 is independent of ε.*

The matrix z_x being of rank 1, there exists an n-tuple of numbers $(c_1, c_2, \cdots, c_n)$ such that $z^i_{x^j}=a^i c_j$ $(i=1, \cdots, p;\ j=1, \cdots, n)$. The linear equation $c_1x^1+\cdots+c_nx^n=0$ has $n-1$ linearly independent solutions, which we can choose normal and orthogonal; we designate them by $\xi_i=(\xi^1_i, \cdots, \xi^n_i)$ $(i=2, \cdots, n)$. Let ξ_1 be chosen normal and ortho-

gonal to all the others; then the system $\bar{\xi}_i$ $(i = 1, \cdots, n)$ can be used as the axes of a rectangular coördinate system. If the coördinates of a point be $(\xi^1, \xi^2, \cdots, \xi^n)$ in the new system and $(x^1, \cdots, x^n)$ in the old, then $\xi^i = \sum_j A^i_j x^j$, A^i_j being a matrix of constant elements. Construct the polyhedron Π_ε:

$$\frac{1}{\varepsilon}|\xi^1| + |\xi^2| + \cdots + |\xi^n| \leqq 1,$$

and the prism P:

$$|\xi^2| + \cdots + |\xi^n| \leqq 1, \qquad |\xi^1| \leqq 1.$$

In Π_ε we define $\omega^i(x)$ as equal to $\sum_k a^i c_k x^k$; if the coördinates of x be $(\xi^1, \cdots, \xi^n)$, this reduces to $(a^i \sum_j c_j \bar{\xi}^j_1)\xi^1$. To each point $(\xi^1, \xi^2, \cdots, \xi^n)$ in $P - \Pi_\varepsilon$ there corresponds a point $(\bar{\xi}^1, \xi^2, \cdots, \xi^n)$ on the boundary of Π_ε, for which $\bar{\xi}^1$ has the same sign as ξ^1. We define $\omega^i(x)$ at the point $(\xi^1, \cdots, \xi^n)$ as equal to $\left(\dfrac{1-\xi^1}{1-\bar{\xi}^1}\right)(a^i \sum_j c_j \bar{\xi}^j_1)\bar{\xi}^1$ if ξ^1 be positive, and as $\left(\dfrac{1+\xi^1}{1-\bar{\xi}^1}\right)(a^i \sum_j c_j \bar{\xi}^j_1)\bar{\xi}^1$ if ξ^1 be negative. $\bar{\xi}^1$ is easily found to be equal to $\pm\varepsilon\left(1 - \sum_{j=2}^n |\xi^j|\right)$, the sign agreeing with that of ξ^1. Consequently $\omega^i(x)$ vanishes on the whole boundary of P, and on the boundary of Π_ε the two definitions are in accord; also $\omega^i(x)$ is continuous on all of P.

We see immediately that $\omega^i_{x^j}(x) = z^i_{x^j}$ in Π_ε. By direct differentiation, recalling that $|\bar{\xi}^1| \leqq \varepsilon$, we show the existence of a K_1 such that in $P - \Pi_\varepsilon$ all the partials of $\omega^i(x)$ are less in absolute value than $K_1\varepsilon$. To avoid excessive notation, we will assume K_1 chosen larger than $\|z_x\|$.

Let now Q be the smallest cube $-b \leqq x \leqq b$ containing P. In $Q - P$ define $\omega(x) \equiv 0$. The $\omega(x)$ thus defined satisfies the conditions of the conclusion of the lemma.

If Q' be a cube $\alpha^i \leqq x^{i'} \leqq \beta^i$ whose side is ϱ times the side of Q, we will say that we have mapped Q on Q' when we have set up a correspondence $x^{i'} = \varrho x^i + h^i$ between their points and defined a set of functions $\omega'(x)$ on Q' such that $\omega'(x') = \varrho\,\omega(x)$. It is obvious that the partial derivatives at corresponding points of Q and Q' are equal.

5. Let A be a closed point set of $n+p$-dimensional space, $\mathfrak{K}$ a class of admissible n-spreads lying in A, and $S: (x, \zeta(x))$ (x on D) a spread of the class $\mathfrak{K}$. Let us designate by L the set consisting of $\bar{D}$ plus the set of all points x such that (1) x is interior to D; (2) x is a point of indifference of S with respect to $\mathfrak{K}$ and A; (3) in some neighborhood of x, S satisfies some Lipschitz condition. We can then state:

THEOREM I. *If S be a minimizing n-spread for $F(S)$ in the class $\mathfrak{K}$, then the set of points x_0 of L for which there exists a matrix of numbers z_x of rank $\leqq 1$ such that $E(x_0, \zeta(x_0), \zeta_x(x_0), \zeta_x(x_0)+z_x)<0$ must have measure zero.*

Denote the set of all such x_0's by the letter N. We will assume $m(N)>0$, and arrive at a contradiction.

Suppose that x_0 is a point of N, and z_x a matrix of rank 1 such that $E(x_0, \zeta(x_0), \zeta_x(x_0), \zeta_x(x_0)+z_x)<0$. (Clearly no such matrix of rank 0 can exist.) We easily see that we can construct a matrix $\bar{z}_x$ of rational numbers of rank 1 such that each element of $\bar{z}_x$ differs by an arbitrarily small quantity from the corresponding element of z_x; and since E is continuous in all its arguments, we can find such a matrix $\bar{z}_x$ for which $E(x_0, \zeta(x_0), \zeta_x(x_0), \zeta_x(x_0)+\bar{z}_x)<0$, and hence less than some negative rational $-k$. Let $z_{1x}, z_{2x}, \cdots$ be an ordering of all matrices of rank 1 with rational elements and $k_1, k_2, \cdots$ an ordering of all positive rational numbers. If we let N_{ij} be the set of all points x of L such that $E(x, \zeta(x), \zeta_x(x), \zeta_x(x)+z_{ix})<-k_j$, we see from the above that each point of N is included in some N_{ij}, hence $N=\sum_{ij} N_{ij}$. Now if $m(N_{ij})=0$ for every i, j, we would have $m(N)=0$; since we have supposed $m(N)>0$, we must have at least one set N_{qs} such that $m(N_{qs})>0$. We may consider that N_{qs} contains only interior points of D, since $m(\bar{D})=0$.

Let x_0 be a point of N_{qs} at which N_{qs} has metric density[7] 1. Since x_0 belongs to L, we can find a cube $(x_0)_a$ $(x_0^i-a \leqq x^i \leqq x^i+a)$ in which $\zeta(x)$ satisfies some Lipschitz condition, say of constant K, and moreover if $(x, z(x))$ is admissible and $\|z(x)-\zeta(x)\|<a$ on $(x_0)_a$ and is elsewhere zero, then $(x, z(x))$ is in $\mathfrak{K}$. By the theorem of the mean

$$E(x, \zeta(x), \zeta_x(x), \zeta_x(x)+\bar{z}_x) = \bar{z}_x f_{z_x z_x}(x, \zeta(x), \zeta_x(x)+\vartheta(x)\bar{z}_x)\bar{z}_x \qquad (0<\vartheta(x)<1).$$

If now each element of $\bar{z}_x$ be less in absolute value than $K_1\gamma$ $(0<\gamma\leqq 1)$, the $f_{z_x z_x}$ are bounded (since they are continuous and the arguments lie in a bounded closed set), so that $|E(x, \zeta(x), \zeta_x(x), \zeta_x(x)+\bar{z}_x)| \leqq K_4\gamma^2$, K_4 some constant, for all x in $(x_0)_a$. The set N_{qs} having metric density 1 at x_0, we can find a positive $b\leqq 2a$ such that in the cube $B: x_0^i-\frac{b}{2}\leqq x^i \leqq x_0^i+\frac{b}{2}$ we have $m(B\cdot N_{qs})/m(B)\geqq 1-\delta$; δ being the smaller of the numbers $K_2\epsilon/2$, $K_2^2 k_s^2/8K_4(K_4+k_s)$. Let ϵ represent $K_2 k_s/2K_4$ (we can suppose K_4 large enough so that $\epsilon<\frac{1}{2}$). Now in Lemma 2 we let z_x be z_{qx} and obtain the functions $\omega(x)$. Map Q on B to obtain $\omega_1(x)$ on B; define

[7] Hobson, loc. cit., vol. I, p. 181.

$\omega_1(x) \equiv 0$ on $D-B$. Divide B into 2^n equal cubes and on each of these map Q, defining a system of functions $\omega_2(x)$ on B; define $\omega_2(x) \equiv 0$ on $D-B$. Continue thus to obtain a sequence $\omega_1, \omega_2, \cdots$ of functions defined on D, all satisfying the same Lipschitz condition on D, such that $\lim_{k\to\infty} \omega_k(x) = 0$ uniformly on D. If for each k we designate by $\Pi_{\varepsilon k}$ the sum of the 2^{kn} sets obtained from Π_ε by the mapping, clearly $m(\Pi_{\varepsilon k}) = K_2 \varepsilon \cdot m(B)$ for all k. Defining now $z_k(x) \equiv \zeta(x) + \omega_k(x)$, we have $\lim_{k\to\infty} S_k = S$, and for all sufficiently large values of k, S_k is in the class $\mathfrak{K}$.

By Lemma 1 we have

$$\overline{\lim_{k\to\infty}}\, [F(S_k) - F(S_0)] = \overline{\lim_{k\to\infty}}\, [F(S_k(B)) - F(S_0(B))]$$

$$= \overline{\lim_{k\to\infty}} \int_B E(x, \zeta(x), \zeta_x(x), z_{qx}(x))\, dx$$

$$= \overline{\lim_{k\to\infty}} \left\{ \int_{\Pi_{\varepsilon k} - N_{qs}} + \int_{\Pi_{\varepsilon k}\cdot N_{qs}} + \int_{B-\Pi_{\varepsilon k}} E(x, \zeta(x), \zeta_x(x), z_{qx}(x))\, dx. \right.$$

The measure of $\Pi_{\varepsilon k} - N_{qs}$ is at most δb^n, and the integrand is less than K_4; hence the first integral is less than $\delta K_4 b^n$. In the third integral the integrand is at most $K_4 \varepsilon^2$, so that the third integral is less than $b^n K_4 \varepsilon^2$. The set $\Pi_{\varepsilon k} \cdot N_{qs}$ has measure at least $K_2 \varepsilon b^n - \delta b^n$, and the integrand is less than $-k_s$; hence the second integral is less than $-k_s b^n (K_2 \varepsilon - \delta)$. Adding and replacing ε and δ by their values specified above, we find the sum less than $-b^n K_2^2 k_s^2 / 8 K_4 < 0$. Hence for an infinitude of values of k we have $F(S_k) < F(S)$, contradicting the hypothesis that S is a minimizing spread for $F(S)$ in $\mathfrak{K}$.

Corollary. *If A be closed, and S; $(x, \zeta(x))$ (x on D) a minimizing spread for $F(S)$ in a class $\mathfrak{K}$ of admissible curves, then at every point x_0 which is a point of indifference of S with respect to $\mathfrak{K}$ and A and at which all the partial derivatives $\zeta_x(x)$ are continuous we must have $E(x_0, \zeta(x_0), \zeta_x(x_0), \zeta_x(x_0) + z_x) \geqq 0$ for every matrix z_x of rank $\leqq 1$.*

For since z_x is continuous at x_0, $\|z_x\|$ is bounded in some neighborhood of x_0, and therefore S satisfies some Lipschitz condition. Now if $E(x_0, \zeta(x_0), \zeta_x(x_0), \zeta_x(x_0) + z_x) < 0$, it would remain negative in some neighborhood of x_0, contradicting Theorem I.

6. Let us say that $F(S)$ satisfies condition (a) on a spread $(x, \zeta(x))$ if there exist constants M, K, β such that for every x for which $\|\zeta_x(x)\| > M$, every z_x such that $\|z_x\| > M$, and every z such that (x, z) is a point of A lying in a β-neighborhood of $(x, \zeta(x))$ the following relations hold:

1) $|f(x, \zeta(x), z_x)| \leqq K_P |f(x, \zeta(x), \zeta_x(x))|$ if $\|z_x - \zeta_x(x)\| \leqq P$ (where P is arbitrary, and $K_P > 0$ depends on P).

2) $|f_{z^i}(x, z, z_x)| \leqq K \cdot |f(x, \zeta(x), z_x)|$ $(i = 1, \cdots, p)$.

3) Each partial derivative of the set $f_{z_x z_x}(x, \zeta(x), z_x)$ has absolute value less than $K \cdot |f(x, \zeta(x), z_x)|$.

REMARK. All but four of the examples of integrals in ordinary form given in Tonelli's "Fondamenti di Calcolo delle Variazione" either satisfy condition (a) on every admissible curve, or else fail to satisfy it only at points at which the E-function is identically zero. The four exceptions are $(x^2+y^2)y'^2+\sqrt{1+y'^2}$, $(x^2-y^2)y'^2+\sqrt{1+y'^2}$, for which 2) fails to be satisfied at the origin and on $y = \pm x$ respectively; and $y'^2+e^{y^2y'^2}$, $e^{y^2y'^2}$, which fail to satisfy 1) and 3).

LEMMA 3. *If the spreads $S_0 : (x, z_0(x))$ and $S_k : (x, z_k(x))$ are all defined on a parallelopiped Q, and S_0 is admissible, and $\|z_{kx} - z_{0x}\| \leqq N$ (a constant) for all k and all x in Q, and $\lim_{k\to\infty} z_k(x) = z_0(x)$ uniformly on Q, and $F(S)$ satisfies condition (a) on S_0, then*

$$\lim_{k\to\infty}\left[F(S_k) - F(S_0) - \int_Q E(x, z_0(x), z_{0x}(x), z_{kx}(x))\,dx\right] = 0.$$

Each spread S_k is admissible; for, first, $z_k(x) = z_0(x) + [z_k(x) - z_0(x)]$, and both functions on the right are absolutely continuous in each variable separately; and second,

$$\begin{aligned}
&|f(x, z_k(x), z_{kx}(x))| \\
&\leqq |f(x, z_0(x), z_{kx}(x))| + |(z_k(x) - z_0(x)) f_z(x, z_0 + \vartheta(z_k - z_0), z_{kx})| \\
&\leqq \{1 + \sum_i K\,|z_k^i(x) - z_0^i(x)|\}\,|f(x, z_0(x), z_{kx}(x))| \\
&\leqq \{1 + \sum_i K \cdot |z_k^i(x) - z_0^i(x)|\} \cdot K_N \cdot |f(x, z_0(x), z_{0x}(x))|
\end{aligned}$$

so that $\int_Q f(x, z_k(x), z_{kx}(x))\,dx$ exists.

Now write

$$\begin{aligned}
F(S_k) - F(S_0) &- \int_Q E(x, z_0(x), z_{0x}(x), z_{kx}(x))\,dx \\
&= \int_Q [f(x, z_k(x), z_{kx}(x)) - f(x, z_0(x), z_{kx}(x))]\,dx \\
&\qquad + \int_Q (z_{kx} - z_{0x}) f(x, z_0(x), z_{0x}(x))\,dx.
\end{aligned}$$

As in Lemma 1, the last integral has the limit zero. To show that the first integral also tends to zero, divide Q into the set E_1, on which $\|z_{0x}(x)\| \leqq M+N$, and E_2, on which $\|z_{0x}(x)\| > M+N$. On the closed set $[(x, z)$ in A, $\|z_x\| \leqq M+2N]$, the function f is continuous and the

arguments bounded, so that the integral over E_1 tends to zero. On E_2, by condition (a), we have

$$\begin{aligned}&\int_{E_2} |f(x, z_k(x), z_{kx}(x)) - f(x, z_0(x), z_{kx}(x))|\, dx \\ &\leqq \int_{E_2} |z_k(x) - z_0(x)| \cdot |f_z(x, z_0(x) + \vartheta(x)(z_k - z_0), z_{kx}(x))|\, dx \\ &\leqq p \cdot \max|z_k(x) - z_0(x)| \cdot \int_{E_2} K \cdot |f(x, z_0(x), z_{kx}(x))|\, dx \\ &\leqq p \cdot \max|z_k(x) - z_0(x)| \cdot K \cdot K_N \cdot \int_{E_2} |f(x, z_0(x), z_{0x}(x))|\, dx,\end{aligned}$$

which tends to zero, establishing the lemma.

7. Corresponding to the surface $S: (x, \zeta(x))$ (x on D) let us define L_1 as the set of all points of $\bar{D}$, plus the set of all points x_0 interior to D such that a) either in some neighborhood of x_0 S satisfies some Lipschitz condition or else there exists some neighborhood $(x_0)_a$ of x_0 such that $F(S)$ satisfies condition (a) on $(x, \zeta(x))$ (x on $(x_0)_a$); and b) x_0 is a point of indifference of S with respect to $\mathfrak{K}$ and A.

We can then state the following theorem, which includes Theorem I:

THEOREM II. *If $S: (x, \zeta(x))$ (x on D) be a minimizing spread for $F(S)$ in $\mathfrak{K}$, then the set of points x_0 of L_1 for which there exists a matrix of numbers z_x of rank $\leqq 1$ such that $E(x_0, \zeta(x_0), \zeta_x(x_0), \zeta_x(x_0) + z_x) < 0$ must have measure zero.*

We proceed to obtain N_{qs}, k_s, z_q as in the proof of Theorem I. By Theorem I the set of points of $L \cdot N_{qs}$ has measure zero; hence for almost all points x_0 of N_{qs} there exists a neighborhood $(x_0)_a$ such that on $(x, \zeta(x))$ (x on $(x_0)_a$) condition (a) is satisfied; moreover this a can be taken smaller than the ϵ occurring in the definition of points of indifference. Let N_1 be a closed set contained in $N_{qs} \cdot (L_1 - L)$ and of measure greater than $\frac{1}{2} m(N_{qs})$. A finite set of the neighborhoods $(x_0)_a$ cover N_1; let this set be called A_1. Suppose now that $\bar{z}_x$ is such that each element is less than $K_1 \gamma$ $(0 < \gamma \leqq 1)$; then for all x in A_1 we have

$$\begin{aligned}&|E(x, \zeta(x), \zeta_x(x), \zeta_x(x) + \bar{z}_x)| \\ &\quad = |\bar{z}_x f_{z_x z_x}(x, \zeta(x), \zeta_x(x) + \vartheta(x)\bar{z}_x)\bar{z}_x| \qquad (0 < \vartheta(x) < 1) \\ &\quad \leqq n^2 p^2 K_1^2 \gamma^2 K |f(x, \zeta(x), \zeta(x) + \vartheta(x)\bar{z}_x)| \\ &\quad \leqq n^2 p^2 K_1^2 \gamma^2 K K_{K_1} |f(x, \zeta(x), \zeta_x(x))|.\end{aligned}$$

Hence there exists a K_5 such that for every subset M of A_1 we have

$$\left|\int_M E(x, \zeta(x), \zeta_x(x), \zeta_x(x) + \bar{z}_x)\, dx\right| \leqq K_5 \gamma^2;$$

moreover we can choose a positive $\delta < \dfrac{K_2 \varepsilon m(N_{qs})}{4(1+\varepsilon K_2)}$ such that over every subset M of A_1 of measure less than δ we have

$$\left| \int_M E(x, \zeta(x), \zeta_x(x), \zeta_x(x) + \bar{z}_x)\, dx \right| \leqq k_s^2 K_2^2 m(N_{qs})^2 / 128 K_5.$$

We now construct[8] a finite set of non-overlapping cubes $B_1, \cdots, B_s$ lying in A_1 such that $N_1 = \sum B_j + e' - e''$, where e' and e'' have measure less than δ. Construct the cube Q of Lemma 2, using z_{qx} for the z_x and $k_s K_2 m(N_{qs})/8K_5$ for the ε of the lemma. On each B_j map Q to obtain $\omega_1(x)$; divide each B_j into 2^n equal cubes and on each map Q to obtain $\omega_2(x)$, and so on. On $D - B$ each $\omega_k(x)$ is defined as identically zero. The sum of all the point-sets corresponding to Π_ε in the kth stage of the process will be called $\Pi_{\varepsilon k}$. The set $\Pi_{\varepsilon k} - N_1$ is contained in $\sum B_j - N_1$, which is contained in e'', and so has measure less than δ. Also $\Pi_{\varepsilon k} \cdot N_1$ has measure greater than $\varepsilon K_2 m(\sum B_j) - \delta > \varepsilon K_2 (m(N_1) - \delta) - \delta > \frac{1}{4} \varepsilon K_2 m(N_{qs})$.

Now setting $z_k(x) = \zeta(x) + \omega_k(x)$, we have by Lemma 3:

$$\begin{aligned} \overline{\lim_{k\to\infty}}\, F(S_k) - F(S_0) &= \overline{\lim_{k\to\infty}} \int_D E(x, \zeta(x), \zeta_x(x), z_{kx}(x))\, dx \\ &= \overline{\lim_{k\to\infty}} \left\{ \int_{\Pi_{\varepsilon k} \cdot N_1} + \int_{\Pi_{\varepsilon k} - N_1} + \int_{\Sigma B_j - \Pi_{\varepsilon k}} \right\} E(x, \zeta(x), \zeta_x(x), z_{kx}(x))\, dx. \end{aligned}$$

The third integral is less than $K_5 \varepsilon^2$; the second integral is less than $k_s^2 K_2^2 m(N_{qs})^2 / 128 K_5$; the first integral is less than $-k_s \cdot \frac{1}{4} \varepsilon K_2 m(N_{qs})$. Setting $k_s K_2 m(N_{qs})/8K_5$ in place of ε and adding, we find that the right member is always less than $-k_s^2 K_2^2 m(N_{qs})^2 / 128 K_5 < 0$; contradicting the hypothesis that S is a minimizing spread for $F(S)$ in $\mathfrak{K}$.

8. We will now deduce two important corollaries of Theorem II. In order to avoid confusion we will drop the matrix notation and write out each term separately.

COROLLARY 1. *Let A be a closed point-set of $n+1$-dimensional space, and $\mathfrak{K}$ a class of admissible curves in A. Let C: $(x, z_0^1(x), \cdots, z_0^n(x))$ (x on $[a, b]$) be a minimizing curve for $F(C)$ in $\mathfrak{K}$. Let L_1 be defined as in § 7. Then for almost all points x_0 of L_1*

$$E(x, z_0^1(x), \cdots, z_0^n(x), z_{0x}^1(x), \cdots, z_{0x}^n(x), \bar{z}^1, \cdots, \bar{z}^n) \geqq 0$$

for all numbers $\bar{z}^1, \cdots, \bar{z}^n$.

[8] De la Vallée Poussin, Cours d'Analyse, 3ième ed., p. 63. The extension to n dimensions is obvious.

COROLLARY 2. *Let A be a closed point-set of $n+1$-dimensional space, and $\mathfrak{K}$ a class of admissible n-spreads in A. Let S: $(x^1, x^2, \cdots, x^n, z(x^1, \cdots, x^n))$ (x on D) be a minimizing n-spread for $F(S)$ in $\mathfrak{K}$. Let L_1 be defined as in § 7. Then for almost all points $(x^1, \cdots, x^n)$ of L_1*

$$E(x^1, \cdots, x^n, z(x^1 \cdots x^n), z_{x^1}(x^1,, \cdots, x^n), \cdots, z_{x^n}(x^1, \cdots, z^n), \bar{z}_1, \cdots, \bar{z}_n) \geqq 0$$

for all numbers $\bar{z}_1, \cdots, \bar{z}_n$.

We have merely to notice that the matrix $\bar{z}_x$ has only one row or one column, and hence is of rank $\leqq 1$.

9. In the following theorems we shall mean by $f_{ij;kl}$ the second partial derivative of $f(x, z, z_x)$ with respect to $z^i_{x^j}$ and $z^k_{x^l}$. The set L_1 will have the same meaning as in § 7.

Retaining the notations of Theorem II, we have by the theorem of the mean

$$\sum_{i,j,k,l} z^i_{x^j} f_{i,j;k,l}(x_0, \zeta(x_0), \zeta_x(x_0) + \vartheta(x) \cdot z_x)\, z^k_{x^l} \qquad (0 < \vartheta(x) < 1)$$

for almost all points x_0 of L_1, z_x being of rank $\leqq 1$. Now replace z_x by $r z_x$, and let r approach zero. Since $f_{ij;kl}$ is continuous in all arguments, it follows that

$$\sum_{i,j,k,l} z^i_{x^j} f_{ij;kl}(x_0, \zeta(x_0), \zeta_x(x_0))\, z^k_{x^l} \geqq 0.$$

From this we have the analogue of Legendre's condition;

THEOREM III. *If S: $(x, \zeta(x))$ be a minimizing spread for $F(S)$ in $\mathfrak{K}$, then for almost all points x_0 of L_1 the quadratic form*

$$\sum_{i,j,k,l} z^i_{x^j} f_{i,j;k,l}(x_0, \zeta(x_0), \zeta_x(x_0))\, z^k_{x^l}$$

is non-negative for all numbers $z^i_{x^j}$ whose matrix is of rank $\leqq 1$.

COROLLARY 1. *If C: $(x, \zeta^1(x), \cdots, \zeta^n(x))$ be a minimizing curve for $F(C)$ in the class $\mathfrak{K}$, then for almost all points x_0 of L_1 the quadratic form*

$$\sum_{i,k} z^i f_{z^i_x z^k_x}(x_0, \zeta(x_0), \zeta_x(x_0))\, z^k$$

is non-negative, for all real values of $z^1, \cdots, z^n$. In particular, the form is non-negative whenever x_0 is a point of indifference near which the $\zeta^i_x(x)$ are continuous.

COROLLARY 2. *If S: $(x^1, \cdots, x^n, \zeta(x^1, \cdots, x^n))$ be a minimizing n-spread for $F(S)$ in $\mathfrak{K}$, then for almost all points x_0 of L_1 the quadratic form*

$$\sum_{i,k} z_i f_{z_{x^i} z_{x^k}}(x^1_0, \cdots, x^n_0, \zeta(x^1_0, \cdots, x^n_0), \zeta_{x^1}, \cdots, \zeta_{x^n})\, z_k$$

is non-negative for all real values of $z_1, \cdots, z_n$. *In particular, the form is non-negative whenever* x_0 *is a point of indifference near which the* $\zeta_{x^j}(x)$ *are continuous.*

10. Let us define as usual the expression "$F(S)$ is positive quasi-regular on A" to mean that for all (x, z) in A and all z_x, $\bar{z}_x$ it is true that $E(x, z, z_x, \bar{z}_x) \geqq 0$. Then by use of Lemma 1 we can easily prove[9]

THEOREM IV. *If* $F(S)$ *is positive quasi-regular on* A *and all the spreads of the class* $\mathfrak{K}$ *of admissible spreads in* A *satisfy the same Lipschitz condition, then* $F(S)$ *is lower semi-continuous on* $\mathfrak{K}$.

For let S_0 be any spread of $\mathfrak{K}$, and let $\{S_k\}$ be a sequence of spreads such that $\lim_{k\to\infty} S_k = S_0$. Then

$$\varliminf_{k\to\infty} [F(S_k) - F(S_0)] = \varliminf_{k\to\infty} \int_{D_0} E(x, z_0(x), z_{0x}(x), z_{kx}(x))\, dx \geqq 0,$$

establishing the theorem.

[9] Cf. Tonelli, *Sur la semi-continuité des intégrales doubles du calcul des variations*, Acta Mathematica, vol. 53 (1929), p. 325. Tonelli considers the case $n = 2$, $p = 1$, and develops a theorem on semi-continuity far more elegant than the above. This paper, previously overlooked by me, was kindly called to my attention by the referee.

NOTES ON THE GAMMA-FUNCTION.[1]

BY G. RASCH.

1. **Introduction of the Γ-function.** The Γ-function may be defined in several ways. The following seems to be a very natural one.

A solution of the difference equation

$$f(z+1) = zf(z) \tag{1}$$

which is a regular analytic function at $z = 1$ will have simple poles at $z = 0, -1, -2, \cdots$. The canonical product which vanishes at these points is

$$h(z) = z\prod_1^\infty \left(1+\frac{z}{\nu}\right)e^{-z/\nu};$$

it is evident that this function satisfies the difference equation

$$h(z+1) = \frac{e^{-C}}{z}\cdot h(z),$$

where C designs the constant of Euler, because

$$h(z+1) = (z+1)\prod_1^\infty \left\{\left(1+\frac{z}{\nu+1}\right)e^{-z/\nu}\cdot\left(1+\frac{1}{\nu}\right)e^{-1/\nu}\right\}$$
$$= \frac{h(1)}{z}\cdot h(z),$$

and

$$\log h(1) = \sum_1^\infty\left(\log\left(1+\frac{1}{\nu}\right)-\frac{1}{\nu}\right) = \lim_{n\to\infty}\left(\log n - 1 - \cdots - \frac{1}{n-1}\right) = -C.$$

Consequently

$$\varphi(z) = h(z)f(z)$$

must be a solution of the equation

$$\varphi(z+1) = e^{-C}\varphi(z)$$

which is satisfied by

$$\varphi(z) = e^{-Cz}.$$

Hence

$$f(z) = \pi(z)e^{-Cz} : h(z)$$

[1] Received October 20, 1930.

represents the general solution of (1), when $\pi(z)$ denotes a periodical function with the period 1. In particular

$$e^{-Cz} : h(z) \tag{2}$$

will be a solution; this function is called the *Γ-function.*

2. **Trigonometric series.** For the purpose of forming the Fourier expansion of $\log \Gamma(x)$ in the interval (0, 1) we shall evaluate the integrals

$$\alpha_n = \int_0^1 \log \Gamma(t)\, e^{2\pi i n t}\, dt. \tag{1}$$

This may be done by means of the multiplication theorem

$$\sum_{\mu=0}^{m-1} \log \Gamma\left(\frac{x+\mu}{m}\right) = \log \Gamma(x) - \left(x - \frac{1}{2}\right) \log m + (m-1) \log \sqrt{2\pi}. \tag{2}$$

First, let $n = 0$. m being any natural number we have

$$\begin{aligned}\alpha_0 &= \frac{1}{m}\int_0^m \log \Gamma\left(\frac{t}{m}\right) dt = \frac{1}{m}\int_0^1 \sum_0^{m-1} \log \Gamma\left(\frac{t+\mu}{m}\right) dt \\ &= \int_0^1 \left(\frac{1}{m}\log \Gamma(t) - \frac{\log m}{m}\left(t - \frac{1}{2}\right) + \frac{m-1}{m}\log \sqrt{2\pi}\right) dt;\end{aligned}$$

here the right hand member $\to \log \sqrt{2\pi}$ for $m \to \infty$ and thus we have

$$\alpha_0 = \int_0^1 \log \Gamma(t)\, dt = \log \sqrt{2\pi}. \tag{3}$$

Next we suppose $n \neq 0$. Substituting $\frac{t}{n}$ for t we get

$$\alpha_{\pm n} = \frac{1}{n}\int_0^1 \sum_0^{n-1} \log \Gamma\left(\frac{t+\nu}{n}\right) e^{\pm 2\pi i t}\, dt$$

which by means of (2) may be written

$$\left\{\begin{aligned}\alpha_{\pm n} &= \frac{1}{n}\int_0^1 \left(\log \Gamma(t) - \left(t - \frac{1}{2}\right)\log n + (n-1)\log \sqrt{2\pi}\right) e^{\pm 2\pi i t}\, dt \\ &= \frac{1}{n}\alpha_{\pm 1} \mp \frac{\log n}{2\pi i n}.\end{aligned}\right. \tag{4}$$

Moreover it is easily seen that

$$\alpha_1 + \alpha_{-1} = \frac{1}{2}. \tag{5}$$

In fact,

$$\alpha_{-1} = \int_0^1 \log \Gamma(1-t)\, e^{2\pi i t}\, dt$$

and consequently the law of reciprocity shows that

$$\alpha_1 + \alpha_{-1} = \int_0^1 \log \frac{\pi}{\sin \pi t} \cdot e^{2\pi i t}\, dt$$

which by means of an integration by parts easily is evaluated to $\frac{1}{2}$.

The constant

$$b = \frac{\alpha_1 - \alpha_{-1}}{i} \cdot \pi \tag{6}$$

may be determined in various manners, for example by means of the desired Fourier series itself. The coefficient of $\cos 2n\pi x$ is

$$a_n = \alpha_n + \alpha_{-n} = \frac{1}{2n}$$

and that of $\sin 2n\pi x$

$$b_n = \frac{\alpha_n - \alpha_{-n}}{i} = \frac{b + \log n}{\pi n}.$$

Thus we have

$$\log \Gamma(x) = \log \sqrt{2\pi} + \sum_1^\infty \left(\frac{\cos 2n\pi x}{2n} + \frac{b + \log n}{\pi n} \cdot \sin 2n\pi x\right) \tag{7}$$

or

$$\begin{aligned} \log \Gamma(x) = & -\frac{1}{2} \log \sin \pi x + b\left(\frac{1}{2} - x\right) + \frac{1}{2} \log \pi \\ & + \sum_1^\infty \frac{\log n}{\pi n} \cdot \sin 2n\pi x. \end{aligned} \tag{7a}$$

Now let us consider the infinite series $\varphi(x)$ on the right. A partial summation gives

$$\varphi(x) = \sum_1^\infty \log \frac{n+1}{n}\, \psi_n(x) \tag{8}$$

with

$$\psi_n(x) = \sum_{n+1}^\infty \frac{\sin 2\nu\pi x}{\pi \nu},$$

since it can be shown that

$$\log n \cdot \psi_n(x) \to 0, \qquad n \to \infty.$$

In fact

$$\sum_{n+1}^m \frac{\sin 2\nu\pi x}{\pi \nu} = -\int_0^x \frac{\sin(2n+1)\pi t - \sin(2m+1)\pi t}{\sin \pi t}\, dt$$

and consequently[2]

$$\psi_n(x) = \frac{1}{2} - \int_0^x \frac{\sin(2n+1)\pi t}{\sin \pi t}\,dt = \int_x^{1/2} \frac{\sin(2n+1)\pi t}{\sin \pi t}\,dt$$
$$= \frac{1}{2n+1}\left(\frac{\cos(2n+1)\pi x}{\pi \sin \pi x} - \int_x^{1/2} \frac{\cos(2n+1)\pi t \cos \pi t}{\sin^2 \pi t}\,dt\right)$$
$$= O\left(\frac{1}{n}\right).$$

The series obtained by formal differentiation of (8) being uniformly convergent in the interval $(\varepsilon, 1-\varepsilon)$, it represents $\varphi'(x)$ and thus we have

$$(9)\qquad \begin{cases} \Psi(x)\sin \pi x = -b \sin \pi x - \dfrac{\pi}{2}\cos \pi x \\ \qquad\qquad + \displaystyle\sum_1^\infty \log \frac{n}{n+1} \sin(2n+1)\pi x. \end{cases}$$

Giving to x the value $\frac{1}{2}$ we obtain the determination of the desired constant:

$$(10)\qquad \begin{cases} b = -\Psi\left(\dfrac{1}{2}\right) + \displaystyle\sum_1^\infty (-1)^n \log \frac{n}{n+1} \\ \quad = C + \log 4 - \log\left(\dfrac{1}{2}\cdot\dfrac{3}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{4}\cdot \ldots\right) \\ \quad = C + \log 2\pi. \end{cases}$$

Introducing this result in (7a) and (9) we obtain the well known formulas due to Kummer and Lerch.

More generally the coefficients in the Fourier expansion of $\log \Gamma(x)$ in an arbitrary interval (x_0, x_0+1) may be expressed in terms of the logarithmic integral

$$(11)\qquad \mathrm{li}\, e^{-x} = -\int_x^\infty \frac{e^{-t}}{t}\,dt.$$

In fact, for $n \neq 0$

$$(12)\qquad \int_{x_0}^{x_0+1} \log \Gamma(t)\, e^{2n\pi i t}\,dt = \frac{e^{2n\pi i x_0}}{2\pi i n}\log x_0 - \frac{1}{2\pi i n}\int_{x_0}^{x_0+1} \Psi(t)\, e^{2n\pi i t}\,dt$$

and substituting the series

$$\Psi(t) = -C + \sum_0^\infty \left(\frac{1}{\mu+1} - \frac{1}{x+\mu}\right)$$

we get

$$(13)\qquad \begin{aligned} \int_{x_0}^{x_0+1} \Psi(t)\, e^{2n\pi i t}\,dt &= -\lim_{m\to\infty} \sum_{\mu=0}^{m} \int_{x_0}^{x_0+1} \frac{e^{2n\pi i t}}{t+\mu}\,dt \\ &= -\int_{x_0}^{\infty} \frac{e^{2n\pi i t}}{t}\,dt = \mathrm{li}\, e^{2n\pi i x_0}. \end{aligned}$$

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[2] See e. g. K. Knopp, Theorie und Anwendung der unendlichen Reihen, 2. Aufl., Berl. 1924, p. 361 (b) and p. 363 (c).

Moreover, applying the difference equation for $\log \Gamma(x)$ we find Raabe's integral

$$(14)\quad \int_{x_0}^{x_0+1} \log \Gamma(t)\,dt = \alpha_0 + \int_0^{x_0} \log t\,dt = x_0 \log x_0 - x_0 + \log \sqrt{2\pi}.$$

Thus we have found the series of Landsberg[3]

$$(15)\quad \begin{aligned} \log \Gamma(x) &= x_0 \log x_0 - x_0 + \log \sqrt{2\pi} - \sum_{-\infty}^{+\infty}{}' \frac{e^{2\pi inx}}{2\pi in}\cdot(e^{-2\pi inx_0}\cdot\log x_0 - \mathrm{li}\, e^{-2\pi inx_0}) \\ &= \left(x - \frac{1}{2}\right)\log x_0 - x_0 + \log \sqrt{2\pi} + \sum_{-\infty}^{+\infty}{}' \frac{e^{2\pi inx}}{2\pi in}\cdot \mathrm{li}\, e^{-2\pi inx_0} \end{aligned}$$

and further (13) together with

$$\int_{x_0}^{x_0+1} \Psi(t)\,dt = \log x_0$$

gives the expansion

$$(16)\quad \Psi(x) = \log x_0 + \sum_{-\infty}^{+\infty}{}' e^{2\pi inx}\, \mathrm{li}\, e^{-2\pi inx_0}$$

due to Nörlund.

Changing x to $1-x$ and applying the law of reciprocity

$$\Psi(1-x) - \Psi(x) = \pi \cot \pi x,$$

we are lead to an other expansion

$$(17)\quad \Psi(x) = -\pi \cot \pi x + \log x_0 + \sum_{-\infty}^{+\infty}{}' e^{2\pi inx}\cdot \mathrm{li}\, e^{+2\pi inx_0}$$

which by a partial summation may be converted into the form

$$(17\text{a})\quad \left\{\begin{aligned} \Psi(x)\sin \pi x = & -\pi \cos \pi x + \log x_0 \cdot \sin \pi x \\ & + \frac{1}{2i}(e^{-\pi ix}\, \mathrm{li}\, e^{-2\pi ix_0} - e^{\pi ix}\, \mathrm{li}\, e^{2\pi ix_0}) \\ & - \sum_1^{\infty} \frac{1}{2i}(\mathrm{li}\, e^{-2\pi inx_0} - \mathrm{li}\, e^{-2\pi i(n+1)x_0})\cdot e^{-(2n+1)\pi ix} \\ & + \sum_1^{\infty} \frac{1}{2i}(\mathrm{li}\, e^{2\pi inx_0} - \mathrm{li}\, e^{2\pi i(n+1)x_0})\cdot e^{(2n+1)\pi ix}. \end{aligned}\right.$$

In particular $x_0 = 0$ gives the series of Lerch.

3. On the remainder in Stirling's formula. Two integrations by parts in Raabe's integral gives

[3] Σ' is used to imply that $n = 0$ is omitted from the sum.

$$\log \Gamma(z) - \alpha \Psi(z) = \left(z - \alpha - \frac{1}{2}\right) \log z - z + \frac{\alpha}{z} + \log \sqrt{2\pi}$$
$$- \int_0^1 \left(\frac{t^2}{2} + \left(\alpha - \frac{1}{2}\right) t\right) \Psi'(t+z)\, dt,$$

where α designs an arbitrary constant. It follows that

$$\text{(1)} \qquad \omega_1(z) = \log z - \Psi(z) = \frac{1}{z} - \int_0^1 t\, \Psi'(t+z)\, dt$$

and

$$\text{(2)} \qquad \left\{ \begin{aligned} \omega(z) &= \log \Gamma(z) - \left(z - \frac{1}{2}\right) \log z + z - \log \sqrt{2\pi} \\ &= \int_0^1 \frac{t(1-t)}{2} \Psi'(t+z)\, dt \\ &= \int_0^1 \left(t - \frac{1}{2}\right) \Psi(t+z)\, dt. \end{aligned} \right.$$

Now let us investigate

$$\Psi'(z) = \sum_0^\infty \frac{1}{(z+\nu)^2}$$

for $z \to \infty$. Assuming the natural number p to be 0 if $\Re z \geqq 0$ and otherwise so determined that $-p+1 > \Re z \geqq -p$ we have

$$|z+p+\nu|^2 = |z+p|^2 + 2\nu \Re(z+p) + \nu^2 \geqq |z+p|^2 + \nu^2$$

and

$$|z+p-1-\nu|^2 = |z+p-1|^2 - 2\nu \Re(z+p-1) + \nu^2 > |z+p-1|^2 + \nu^2;$$

consequently

$$\Psi'(z) \leqq \sum_0^{p-1} \frac{1}{|z+p-1-\nu|^2} + \sum_0^\infty \frac{1}{|z+p+\nu|^2}$$
$$< \sum_0^\infty \frac{1}{|z+p-1|^2 + \nu^2} + \sum_0^\infty \frac{1}{|z+p|^2+\nu^2}.$$

When z recedes indefinitely from the negative axis $z_1 = z + p$ tends to infinity; m being any natural number, we have

$$\sum_0^\infty \frac{1}{|z_1|^2 + \nu^2} \leqq \sum_0^m \frac{1}{|z_1|^2+\nu^2} + \sum_{m+1}^\infty \frac{1}{\nu^2},$$

so that

$$\overline{\lim_{|z_1| \to \infty}} \sum_0^\infty \frac{1}{|z_1|^2+\nu^2} \leqq \sum_{m+1}^\infty \frac{1}{\nu^2} \to 0, \qquad m \to \infty.$$

Hence

$$\Psi'(z) \to 0$$

when $z \to \infty$ in the manner mentioned above.[4]

[4] This fact Jensen in his posthumous papers derives from the elementary identity

$$\sum_1^\infty \frac{1}{x^2+\nu^2} = \frac{\pi}{2x} - \frac{1}{2x^2} + \frac{\pi}{x} \cdot \frac{1}{e^{2\pi x}-1}.$$

From (1) and (2) it follows immediately that $\omega(z)$ and $\omega_1(z)$ tend to C when $z \to \infty$ in the manner indicated above and thus we have proved Stirling's formula and the corresponding asymptotic formula for $\Psi(z)$.

The expressions (1) and (2) for $\omega_1(z)$ and $\omega(z)$ seem to be very useful for an elementary investigation of these functions. The difference equations

(3) $$\omega(z+1) - \omega(z) = 1 - \left(z + \frac{1}{2}\right) \log\left(1 + \frac{1}{z}\right)$$

and

(4) $$\omega_1(z+1) - \omega_1(z) = \log\left(1 + \frac{1}{z}\right) - \frac{1}{z}$$

are derived at once and from them we further may deduce the series

(5) $$\omega(z) = \sum_0^\infty \left[\left(z + \nu + \frac{1}{2}\right) \log\left(1 + \frac{1}{z+\nu}\right) - 1\right]$$

and

(6) $$\omega_1(z) = \sum_0^\infty \left[\frac{1}{z+\nu} - \log\left(1 + \frac{1}{z+\nu}\right)\right]$$

due to Gudermann and Jensen. But perhaps particular interest ought to be paid to the fact that they lead to very simple proofs and also to interesting generalizations of the double series and the factorial series of Binet.

In fact, it follows from (2) that

$$\omega(x+y) = \int_0^1 \left(t - \frac{1}{2}\right) \Psi(x+y+t)\, dt$$
$$= \sum_0^\infty \frac{\Psi^{(\nu)}(x)}{\nu!} \cdot \int_0^1 \left(t - \frac{1}{2}\right)(t+y)^\nu\, dt,$$

but now we have

$$\int_0^1 \left(t - \frac{1}{2}\right)(t+y)^\nu\, dt = \frac{\nabla y^{\nu+1}}{\nu+1} - \frac{\Delta y^{\nu+2}}{(\nu+1)(\nu+2)}$$

and in particular

$$\int_0^1 \left(t - \frac{1}{2}\right) dt = 0.$$

Accordingly we find

(7) $$\omega(x+y) = \sum_2^\infty \left(\nabla y^\nu - \frac{\Delta y^{\nu+1}}{\nu+1}\right) \cdot \frac{\Psi^{(\nu-1)}(x)}{\nu!}.$$

Since

$$\overline{\lim_{\nu\to\infty}} \left|\frac{\Psi^{(\nu-1)}(x)}{\nu!}\right|^{1/\nu} = \operatorname*{Max}_{\nu=0,1,\dots} \frac{1}{|x+\nu|}$$

and

$$\lim_{\nu\to\infty} \left|\nabla y^\nu - \frac{\Delta y^{\nu+1}}{\nu+1}\right|^{1/\nu} = |y| \quad \text{or} \quad |y+1|$$

the region of convergence is

$$|x+\nu| > \begin{cases} |y| \\ |y+1| \end{cases},$$

so that x must be situated in the part of the plane which is outside the set of circles

$$|x+\nu| = \operatorname{Max}(|y|, |y+1|). \tag{8}$$

As remarkable cases may be noticed

$$y = 0, \qquad x = z: \qquad \omega(z) = \sum_2^\infty (-1)^\nu \cdot \frac{\nu-1}{2\nu(\nu+1)} \cdot \sum_0^\infty \frac{1}{(z+s)^\nu},$$

$$y = -1, \quad x = z+1: \qquad = \sum_2^\infty \frac{\nu-1}{2\nu(\nu+1)} \cdot \sum_1^\infty \frac{1}{(z+s)^\nu},$$

$$y = -\frac{1}{2}, \quad x = z+\frac{1}{2}: \qquad = \sum_1^\infty \frac{4^{-\nu}}{2\nu+1} \cdot \sum_0^\infty \frac{1}{\left(z+s+\frac{1}{2}\right)^{2\nu}}.$$

The last of these series has the greatest possible region of convergence for a constant y, and has been found in the posthumous papers of Jensen; the other two are the well known series of Binet.

In virtue of the classical formula

$$\frac{1}{z} - \frac{1}{z+\varrho} = \sum_0^\infty \frac{\varrho \cdots (\varrho+\nu)}{(z+\varrho)\cdots(z+\varrho+\nu+1)}, \qquad \Re z > 0,$$

it is evident that

$$\Psi(z+\varrho) - \Psi(z) = \sum_0^\infty \frac{1}{\nu+1} \cdot \frac{\varrho \cdots (\varrho+\nu)}{(z+\varrho)\cdots(z+\varrho+\nu)} + \pi(z), \quad \Re z > 0, \tag{9}$$

but since

$$\Psi(z+\varrho) - \Psi(z) = \log\left(1+\frac{\varrho}{z}\right) + o(1) \to 0, \qquad z \to +\infty$$

we must have $\pi(z) \equiv 0$. Substituting this result in

$$\omega(z) = \int_0^1 \left(\frac{1}{2} - t\right)(\Psi(z+\varrho) - \Psi(z+t))\,dt \tag{2a}$$

we find

$$\omega(z) = \sum_0^\infty \frac{1}{\nu+1} \cdot \frac{A_\nu(\varrho)}{(z+\varrho)\cdots(z+\varrho+\nu)}, \qquad \Re z > 0 \tag{10}$$

where

$$A_\nu(\varrho) = \int_0^1 \left(\frac{1}{2} - t\right)(\varrho - t)\cdots(\varrho - t + \nu)\,dt. \tag{10a}$$

In order to form an analogous series for $\omega_1(z)$ we first write $z-\varrho$ for z; next we differentiate in respect to ϱ and again we write $z+\varrho$ for z. Thus we find

(11) $$\omega_1(z) = \sum_0^\infty \frac{1}{\nu+1} \cdot \frac{C_\nu(\varrho)}{(z+\varrho)\cdots(z+\varrho+\nu)}, \quad \Re z > 0$$

where the coefficients may be written in the form

(11a) $$C_\nu(\varrho) = \varrho\cdots(\varrho+\nu) - \int_0^1 (\varrho-t)\cdots(\varrho-t+\nu)\,dt.$$

The series cannot converge in a greater half-plane than $\Re z > 0$ since the functions represented have logarithmic singularities at $z = 0$.

The coefficients can be expressed in terms of Nörlund's generalized Bernoullian polynomials.[5] In fact

$$B_n^{(n+1)}(x) = (x-1)\cdots(x-n),$$
$$B_n^{(n)}(x) = \int_x^{x+1} (t-1)\cdots(t-n)\,dt$$

and accordingly

$$(-1)^\nu A_\nu(\varrho) = B_{\nu+2}^{(\nu+2)}(2-\varrho) + \left(\varrho - \frac{3}{2}\right) B_{\nu+1}^{(\nu+1)}(1-\varrho)$$
$$= B_{\nu+2}^{(\nu+2)}(1-\varrho) + \left(\varrho + \nu + \frac{1}{2}\right) B_{\nu+1}^{(\nu+1)}(1-\varrho),$$
$$(-1)^{\nu+1} C_\nu(\varrho) = B_{\nu+1}^{(\nu+2)}(1-\varrho) - B_{\nu+1}^{(\nu+1)}(1-\varrho).$$

In particular[6]

$$(-1)^{\nu+1} A_\nu(0) = \frac{B_{\nu+2}^{(\nu+1)}}{\nu+1} + \frac{2\nu+1}{2\nu} B_{\nu+1}^{(\nu)}, \qquad (-1)^{\nu+1} C_\nu(0) = \frac{B_{\nu+1}^{(\nu)}}{\nu},$$

$$(-1)^{\nu+1} A_\nu(1) = \frac{B_{\nu+2}^{(\nu+1)}}{\nu+1} + \frac{1}{2} B_{\nu+1}^{(\nu+1)} = -B_{\nu+2}^{(\nu+2)} - \left(\nu + \frac{3}{2}\right) B_{\nu+1}^{(\nu+1)},$$

$$(-1)^{\nu+1} C_\nu(1) = (\nu+1)! - B_{\nu+1}^{(\nu+1)}.$$

[5] N. E. Nörlund, Differenzenrechnung, Berlin 1924, Sechstes Kapitel, in particular § 5.
[6] Cf. N. E. Nörlund, l. c., p. 244.

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ON THE ABSOLUTE CONVERGENCE OF DIRICHLET SERIES.[1]

By H. F. Bohnenblust and Einar Hille.

The problem of the absolute convergence for Dirichlet series $\sum a_n e^{-\lambda_n s}$ deals with the relative position of the abscissa σ_a of absolute convergence and the abscissae σ_u of uniform convergence and σ_b of boundedness and regularity. For ordinary Dirichlet series $\sum a_n n^{-s}$ we have $\sigma_u = \sigma_b$.

In order to find an upper bound for the difference $\sigma_a - \sigma_u$, H. Bohr established in a paper published in 1913[2] a connection between the behavior of ordinary Dirichlet series and of power series in an infinite number of variables. Writing n as a product of prime numbers, the Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

can be written as a power series

$$c + \sum_{i_1=1}^{\infty} c_{i_1} x_{i_1} + \sum_{i_1 \leqq i_2 = 1}^{\infty} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots \tag{2}$$

in the variables $x_n = p_n^{-s}$, where p_n is the n-th prime number. The coefficient $c_{i_1 i_2 \cdots i_m}$ of (2) is equal to the coefficient a_n of (1), whose index n is equal to $p_{i_1} p_{i_2} \cdots p_{i_m}$. Bohr showed that, though actually functions of a single variable s, the variables $x_n = p_n^{-s}$ behave in many ways as if they were independent of one another. This is due to the linear independence of the quantities $\log p_n$.

The power series (2) will be said to be *bounded* in the domain $(G): |x_n| \leqq G_n$; where the G_n are non negative numbers, if

1° for every integer m, the m-truncated power series

$$c + \sum_{i_1=1}^{m} c_{i_1} x_{i_1} + \sum_{i_1 \leqq i_2 = 1}^{m} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots \tag{3}$$

obtained from (2) by putting $x_{m+1} = x_{m+2} = \cdots = 0$ is *absolutely convergent* in the domain (G), and

[1] Received March 10, 1931.—Presented to the American Mathematical Society, Dec. 31, 1930, (abstract nos. 37-1-91, 95). A short account of the main results of this paper appeared in the Comptes Rendus, t. **192**, pp. 30–32, séance du 5 janvier 1931.

[2] H. Bohr, Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der Dirichletschen Reihen $\sum a_n n^{-s}$, Gött. Nachr. (1913), p. 441–488.

2° if there exists an upper bound H, such that for every x_n of (G) and every m, the truncated power series (3) is in absolute value $\leqq H$.[3]

By means of this definition, Bohr was able to establish:[4]

THEOREM A: *Let σ_u be the abscissa of uniform convergence of the Dirichlet series* (1), *then the associated power series* (2) *is bounded in every domain* $|x_n| \leqq G_n = p_n^{-(\sigma_u+\delta)}$, *where δ is any arbitrarily small positive number,* and the converse

THEOREM B: *If the power series* (2) *is bounded in the domain* $|x_n| \leqq G_n = p_n^{-\sigma_0}$, *then* $\sigma_u \leqq \sigma_0$.

The problem of the absolute convergence of (1) is thus reduced to a problem on power series in an infinite number of variables.

If S is the least upper bound of all positive numbers α, such that every power series (2) bounded in (G) is absolutely convergent in (G'): $|x_n| \leq G_n' = \varepsilon_n G_n$, whenever $0 < \varepsilon_n < 1$ and $\sum \varepsilon_n^\alpha$ converges, then

THEOREM C:[5] *The maximal width of the strip in which a Dirichlet series is uniformly, but non-absolutely convergent is* $= 1/S$.

Bohr showed that $S \geqq 2$, but could not prove the finiteness of S. In the same year, Toeplitz[6] settled this question by showing that $S \leqq 4$. There exist Dirichlet series for which the width $\sigma_a - \sigma_u$ of the strip of uniform but non-absolute convergence is arbitrarily close to $\frac{1}{4}$. His examples were constructed by means of quadratic forms in the x_n; the coefficients of which are essentially the elements of orthogonal (not normalized) matrices, and have the values ± 1.

For general Dirichlet series $\sum a_n e^{-\lambda_n s}$ we have to consider the abscissa σ_b of regularity and boundedness. Hardy, Carlson and Neder proved[7]

$$\sigma_a - \sigma_b \leqq \frac{D}{2},$$

where

$$D = \overline{\lim_{n \to \infty}} \frac{\log n}{\lambda_n}; \tag{4}$$

while Neder showed that this result is the best possible:

[3] D. Hilbert, Wesen und Ziele einer Analysis der unendlich vielen unabhängigen Variablen, Palermo Rend., **27**, (1909), p. 59–74.

[4] H. Bohr, loc. cit. Theorems VII and VIII, p. 472 and p. 475.

[5] H. Bohr, loc. cit. Theorem IX, p. 477.

[6] O. Toeplitz, Über eine bei den Dirichletschen Reihen auftretende Aufgabe aus der Theorie der Potenzreihen von unendlich vielen Veränderlichen, Göttinger Nachrichten (1913), p. 417–432.

[7] G. H. Hardy, The application of Abel's method of summation to Dirichlet series, Quarterly Journal, **47** (1916), p. 176–192.—F. Carlson, Sur les séries de Dirichlet, Comptes Rendus, t. **172** (1921), p. 838.—L. Neder, Zum Konvergenzproblem der Dirichletschen Reihen beschränkter Funktionen, Mathematische Zeitschrift, **14** (1922), p. 149–158.

For every $D \geqq 0$, there exist types satisfying (4) for which $\sigma_a - \sigma_b = \frac{D}{2}$. In the case $D = 1$ however, the type considered by Neder is not the type $\lambda_n = \log n$ of ordinary Dirichlet series.

In a different connection Littlewood[8] has considered bounded bilinear forms in an infinite number of variables. He obtained necessary conditions for boundedness and showed that they are the best of their kind.

In this paper we generalize in the two first sections the conditions of Littlewood to cover m-linear forms in an infinite number of variables and in the third section to cover symmetrical linear forms and m-ic forms. Then we apply the result to power series in an infinite number of variables (section 4) and to Dirichlet series (section 5). We shall prove that if σ_u is the abscissa of uniform convergence of the Dirichlet series (1), the series $\sum b_n n^{-s}$, where

$$b_n = \begin{cases} a_n & \text{when } n \text{ does not contain more than } m \text{ prime factors (the same} \\ & \text{or different),} \\ 0 & \text{otherwise,} \end{cases}$$

has an abscissa of absolute convergence $\leqq \sigma_u + \frac{m-1}{2m}$, and also that these inequalities are the best of their kind. Taking $m = 2$, we see that Toeplitz obtained the largest width $\sigma_a - \sigma_u$ possible for Dirichlet series associated with quadratic forms.

Then combining forms of different degrees, we shall prove in section 6 the main result

THEOREM VII. *For any given σ in the interval $0 \leqq \sigma \leqq \frac{1}{2}$, there exist ordinary Dirichlet series for which the width of the strip of uniform, but non-absolute convergence is exactly equal to σ.*

This theorem cannot be generalized to all types of Dirichlet series. However Bohr[9] has extended his results to cover certain types of Dirichlet series. Accordingly we are able to extend (section 7) our results for certain types and we obtain at the same time new examples proving Neder's result.

Throughout this paper we shall write the power series associated with (1) not in the form (2), where the summations are subject to the conditions $i_1 \leqq i_2 \leqq \cdots \leqq i_m$, but in the symmetrical form

$$c + \sum_{i_1=1}^{\infty} c_{i_1} x_{i_1} + \sum_{i_1 i_2=1}^{\infty} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots,$$

whose coefficients are the old ones divided by binomial factors.

[8] J. E. Littlewood, On bounded bilinear forms, Quarterly Journal of Mathematics (Oxford series), 1 (1930), p. 164–174.

[9] H. Bohr, Zur Theorie der allgemeinen Dirichletschen Reihen, Mathematische Annalen, **79** (1919), p. 136–156.

1. **Bounded m-linear forms in an infinite number of variables.** Let us consider the m-linear form

$$L(x^{(1)}, x^{(2)}, \cdots, x^{(m)}) \equiv \sum a_{i_1 i_2 \cdots i_m} x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_m}^{(m)}, \tag{1.1}$$

where the indices $i_1, i_2, \cdots, i_m$ independently assume all positive integral values. The coefficients $a_{i_1 \cdots i_m}$ and the variables $x_{i_\nu}^{(\nu)}$ are complex. The vector

$$x^{(\nu)} \equiv \{x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_n^{(\nu)}, \cdots\}$$

is said to belong to the domain (G_0), if for all values of n

$$|x_n^{(\nu)}| \leqq 1.$$

DEFINITION: *An m-linear form $L(x^{(1)}, \cdots, x^{(m)})$ is said to be bounded by H in the domain (G_0) if all the truncated forms*

$$\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_m=1}^{N_m} a_{i_1 i_2 \cdots i_m} x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_m}^{(m)}$$

are absolutely $\leqq H$ in (G_0).[10]

We can evidently suppose all the N_ν to be equal, say equal to N. In order to generalize the Littlewood inequalities we first introduce the following notations

$$\varrho = \frac{2m}{m+1},$$

$$S = \left[\sum_{i_1, \cdots, i_m=1}^{\infty} |a_{i_1 \cdots i_m}|^\varrho\right]^{\frac{1}{\varrho}}, \qquad T_{i_\nu}^{(\nu)} = [\sum |a_{i_1 \cdots i_m}|^2]^{\frac{1}{2}},$$

where the last sum is to be extended over $i_1, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_m$ from 1 to ∞; and finally

$$T^{(\nu)} = \sum_{i_\nu=1}^{\infty} T_{i_\nu}^{(\nu)}.$$

These values S, $T_{i_\nu}^{(\nu)}$, $T^{(\nu)}$ may of course be equal to ∞.

The first step in establishing the necessary conditions for boundedness is to prove the inequality

$$S^\varrho \leqq \sum_{\nu=1}^{m} T^{(\nu)\varrho}, \tag{1.2}$$

which shows in particular that S is finite if all the $T^{(\nu)}$ are finite. If one of the $T^{(\nu)}$ is divergent, there is nothing to prove. We may therefore suppose all the $T^{(\nu)}$ and hence all the $T_{i_\nu}^{(\nu)}$ to be finite. We can also assume for the proof that all the coefficients $a_{i_1 \cdots i_m}$ are real and non-

[10] This definition is equivalent to Hilbert's, since the truncated forms, containing only a finite number of terms, are absolutely convergent.

negative and that for every fixed ν, $T_{i_\nu}^{(\nu)}$ is a non-increasing function of i_ν. Should the last not be the case initially, the following permutation of the indices $i_1, \cdots, i_m$ will ensure it: Consider first the quantities $T_{i_1}^{(1)}$. A permutation of the i_1 interchanges the $T_{i_1}^{(1)}$, but does not alter $T^{(1)}$, nor S, nor any $T_{i_\nu}^{(\nu)}$ with a $\nu \geqq 2$. Since $T_{i_1}^{(1)} \to 0$, when $i_1 \to \infty$, we can rearrange the $T_{i_1}^{(1)}$ in non-increasing order of magnitude. Proceeding similarly for $\nu = 2, \cdots, m$ we obtain the desired order. Hence for all $\nu = 1, 2, \cdots, m$ and all $i_\nu = 1, 2, \cdots$

$$i_\nu T_{i_\nu}^{(\nu)} \leqq T^{(\nu)}. \tag{1.3}$$

Using the notation $\sum_{(i_\nu)}^{(\nu)}$ to indicate that ν and i_ν are fixed and that the summation is to be extended

1° over those values of $i_1, \cdots, i_{\nu-1}$, which do not exceed i_ν, and

2° over those values of $i_{\nu+1}, \cdots, i_m$ which are smaller than i_ν, we have

by Hölder's inequality with $p = \dfrac{2}{2-\varrho} = m+1$, $q = \dfrac{2}{\varrho} = \dfrac{m+1}{m}$

$$\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{\varrho} \leqq \left(\sum\nolimits_{(i_\nu)}^{(\nu)} 1\right)^{\frac{1}{m+1}} \cdot \left(\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{2}\right)^{\frac{\varrho}{2}}$$
$$\leqq i_\nu^{\frac{m-1}{m+1}} \cdot \left(\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{2}\right)^{\frac{\varrho}{2}}.$$

Extending the summation in the last sum over the range $i_1, \cdots, i_{\nu-1}$, $i_{\nu+1}, \cdots, i_m = 1, 2, \cdots$ increases the right hand side; hence

$$\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{\varrho} \leqq i_\nu^{\frac{m-1}{m+1}} \cdot T_{i_\nu}^{(\nu)\frac{\varrho}{2}}$$

or

$$\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{\varrho} \leqq \left(i_\nu T_{i_\nu}^{(\nu)}\right)^{\frac{m-1}{m+1}} T_{i_\nu}^{(\nu)}.$$

Applying (1.3) we obtain

$$\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{\varrho} \leqq \left(T^{(\nu)}\right)^{\frac{m-1}{m+1}} \cdot T_{i_\nu}^{(\nu)}. \tag{1.4}$$

On the other hand, as there is always a last largest number in a finite sequence of positive integers, every term in the infinite sum S^ϱ appears in one and only one $\sum_{(i_\nu)}^{(\nu)}$; hence

$$S^\varrho = \sum_{\nu=1}^{m}\left(\sum_{i_\nu=1}^{\infty}\left(\sum\nolimits_{(i_\nu)}^{(\nu)} a_{i_1\cdots i_m}^{\varrho}\right)\right)$$

and by (1.4)

$$S^\varrho \leqq \sum_{\nu=1}^{m}\left(\sum_{i_\nu=1}^{\infty} T^{(\nu)\frac{m-1}{m+1}}\, T_{(i_\nu)}^{(\nu)}\right) = \sum_{\nu=1}^{m} T^{(\nu)\varrho},$$

which is the inequality (1.2).

We now proceed to the second step. Supposing that H is an upper bound of the form (1.1), we are going to compare the series $T^{(\nu)}$ with H and prove the existence of a constant A (determined by m), such that

$$T^{(\nu)} \leqq A \cdot H, \tag{1.5}$$

for all forms (1.1). The proof of these inequalities is based upon the following lemma:[11]

Let $\lambda > 0$ and let $M_\lambda(f)$ denote the λ-th root of the mean value of the numbers $|f|^\lambda$ with respect to a set of discrete or continuous variables.

LEMMA 1: *Suppose that* $M_2(f) \geqq T$, $M_4(f) \leqq A' T$. (The A's are absolute constants, while T may depend on the function f). *Then will*

$$A'' \cdot T \leqq M_1(f) \leqq A''' \cdot T.$$

We shall apply this lemma, taking for f the function

$$f \equiv \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

with complex coefficients $a_{i_1 \cdots i_m}$, of m points $P_\nu(x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_n^{(\nu)})$. The mean value is the mean value of f, when each point P_ν runs independently through the vertices $(\pm 1, \pm 1, \cdots, \pm 1)$ of the unit cube in the n-dimensional euclidean space.

We first evaluate $M_2(f)$. Evidently

$$\begin{aligned} M_2^2(f) &= \frac{1}{2^{n \cdot m}} \sum_P \left(\sum_{i=1}^{n} a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \right) \left(\sum_{j=1}^{n} \bar{a}_{j_1 \cdots j_m} x_{j_1}^{(1)} \cdots x_{j_m}^{(m)} \right) \\ &= \frac{1}{2^{n \cdot m}} \sum_{i,j} \left(a_{i_1 \cdots i_m} \bar{a}_{j_1 \cdots j_m} \sum_P x_{i_1}^{(1)} \cdots x_{j_m}^{(m)} \right). \end{aligned}$$

But since

$$\sum_{P_\nu} x_{i_\nu}^{(\nu)} x_{j_\nu}^{(\nu)} = 2^n \cdot \delta_{i_\nu, j_\nu},$$

it follows that

$$\sum_P x_{i_1}^{(1)} \cdots x_{j_m}^{(m)} = 2^{n \cdot m} \, \delta_{i_1 j_1} \cdots \delta_{i_m j_m},$$

and we obtain

$$M_2^2(f) = \sum_{i_1, \cdots, i_m = 1}^{n} | a_{i_1 \cdots i_m}^2 |,$$

i. e.

$$M_2(f) = \left\{ \sum_{i_1, \cdots, i_m = 1}^{n} | a_{i_1 \cdots i_m}^2 | \right\}^{\frac{1}{2}}. \tag{1.6}$$

[11] J. E. Littlewood, loc. cit. p. 169–170.

We next evaluate $M_4(f)$. We have

$$M_4^4(f) = \frac{1}{2^{n\cdot m}} \sum_{P,i,j,k,l} a_{i_1\cdots i_m} a_{j_1\cdots j_m} \bar{a}_{k_1\cdots k_m} \bar{a}_{l_1\cdots l_m} x_{i_1}^{(1)} \cdots x_{l_m}^{(m)}$$

$$= \frac{1}{2^{n\cdot m}} \sum_{i,j,k,l,} \left\{ a_{i_1\cdots i_m} \cdots \bar{a}_{l_1\cdots l_m} \sum_{P_1} x_{i_1}^{(1)} \cdots x_{l_1}^{(1)} \cdots \sum_{P_m} x_{i_m}^{(m)} \cdots x_{l_m}^{(m)} \right\}.$$

But the sum $\sum_{P_\nu} x_{i_\nu}^{(\nu)} \cdots x_{l_\nu}^{(\nu)}$ is equal to 2^n when either

$$\begin{aligned} &\alpha) \quad i_\nu = j_\nu \text{ and } k_\nu = l_\nu, \\ \text{or } &\beta) \quad i_\nu = k_\nu \text{ and } j_\nu = l_\nu, \\ \text{or } &\gamma) \quad i_\nu = l_\nu \text{ and } j_\nu = k_\nu, \end{aligned} \tag{1.7}$$

and equal to zero otherwise. Hence

$$M_4^4(f) = \sum a_{i_1\cdots i_m} \cdots \bar{a}_{l_1\cdots l_m} \leqq \sum | a_{i_1\cdots i_m} | \cdots | a_{l_1\cdots l_m} |,$$

where these sums are to be extended over all possible combinations (1.7) for all ν's. The combinations (1.7) are not mutually exclusive; but counting each of the overlapping terms separately on each occurence, we increase the right hand side and obtain

$$M_4^4(f) \leqq \sum_{\alpha,\cdots,\alpha,\alpha} + \sum_{\alpha,\cdots,\alpha,\beta} + \cdots + \sum_{\gamma,\cdots,\gamma,\gamma}. \tag{1.8}$$

These sums are to be understood as follows: for the indices $i_\nu, j_\nu, k_\nu, l_\nu$ we take the combination α, β, or γ in (1.7) according to the letter in the ν-th place under the sign Σ. Then we sum over all n^{2m} terms thus obtained. The right hand side of (1.8) contains exactly 3^m sums. Each of them is of the form

$$\sum_{\alpha,\cdots,\alpha,\beta,\cdots,\beta,\gamma,\cdots,\gamma} | a_{i_1\cdots i_m} | \, | a_{j_1\cdots j_m} | \, | a_{k_1\cdots k_m} | \, | a_{l_1\cdots l_m} |, \tag{1.9}$$

where the combination α occurs ω_1, the combination β ω_2 and the combination γ ω_3 times and $\omega_1 + \omega_2 + \omega_3 = m$. Let the indices $i_\alpha (= 1, 2, \cdots, n^{\omega_1})$, $i_\beta (= 1, 2, \cdots, n^{\omega_2})$ and $i_\gamma (= 1, 2, \cdots, n^{\omega_3})$ replace respectively the indices $i_1, i_2, \cdots, i_{\omega_1}$; $i_{\omega_1+1}, \cdots, i_{\omega_1+\omega_2}$; $i_{\omega_1+\omega_2+1}, \cdots, i_m$, then the sum (1.9) can be written in the form

$$\sum_{\substack{i_\alpha, i_\beta, i_\gamma \\ j_\alpha, j_\beta, j_\gamma}} | a_{i_\alpha i_\beta i_\gamma} | \, | a_{i_\alpha j_\beta j_\gamma} | \, | a_{j_\alpha i_\beta j_\gamma} | \, | a_{j_\alpha j_\beta i_\gamma} |$$

$$= \sum_{i_\alpha i_\beta j_\alpha j_\beta} \left(\sum_{i_\gamma} | a_{i_\alpha i_\beta i_\gamma} | \, | a_{j_\alpha j_\beta i_\gamma} | \right) \left(\sum_{i_\gamma} | a_{j_\alpha i_\beta i_\gamma} | \, | a_{j_\alpha j_\beta i_\gamma} | \right)$$

and using Cauchy's inequality

$$\leqq \sum_{i_\alpha i_\beta j_\alpha j_\beta} \left\{\sum_{i_\gamma} | a^2_{i_\alpha i_\beta i_\gamma} |\right\}^{\frac{1}{2}} \left\{\sum_{i_\gamma} | a^2_{j_\alpha j_\beta i_\gamma} |\right\}^{\frac{1}{2}} \left\{\sum_{i_\gamma} | a^2_{j_\alpha i_\beta i_\gamma} |\right\}^{\frac{1}{2}} \left\{\sum_{i_\gamma} | a^2_{i_\alpha j_\beta i_\gamma} |\right\}^{\frac{1}{2}}$$

$$= \sum_{i_\alpha j_\alpha} \left\{\sum_{i_\beta} \left\{\sum_{i_\gamma} | a^2_{i_\alpha i_\beta i_\gamma} |\right\}^{\frac{1}{2}} \left\{\sum_{i_\gamma} | a^2_{j_\alpha i_\beta i_\gamma} |\right\}^{\frac{1}{2}}\right\}^2.$$

Applying Cauchy's inequality a second time, we see that (1.4) is

$$\leqq \sum_{i_\alpha j_\alpha} \Big(\sum_{i_\beta i_\gamma} | a^2_{i_\alpha i_\beta i_\gamma} |\Big)\Big(\sum_{i_\beta i_\gamma} | a^2_{j_\alpha i_\beta i_\gamma} |\Big)$$

$$= \left\{\sum_{i_\alpha i_\beta i_\gamma} | a^2_{i_\alpha i_\beta i_\gamma} |\right\}^2 = \left\{\sum_{i_1 \cdots i_m} | a^2_{i_1 \cdots i_m} |\right\}^2.$$

Since this result is independent of the choice of the combinations, the same evaluation applies for every term of (1.8). Hence

$$M_4^4(f) \leqq 3^m \Big(\sum_{i_1 \cdots i_m} | a^2_{i_1 \cdots i_m} |\Big)^2,$$

or

$$M_4(f) \leqq 3^{\frac{m}{4}} \Big(\sum_{i_1 \cdots i_m} | a^2_{i_1 \cdots i_m} |\Big)^{\frac{1}{2}}. \tag{1.10}$$

The integer m being fixed, $3^{\frac{m}{4}}$ is an absolute constant, and the hypotheses of Littlewood's lemma are satisfied with

$$T = \left\{\sum_{i_1 \cdots i_m} | a^2_{i_1 \cdots i_m} |\right\}^{\frac{1}{2}}.$$

There exist therefore two constants A'' and A''' (depending only on m) such that

$$A'' \cdot (\sum | a |^2)^{\frac{1}{2}} \leqq M_1(f) \leqq A''' (\sum | a |^2)^{\frac{1}{2}}. \tag{1.11}$$

We can now prove (1.5) without difficulty. The maximum of every truncated form

$$\sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}$$

of a form (1.1) in the domain (G_0) is equal to the maximum of

$$\sum_{i_1=1}^{n} \left| \sum_{i_2 \cdots i_m = 1}^{n} a_{i_1 \cdots i_m} x^{(2)}_{i_2} \cdots x^{(m)}_{i_m} \right|$$

in the same domain. This maximum is larger or at least not less than any mean value taken with respect to vectors of the domain (G_0). We

can apply (1.11) to every term,[12] and if H is an upper bound of (1.1) we obtain

$$H \geqq A^* \sum_{i_1=1}^{n} \left\{ \sum_{i_2 \cdots i_m=1}^{n} | a^2_{i_1 \cdots i_m} | \right\}^{\frac{1}{2}}$$

for every n, and therefore

$$H \geqq A^* \sum_{i_1=1}^{\infty} \left\{ \sum_{i_2 \cdots i_m=1}^{\infty} | a^2_{i_1 \cdots i_m} | \right\}^{\frac{1}{2}} = A^* \cdot T^{(1)},$$

and similarly for all other $T^{(\nu)}$:

$$T^{(\nu)} \leqq A \cdot H \tag{1.5}$$

and using (1.2)

$$S \leqq A_1 \cdot H. \tag{1.12}$$

This proves

THEOREM I. *In order that an m-linear form* (1.1) *be bounded by H in the domain* (G_0), *it is necessary that* $T^{(1)}, T^{(2)}, \cdots, T^{(m)}$ *and S should all be less than* $A \cdot H$, *where A is a constant depending only on m.*

2. **Theorem I in the first section is the best result of its kind.** As in the case $m = 2$, the proof is based upon the construction for every n of m-linear forms

$$\sum_{i_1, \cdots, i_m=1}^{n} a_{i_1 \cdots i_m} x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}, \tag{2.1}$$

for which

$$\left\{ \sum_{i_1, \cdots, i_m=1}^{n} | a_{i_1 \cdots i_m} |^{\frac{2m}{m+1}} \right\}^{\frac{m+1}{2m}} \geqq A_m \cdot H_n. \tag{2.2}$$

The value A_m is determined by m, independently of n; and H_n is the maximum of the absolute value of the form (2.1) in the domain (G_0). We can express these results by means of the series S, $T^{(\nu)}$:

THEOREM II. *Given any positive* $t^{(\nu)}_{i_\nu}$ *and* $s_{i_1 \cdots i_m}$, *for which*

$$\lim_{i_\nu \to \infty} t^{(\nu)}_{i_\nu} = \lim_{i_1, \cdots, i_m \to \infty} s_{i_1 \cdots i_m} = \infty,$$

there exist bounded forms for which

$$\sum t^{(\nu)}_{i_\nu} T^{(\nu)}_{i_\nu} \quad \textit{and} \quad \sum s_{i_1 \cdots i_m} | a_{i_1 \cdots i_m} |^{\frac{2m}{m+1}}$$

are divergent.

We turn first to the construction of the special forms (2.1). Let $\| a_{rs} \|$ be an n-rowed square matrix satisfying the conditions

[12] We use (1.11) for $(m-1)$ instead of m.

$$(2.3)\qquad \begin{cases} \sum_{t=1}^{n} a_{rt}\,\bar{a}_{st} = n\cdot\delta_{rs} \\ |a_{rs}| = 1. \end{cases}$$

Examples of such matrices have been given by Toeplitz[13] and Littlewood.[14] The simplest is

$$a_{rs} = e^{2\pi i \frac{r\cdot s}{n}} \qquad (r, s = 1, 2, \cdots, n).$$

From any two matrices satisfying (2.3) $\|a_{rs}\|$, $\|b_{pq}\|$ (with possibly not the same number of rows) we can construct a third by "substituting" the second into the first one:

$$\left\| \begin{matrix} a_{11}\,\|b_{pq}\| & \cdots & a_{1n}\,\|b_{pq}\| \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ a_{n1}\,\|b_{pq}\| & \cdots & a_{nn}\,\|b_{pq}\| \end{matrix} \right\|;$$

the resulting array we regard as a new matrix, whose number of rows will be equal to the product of the number of rows of $\|a_{rs}\|$ and $\|b_{pq}\|$.

Let $\|a_{rs}\|$ be any n-rowed matrix satisfying (2.3) and consider the form

$$(2.4)\qquad \sum_{i_1,\cdots i_m=1}^{n} a_{i_1 i_2}\cdot a_{i_2 i_3}\cdots a_{i_{m-1} i_m}\, x^{(1)}_{i_1}\cdots x^{(m)}_{i_m}.$$

The coefficients of this form are all equal to one in absolute value, hence

$$T^{(\nu)} = S = n^{\frac{m+1}{2}}.$$

We now proceed to evaluate the maximum H of this form in the domain (G_0). For any vectors $x^{(\nu)}$ in (G_0) we have

$$\left| \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1 i_2}\cdots a_{i_{m-1} i_m}\, x^{(1)}_{i_1}\cdots x^{(m)}_{i_m} \right|$$

$$\leqq \sum_{i_m=1}^{n} \left| \sum_{i_1,\cdots,i_{m-1}=1}^{n} a_{i_1 i_2}\cdots a_{i_{m-1} i_m}\, x^{(1)}_{i_1}\cdots x^{(m-1)}_{i_{m-1}} \right|,$$

which by Cauchy's inequality is

$$\leqq n^{\frac{1}{2}} \left\{ \sum_{i_m=1}^{n} \sum_{\substack{i_1,\cdots,i_{m-1} \\ j_1,\cdots,j_{m-1}}=1}^{n} a_{i_1 i_2}\,\bar{a}_{j_1 j_2}\cdots a_{i_{m-1} i_m}\,\bar{a}_{j_{m-1} i_m}\, x^{(1)}_{i_1}\cdots \bar{x}^{(m-1)}_{j_{m-1}} \right\}^{\frac{1}{2}}$$

$$= n^{\frac{1}{2}} \left\{ \sum_{\substack{i_1,\cdots,i_{m-1} \\ j_1,\cdots,j_{m-1}}=1}^{n} a_{i_1 i_2}\cdots \bar{a}_{j_{m-2} j_{m-1}}\, x^{(1)}_{i_1}\cdots \bar{x}^{(m-1)}_{j_{m-1}} \sum_{i_m=1}^{n} a_{i_{m-1} i_m}\,\bar{a}_{j_{m-1} i_m} \right\}^{\frac{1}{2}}$$

[13] O. Toeplitz, loc. cit. p. 422–424.

[14] J. E. Littlewood, loc. cit. p. 172.

$$= n\left\{\sum_{\substack{i_1,\cdots,i_{m-2}\\ j_1,\cdots,j_{m-2}}}\sum_{i_{m-1}} a_{i_1 i_2}\cdots\bar{a}_{j_{m-2} i_{m-1}}\, x_{i_1}^{(1)}\cdots \bar{x}_{i_{m-1}}^{(m-1)}\right\}^{\frac{1}{2}}$$

$$\leqq n\left\{\sum_{\substack{i_1,\cdots,i_{m-2}\\ j_1,\cdots,j_{m-2}}}\sum_{i_{m-1}} a_{i_1 i_2}\cdots\bar{a}_{j_{m-2} i_{m-1}}\, x_{i_1}^{(1)}\cdots \bar{x}_{j_{m-2}}^{(m-2)}\right\}^{\frac{1}{2}},$$

since the coefficients of $|x_{i_{m-1}}^{(m-1)}|^2$ are $\geqq 0$. Repeating this evaluation for the summation over the other indices, we obtain by induction

$$H \leqq n^{\frac{m+1}{2}}$$

and therefore

$$S = T^{(\nu)} \geqq H,$$

that is (2.2). The proof of Theorem II proceeds now on exactly the same lines as Littlewood's proof for the case $m = 2$.

3. Symmetric m-linear forms and m-ic forms. In the application to Dirichlet series, we shall deal with forms

$$Q(x) \equiv \sum a_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m}$$

of the m-th degree instead of m-linear forms, where the coefficients $a_{i_1\cdots i_m}$ are supposed to be symmetrical. To every such form we associate the m-linear form

$$L(x^{(1)}, x^{(2)}, \cdots, x^{(m)}) \equiv \sum a_{i_1\cdots i_m} x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}$$

and conversely, to every symmetrical linear form there corresponds an m-ic form, obtained by putting all $x^{(\nu)} = x$. Denoting by H, resp. $\mathfrak{H}$, the maxima of $Q(x)$, resp. $L(x^{(1)}, \cdots, x^{(m)})$ for the domain (G_0), we obviously have $H \leqq \mathfrak{H}$. On the other hand the identity

$$\begin{aligned} &L(x^{(1)}, \cdots, x^{(m)}) \\ &\equiv \frac{1}{2^{m-1}\cdot m!}\sum_{\varepsilon}(-1)^{\varepsilon_2+\varepsilon_3+\cdots+\varepsilon_m} Q(x^{(1)}+(-1)^{\varepsilon_2}x^{(2)}+\cdots+(-1)^{\varepsilon_m}x^{(m)}), \end{aligned} \tag{3.1}$$

where the sum is to be extended over $\varepsilon_\nu = 0, 1$; $\nu = 2, 3, \cdots, m$, shows that

$$\mathfrak{H} \leqq \frac{m^m}{m!} H.$$

Hence:

A symmetrical m-linear form is bounded in the domain (G_0) if and only if the corresponding m-ic form is bounded in (G_0).

The identity (3.1) can be proved as follows. We have

$$\sum_{\varepsilon}(-1)^{\varepsilon_2+\cdots+\varepsilon_m} Q(x^{(1)}+(-1)^{\varepsilon_2}x^{(2)}+\cdots+(-1)^{\varepsilon_m}x^{(m)})$$
$$=\sum_{\varepsilon,i,\nu}(-1)^{\varepsilon_2+\cdots+\varepsilon_m+\varepsilon_{\nu_1}+\cdots+\varepsilon_{\nu_m}}\cdot a_{i_1\cdots i_m}x_{i_1}^{(\nu_1)}\cdots x_{i_m}^{(\nu_m)}$$
$$=\sum_{i,\nu}a_{i_1\cdots i_m}x_{i_1}^{(\nu_1)}\cdots x_{i_m}^{(\nu_m)}\sum_{\varepsilon}(-1)^{\varepsilon_2+\cdots+\varepsilon_m+\varepsilon_{\nu_1}+\cdots+\varepsilon_{\nu_m}}.\ [15]$$

Let α_r be the number of ν's equal to r $(r=1,2,\cdots,m)$ for a fixed set of values ν. We have $\alpha_1+\alpha_2+\cdots+\alpha_m=m$, where the α's are non-negative integers. Then

$$\sum_{\varepsilon}(-1)^{\varepsilon_2+\cdots+\varepsilon_{\nu_m}}=\prod_{r=2}^{m}(1+(-1)^{\alpha_r+1})$$
$$(3.2)\qquad =\begin{cases}2^{m-1} \text{ when all } \alpha_r+1 \text{ are even,}\\ 0 \text{ otherwise.}\end{cases}$$

In the first case, since $\alpha_r\geqq 0$, all α_r $(r=2,3,\cdots,m)$ are positive odd integers and therefore

$$\alpha_1+(m-1)+2x=m_1$$

where x is an integer larger than the number of $\alpha_r\geqq 3$. Hence $x=0$, $\alpha_1=\alpha_r=1$ and the sum (3.2) is zero except, when the $\nu_1,\nu_2,\cdots,\nu_m$ are a permutation of $1,2,\cdots,m$, and the proof of (3.1) is completed.

Theorem I therefore holds for m-ic forms; but as the examples we have constructed for Theorem II are not symmetrical, we must refine them in order to show that Theorem I is also the best result of its kind for the symmetrical case. We can still vary the matrix $\|a_{rs}\|$, subject only to conditions (2.3). Let p be a prime number $>m$. Starting from the matrix

$$M_1\equiv\left\|e^{2\pi i\frac{r\cdot s}{p}}\right\|,\qquad (r,s=1,2,\cdots,p),$$

with p rows, we form successively for $\mu=2,3,\cdots$ the matrices M_μ, which are obtained by "substituting" the matrix $M_{\mu-1}$ into M_1:

$$M_\mu=\left\|e^{2\pi i\frac{r\cdot s}{p}}M_{\mu-1}\right\|.$$

The matrix M_μ contains $p^{2\mu}$ elements, each being a root of the equation $z^p-1=0$.

We put

$$\|a_{rs}\|=M_\mu\qquad (r,s=1,2,\cdots,p^\mu)$$

and form for this matrix, the form (2.5)

$$\sum_{i_1,\cdots,i_m=1}^{p^\mu}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{m-1}i_m}x_{i_1}\cdots x_{i_m}.$$

---

[15] Whenever ε_1 appears, it is put equal to 0, as the coefficient of $x^{(1)}$ is always $+1$.

The coefficients of this form are all roots of $z^p - 1 = 0$. It is not symmetrical, but by adding all the forms obtained by permuting the indices $i_1, i_2, \cdots, i_m$ we obtain a symmetrical one. Every coefficient of it is equal to

$$\sum_{k=0}^{p-1} \lambda_k \cdot \zeta_k,$$

where the ζ_k are the roots of $z^p = 1$, and where the λ_k are non negative integers, whose sum $\sum \lambda_k = m!$. Since p is a prime number, there are no relations of the form[16]

$$\sum_{k=0}^{p-1} \lambda_k \zeta_k = 0,$$

except possibly those in which all λ's are equal, say $= \lambda$. In such a case λ satisfies the equation

$$p \cdot \lambda = m!,$$

which is impossible, since p is a prime number larger than m. Thus

No coefficient of our symmetrical form is zero.

Every λ_k being $\leqq m!$, there exists only a finite number of values $\sum \lambda_k \zeta_k$, thus

There are only a finite number of coefficients different from each other.

There exists therefore an $\eta > 0$, such that all coefficients of the form are $\geqq \eta$ in absolute value. Then

$$S \geqq \eta \cdot n^{\frac{m+1}{2}}$$

and therefore

$$S \geqq A_m H_n, \qquad (n = p^\mu). \tag{3.3}$$

It is readily verified, that for the proof of Theorem II it suffices to have forms satisfying (3.3) for all $n = p^\mu$.

Theorem II *remains true for symmetrical and for m-ic forms.*

The coefficients of the form (3.1) possess a certain symmetry, since the permutation

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i_n & i_{n-1} & \cdots & i_1 \end{pmatrix}$$

leaves the coefficients $a_{i_1 \cdots i_m}$ invariant. In order to obtain a symmetrical form it suffices therefore to consider $m!/2$ permutations instead of $m!$. For $m = 3$, this enables us to take $p = 2$ and to construct "best possible" examples with real coefficients, these all have one of the values ± 1, ± 3.

[16] Such a relation with different λ's, would imply the existence of a polynomial $P(z)$ of degree $m - 1$ with rational coefficients, essentially different from the irreducible polynomial $z^{p-1} + z^{p-2} + \cdots + 1$ and having roots in common with it.

4. **Application to power series in an infinite number of variables.** In this section we take the first steps toward the application of the preceding results to ordinary Dirichlet series. Bohr proved the following theorem.[17]

Let G_n be a sequence of positive numbers and

$$P(x_1, x_2, \cdots, x_n, \cdots) \equiv c + \sum_{i=1}^{\infty} c_i x_i + \sum_{i_1, i_2=1}^{\infty} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots$$

a power series in an infinite number of variables, bounded in the domain $|x_n| \leqq G_n$. Let further ε_n be a sequence of positive number such that 1° $0 < \varepsilon_n < 1$ and 2° $\sum \varepsilon_n^2$ is convergent; then the power series P is absolutely convergent in the domain $|x_n| \leqq \varepsilon_n G_n$.

We now prove the following results, which complete Bohr's theorem.

THEOREM III. *If the power series P is bounded in $|x_n| \leqq G_n$: then its m-th polynomial P_m*

$$P_m \equiv c + \sum_{i=1}^{\infty} c_i x_i + \cdots + \sum_{i_1, \cdots, i_m=1}^{\infty} c_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

is absolutely convergent in $|x_n| \leqq \varepsilon_n G_n$, when $\sum \varepsilon_n^{\sigma_m}$ converges, $\sigma_m = \dfrac{2m}{m-1}$.

This exponent σ_m is the best possible one:

THEOREM IV. *There exist polynomials of the m-th degree in an infinite number of variables bounded in $|x_n| \leqq 1$, such that for every $\delta > 0$, the polynomial is non-absolutely convergent for $x_n = \varepsilon_n$ although the series $\sum \varepsilon_n^{\sigma_m + \delta}$ converges.*

Since $\sigma_m \to 2$, when $m \to \infty$ it follows immediately from this theorem that the exponent 2 of Bohr's theorem cannot be increased.

REMARK: Condition 1° of Bohr's theorem, namely $0 < \varepsilon_n < 1$, is not essential to the truth of Theorem III.

Proof of Theorem III. It is obviously sufficient to prove this theorem for the domain (G_0). We show first the truth of

LEMMA 2. *If the power series $P(x_1, \cdots)$ is bounded by H in (G_0); then H is also an upper bound for the form*

$$\sum_{i_1, \cdots i_m=1}^{\infty} c_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}. \tag{4.1}$$

The proof follows the same lines as Bohr's proof in the case $m = 1$. (Theorem V of Bohr). We have to show, that every truncated form

$$\sum_{i_1, \cdots i_m=1}^{N} c_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

[17] H. Bohr, loc. cit. p. 462.

is in absolute value $\leqq H$. Let x_n^* be the set of the variables for which the truncated form takes on a value H^*, which is absolutely the largest. This set exists since the truncated domain $(G_0):|x_n|\leqq 1,\ n\leqq N;\ x_n=0,\ n>N$, is compact. Put $x_n = t\cdot x_n^*$, then the truncated power series is a power series in t, such that the coefficient of t^m equals this value H^*. By Cauchy's evaluation of the coefficients of a power series, we see that this value is absolutely $\leqq H$.

It suffices therefore to prove Theorem III for m-ic forms. Let $\varepsilon_n>0$ be any sequence for which $\sum \varepsilon_n^{\sigma_m}$ converges. By Hölder's inequality with $p=\dfrac{2m}{m-1},\ q=\dfrac{2m}{m+1}$

$$\sum_{i_1,\cdots,i_m=1}^{N} \varepsilon_{i_1}\cdots\varepsilon_{i_m}|c_{i_1\cdots i_m}| \leqq \left(\sum_1^N (\varepsilon_{i_1}\cdots\varepsilon_{i_m})^{\frac{2m}{m-1}}\right)^{\frac{m-1}{2m}}\cdot\left(\sum_1^N |c_{i_1\cdots i_m}|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}$$

$$\leqq E^{\frac{m-1}{2}}\cdot A_m\cdot H,$$

where E is the sum of the series $\sum \varepsilon_n^{\sigma_m}$, A_m a constant determined by m and H an upper bound of the form (4.1). Hence the sum

$$\sum_1^{\infty} |c_{i_1\cdots i_m}|\,\varepsilon_{i_1}\cdots\varepsilon_{i_m}$$

is convergent, which proves Theorem III.

In the case $m=2$, Theorem III shows that the examples given by Toeplitz were the best obtainable from quadratic forms.

Proof of Theorem IV. We use the examples constructed in section 3. The integers m and p being fixed, let $Q_\mu(x)$ be the m-ic form corresponding to $n=p^\mu$. The variables which enter in the different $Q_\mu(x)$ shall be independent of each other; in order to differentiate between them we use a superscript μ:

$$Q_\mu(x^{(\mu)}) \equiv \sum_{i_1,\cdots i_m=1}^{p^\mu} c_{i_1\cdots i_m} x_{i_1}^{(\mu)}\cdots x_{i_m}^{(\mu)}.$$

Let

$$Q(x) \equiv \sum_{\mu=1}^{\infty} \frac{1}{\mu^2}\, p^{-\mu\cdot\frac{m+1}{2}}\, Q_\mu(x^{(\mu)}). \tag{4.2}$$

This form $Q(x)$ is an m-ic form in the variables

$$\{x_n\} \equiv \{x_1^{(1)},\cdots,x_p^{(1)};\ x_1^{(2)},\cdots,x_{p^2}^{(2)};\ x_1^{(3)},\cdots\}.$$

It is bounded in the domain (G_0), because the maxima of the forms Q_μ (truncated or not) are of order of magnitude $p^{\mu\cdot\frac{m+1}{2}}$, and the series $\sum\dfrac{1}{\mu^2}$ is convergent.

To study the absolute convergence, we use the fact, that there exists an $\eta > 0$, such that all coefficients of the forms Q_μ are in absolute value $\geqq \eta$. The m-ic form $Q(x)$ will therefore be non-absolutely convergent for a set of values $x_n^{(\mu)}$ if

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu^2} p^{-\mu \cdot \frac{m+1}{2}} \left\{ \sum_{n=1}^{p^\mu} |x_n^{(\mu)}| \right\}^m \tag{4.3}$$

diverges.

Given an arbitrarily small number $\delta > 0$, p^δ is larger than one and we can choose $k < 1$, so that

$$h = p^\delta \cdot k^{\frac{m-1}{2} \cdot (1-\delta)} > 1.$$

We then take all the variables $x_n^{(\mu)}$ of the same form Q_μ equal to each other,

$$x_n^{(\mu)} = (k \cdot p^{-1})^{\mu \cdot \frac{m-1}{2m}(1-\delta)}.$$

The series $\sum x_n^{\frac{2m}{m-1} \frac{1}{1-\delta}}$ is thus equal to

$$\sum_{\mu=1}^{\infty} \sum_{n=1}^{p^\mu} (x_n^{(\mu)})^{\frac{2m}{m-1} \cdot \frac{1}{1-\delta}} = \sum_{\mu=1}^{\infty} k^\mu,$$

which converges since $k < 1$.

But for these special values $x_n^{(\mu)}$ the expression (4.3) diverges. It is equal to

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \left(p^\delta k^{\frac{m-1}{2}(1-\delta)} \right)^\mu = \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} h^\mu$$

and the last series is divergent, since $h > 1$. When $\delta \to 0$

$$\frac{2m}{m-1} \cdot \frac{1}{1-\delta} \to \frac{2m}{m-1};$$

we have therefore constructed a form in the variables x_n, bounded in (G_0), which for any given $\delta' > 0$, is non-absolutely convergent for a certain set of values x_n with convergent $\sum x_n^{\sigma_m + \delta'}$; completing thus the proof of Theorem IV.

5. **Applications to ordinary Dirichlet series.** We now apply the results of the preceeding section to ordinary Dirichlet series, with the help of Bohr's Theorems A, B, and C, mentioned in the introduction. We proved $S = 2$, hence by Theorem C:

There exist ordinary Dirichlet series for which the widths of their strips of uniform, non-absolute convergence are arbitrarily close to $\frac{1}{2}$.

Consider now Theorems III and IV; applying them to the theory of Dirichlet series we shall prove

THEOREM V. *Let σ_u be the abscissa of uniform convergence of the Dirichlet series*

$$\sum_{n=1}^{\infty} a_n n^{-s}. \tag{5.1}$$

If we replace by zero those terms for which n, decomposed into a product of prime numbers, contains more than m factors, then the new series is absolutely convergent in the half plane

$$\sigma > \sigma_u + \frac{m-1}{2m},$$

and

THEOREM VI. *There exist ordinary Dirichlet series with $a_n = 0$, when n contains more than m prime factors, which converge uniformly, but nonabsolutely in strips whose widths are exactly equal to* $\frac{m-1}{2m}$:

$$\sigma_a - \sigma_u = \frac{m-1}{2m}.$$ [18]

When $m = 1$, we obtain a result proved by Bohr.[19]

The proof of these two theorems proceeds in the same way as Bohr's proof of Theorem C.

Let σ_u be the abscissa of uniform convergence of (5.1). Then by Theorem A of Bohr the power series corresponding to (5.1) is bounded in the domain $|x_n| \leqq p_n^{-\sigma_u-\delta}$, for every $\delta > 0$. Hence by Theorem III, and since

$$\sum_{n=1}^{\infty} p_n^{-1-\delta\cdot\frac{2m}{m-1}}$$

is convergent, the m-th polynomial is absolutely convergent in the domain $|x_n| \leqq p_n^{-\sigma_u-\frac{m-1}{2m}-2\delta}$. By Theorem B of Bohr, the Dirichlet series associated with this m-th polynomial is absolutely convergent in

$$\sigma > \sigma_u + \frac{m-1}{2m} + 2\delta,$$

which proves Theorem V.[20]

[18] If the abscissa of uniform convergence is $\leqq 0$, then as Hardy and Carlson (cf. introduction) have proved, the series $\sum a_n^2$ is convergent. Theorem V states essentially that $\sum a_n^{2m/(m+1)}$ converges, if the summation is extended only over those indices n, which contain no more than m prime factors. Theorem VI shows that this exponent $2m/(m+1)$ is the best possible.

[19] H. Bohr, loc. cit. p. 468.

[20] It is interesting to interpret Lemma 2 as a result concerning Dirichlet series. Putting certain coefficients of a Dirichlet series equal to zero can change the position of the abscissa σ_u of uniform convergence. Lemma 2 states that σ_u is never shifted to the right, when all coefficients, whose indices contain more than m prime factors, are replaced by zero.

In order to prove Theorem VI, we consider the Dirichlet series associated with the example (4.2). By Theorem B, the abscissa of uniform convergence is $\leqq 0$. On the other hand, there exists for any $\delta' > 0$, at least one set of real, non-increasing values x_n with convergent

$$\sum_{n=1}^{\infty} x_n^{\frac{2m}{m-1}+\delta'},$$

such that the form (4.2) is non-absolutely convergent for these x_n. They are non-increasing, hence

$$x_n^{\frac{2m}{m-1}+\delta'} = O\left(\frac{1}{n}\right)$$

as $n \to \infty$, or

$$x_n = O\left(n^{-\frac{1}{\frac{2m}{m-1}+\delta'}}\right).$$

But p_n being the n-th prime number, we know that for any $\delta'' > 0$

$$p_n = O(n \cdot \log n) = o(n^{1+\delta''});$$

hence

$$n^{-(1+\delta'')} = o(p_n^{-1})$$

and

$$x_n = o\left(p_n^{-\frac{m-1}{2m}+\delta}\right)$$

for every $\delta > 0$. Thus there exist a constant A, such that

$$x_n \leqq A \cdot p_n^{-\frac{m-1}{2m}+\delta}.$$

This shows that the Dirichlet series associated with (4.2) is non-absolutely convergent in any half-plane

$$\sigma \geqq \frac{m-1}{2m} - \delta \qquad (\delta > 0)$$

and therefore

$$\sigma_a \geqq \frac{m-1}{2m}.$$

Since $\sigma_u \leqq 0$

$$\sigma_a - \sigma_u \geqq \frac{m-1}{2m},$$

but by Theorem V, the left hand side cannot exceed $\frac{m-1}{2m}$, hence

This is not true in general. Given any Dirichlet series for which $\sigma_a - \sigma_u$ is different from zero, there exist coefficients, such that putting these equal to zero shifts σ_u to the right by the difference $\sigma_a - \sigma_u$.

$$\sigma_u = 0, \quad \sigma_a = \frac{m-1}{2m}, \quad \sigma_a - \sigma_u = \frac{m-1}{2m}.$$

6. **Solution of the main problem.** We give first an example of an ordinary Dirichlet series, for which $\sigma_a - \sigma_u = \frac{1}{2}$.

In the preceding sections we have only shown, that the inequality $\sigma_a - \sigma_u \leqq \frac{1}{2}$ cannot be replaced by a better one. But it is now easy to construct a Dirichlet series for which the width of the strip of uniform, but non-absolute convergence is exactly equal to $\frac{1}{2}$.

We start from the examples (4.2)

$$Q_m(x) \equiv \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} p^{-\mu \cdot \frac{m+1}{2}} Q_\mu(x^{(\mu)}),$$

where the index m refers to the degree of the form. Dividing each of these forms by its upper bound, we obtain new forms $Q_m^*(x)$ bounded by 1 in the domain (G_0). Let q_m be any sequence of positive numbers, such that the series $\sum q_m$ is convergent. Put

$$P(x) \equiv \sum_{m=1}^{\infty} q_m Q_m^*(x).$$

The power series P is obviously bounded by $\sum q_m$ in (G_0), the abscissa of uniform convergence of the associated Dirichlet series is non-positive

$$\sigma_u \leqq 0.$$

On the other hand, the forms Q_m^* have no term in common, because of their different degrees. Hence the power series is non-absolutely convergent as soon as one of the forms Q_m^* is non-absolutely convergent. This implies

$$\sigma_a \geqq \frac{1}{2},$$

because, if $\sigma_a < \frac{1}{2}$, we can find an m large enough, such that the corresponding Q_m^* would be non-absolutely convergent for $x_n = p_n^{-\sigma_0}$, where $\sigma_a < \sigma_0 < \frac{1}{2}$. But since $\sigma_u \leqq 0$ and since the difference $\sigma_a - \sigma_u \leqq \frac{1}{2}$, it must be exactly $= \frac{1}{2}$:

$$\sigma_a = \frac{1}{2}; \qquad \sigma_u = 0.$$

We turn now to the proof of the main theorem:

THEOREM VII. *For any given σ, in the interval $0 \leqq \sigma \leqq \frac{1}{2}$, there exist ordinary Dirichlet series for which the width of the strip of uniform, but non-absolute convergence is exactly equal to σ.*[21]

The preceding example proved this theorem in the case $\sigma = \frac{1}{2}$, we therefore can suppose $\sigma < \frac{1}{2}$. We determine m such that

[21] Cf. Remark at the end of the paper.

$$\frac{m-1}{2m} \geqq \sigma$$

and Theorem VII will be proved, if we can show

THEOREM VIII. *There exist ordinary Dirichlet series associated with m-ic forms, for which* $\sigma_a - \sigma_u = \sigma$, *for every* σ *in the interval* $0 \leqq \sigma \leqq \frac{m-1}{2m}$.

We take up again the m-ic forms

$$Q_\mu(x^{(\mu)}) \equiv \sum_{i_1, \cdots i_m = 1}^{p^\mu} c_{i_1 \cdots i_m} x_{i_1}^{(\mu)} \cdots x_{i_m}^{(\mu)},$$

which were used to build up the example (4.2). By means of a parameter u, $0 \leqq u \leqq 1$, we are going to deform them into the worst examples, where all the coefficients are positive. We put

$$Q_\mu(x^{(\mu)}, u) \equiv \sum_1^{p^\mu} \varepsilon_{i_1 \cdots i_m}(u)\, c_{i_1 \cdots i_m} x_{i_1}^{(\mu)} \cdots x_{i_m}^{(\mu)},$$

and suppose that the functions $\varepsilon_{i_1 \cdots i_m}(u)$ are continuous in u; equal to one in absolute value, and satisfy the boundary conditions

$$\varepsilon_{i_1 \cdots i_m}(0) = 1 \text{ and } \varepsilon_{i_1 \cdots i_m}(1) = \frac{|c_{i_1 \cdots i_m}|}{c_{i_1 \cdots i_m}}.$$

Let $H_\mu(u)$ denote the maximum of the form $Q_\mu(u)$ in the domain (G_0). It is readily seen that for all μ, $H_\mu(u)$ is a continuous function of u with the following properties:

There exist absolute constants A_1, A_2, A_3 such that

1° for all values u and all $n = p^\mu$

$$H_\mu(u) \geqq A_1 \cdot n^{\frac{m+1}{2}}. \quad \text{(Theorem I)}$$

2° for $u = 0$ and all $n = p^\mu$

$$H_\mu(0) \leqq A_2\, n^{\frac{m+1}{2}}. \quad \text{(Theorem II)}$$

3° for $u = 1$ and all $n = p^\mu$

$$H_\mu(1) \geqq A_3\, n^m.$$ [22]

We have $A_1 \leqq A_2$ and can suppose $A_1 \leqq A_3$.

LEMMA 3. *There exist two constants* B_1 *and* B_2 *such that for every* τ, $\frac{m+1}{2} \leqq \tau \leqq m$, *and every* μ, *there exists a* u_μ *for which*

$$B_1 \cdot n^\tau \leqq H_\mu(u_\mu) \leqq B_2 \cdot n^\tau.$$

[22] This follows from the fact that there exists an $\eta \geqq 0$, such that $|c_{i_1 \cdots i_m}| \geqq \eta$.

It is evidently sufficient to prove this lemma for large values of n. Let N be so large that

$$A_2 N^{\frac{m+1}{2}} \leqq A_3 N^m.$$

Then for all $n \geqq N$, we shall prove

$$A_1 n^\tau \leqq H_\mu(u_\mu) \leqq A_2 n^\tau. \tag{6.1}$$

Since (Condition 2)

$$H_\mu(0) \leqq A_2 n^{\frac{m+1}{2}} \leqq \begin{cases} A_3 n^m, \\ A_2 n^\tau, \end{cases}$$

and (Condition 3)

$$H_\mu(1) \geqq A_3 n^m,$$

we can find a u_μ, such that

$$H_\mu(u_\mu) = \operatorname{Min}(A_3 n^m,\ A_2 n^\tau). \tag{6.2}$$

This $H_\mu(u_\mu)$ satisfies (6.1) because

$$A_1 n^\tau \leqq \operatorname{Min}(A_3 n^m,\ A_2 n^\tau).$$

LEMMA 4. *For every ϱ in the interval $1 \leqq \varrho \leqq \dfrac{2m}{m+1}$, and every $n = p^\mu$, there exist m-ic forms in n dimensions, for which*

$$C_1 \cdot n^{\frac{m}{\varrho}} \leqq C_2 H_\mu \leqq \left\{ \sum_{i_1 \cdots i_m = 1}^{n} |C_{i_1 \cdots i_m}|^\varrho \right\}^{\frac{1}{\varrho}} \leqq C_3 H_\mu \leqq C_4 n^{\frac{m}{\varrho}},$$

where the constants C_1, C_2, C_3 and C_4 are independent of n.

It is easily verified that the forms $Q_\mu(x^{(\mu)}, u_\mu)$, where u_μ is determined by (6.2) satisfy these conditions.

We proceed from now on exactly as for the extreme case $\varrho = \dfrac{2m}{m+1}$. We put

$$Q(x) \equiv \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} p^{-\mu \cdot \frac{m}{\varrho}} Q_\mu(x^{(\mu)}, u_\mu)$$

and prove that $Q(x)$ satisfies the following conditions:

1° *If $Q(x)$ is bounded in $|x_n| \leqq G_n$, then it is absolutely convergent in $|x_n| \leqq \varepsilon_n G_n$; when $\sum \varepsilon_n^\varkappa$ converges $\left(\varkappa = \dfrac{\varrho}{\varrho - 1}\right)$.*

2° *$Q(x)$ is bounded in $|x_n| \leqq 1$.*

3° *For every $\delta > 0$, there exist a set of real values x_n with convergent $\sum x_n^{\varkappa+\delta}$, for which $Q(x)$ is non-absolutely convergent.*

The associated Dirichlet series will be an example proving Theorem VIII, if we take $\varrho = \dfrac{1}{1-\sigma}$.

7. Generalization to more general types of Dirichlet series. A theorem similar to Theorem VII cannot be formulated for general Dirichlet series $\sum a_n e^{-\lambda_n s}$. The inequality

$$\sigma_a - \sigma_u \leqq \frac{D}{2}; \qquad D = \overline{\lim} \frac{\log n}{\lambda_n}, \tag{7.1}$$

is not necessarily the best possible one, when the type $\{\lambda_n\}$ is given and not merely the quantity D, as in the case considered by Neder. For the type determined by the Dirichlet series associated with the polynomials of the mth degree, (7.1) can be replaced (Theorem V) by

$$\sigma_a - \sigma_u \leqq \frac{m-1}{m} \cdot \frac{D}{2},$$

the upper limit D being equal to one for this type.

However Bohr[23] extended the results of his paper in the Göttinger Nachrichten to cover certain types of general Dirichlet series and for these types, with a further restriction to prevent the occurrence of examples similar to the one just mentioned, the inequality (7.1) is the best obtainable result.

Suppose that $\{b_n\}$ is a sequence of increasing positive numbers, which tend to ∞ and which are linearly independent in the field of rational numbers. Consider then the values

$$\sum_{\nu=1}^{n} r_\nu b_\nu,$$

for any positive n, and any non-negative integral coefficients r_ν. These values form a sequence, $\{\lambda_n\}$, which can be arranged in increasing order

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

and whose elements λ_n tend to ∞. This sequence can therefore be considered as the *type* of a Dirichlet series. The first sequence $\{b_n\}$ forms an integral base for $\{\lambda_n\}$; the associated function obtained by putting $x_n = e^{-b_n s}$ is a power series in x_n. The methods of proving Theorems A, B of Bohr and our results apply in this case:

For such types the inequality (7.1) *is the best possible result.*

In the construction of examples for which $\sigma_a - \sigma_u = \frac{D}{2}$ it is essential to observe that

$$D(b) = \overline{\lim} \frac{\log n}{b_n} = D(\lambda) = \overline{\lim} \frac{\log n}{\lambda_n},$$

[23] H. Bohr, Zur Theorie der allgemeinen Dirichletschen Reihen, Math. Annalen **79**, (1919), p. 136–156.

which is equivalent to the statement:

The exponent $D(b)$ of convergence of the sequence $\{e^{b_n}\}$ is equal to the exponent $D(\lambda)$ of convergence of the sequence $\{e^{\lambda_n}\}$.

We have obviously $\lambda_n \leqq b_n$ and therefore

$$D(b) \leqq D(\lambda),$$

and it remains only to show that $\sum e^{-\lambda_n \sigma}$ converges when $\sum e^{-b_n \sigma}$ converges. For the subsequence $\{\lambda_n'\}$, obtained by taking only the m first elements of the base, we have

$$\sum e^{-\lambda_n' \sigma} = \sum_{r_1, \cdots r_m = 0}^{\infty} e^{-\sigma \sum_1^m r_\nu b_\nu} = \prod_{\nu=1}^{m} \frac{1}{1 - e^{-\sigma b_\nu}}$$

$$\leqq \prod_{\nu=1}^{m} \left(1 + \frac{e^{-\sigma b_\nu}}{1 - e^{-\sigma b_1}}\right) \leqq \prod_{\nu=1}^{\infty} \left(1 + \frac{e^{-\sigma b_\nu}}{1 - e^{-\sigma b_1}}\right).$$

This last infinite product is convergent, proving thus

$$D(b) \geqq D(\lambda),$$

and therefore

$$D(b) = D(\lambda).$$

We are able now to construct new examples proving Neder's result. Given any non-negative D, there exist Dirichlet series for which

$$\sigma_a - \sigma_u = \frac{D}{2}$$

of all types $\{\lambda_n\}$, obtained from a base $\{b_n\}$, $0 < b_1 < \cdots b_n \cdots \to \infty$, satisfying

$$D = \overline{\lim} \frac{\log n}{b_n}.$$

The simplest example is

$$b_n = \frac{1}{D} \cdot \log n,$$

which gives Dirichlet series of the type $\sum a_n \cdot n^{-\frac{s}{D}}$, obtained from ordinary Dirichlet series by the substitution $\left(s \mid \frac{s}{D}\right)$.

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REMARK (added in proof, May, 1931). As Prof. Bohr pointed out to us, Theorem VII can be proved more simply as follows. Let $f(s)$ be a Dirichlet series whose $\sigma_u = 0$ and whose $\sigma_a = \frac{1}{2}$. If $\zeta(s)$ denotes the Riemann Zeta-function ($\sigma_a = \sigma_u = 1$) and σ any real number $0 \leqq \sigma \leqq \frac{1}{2}$, then $f(s) + \zeta(s + 1 - \sigma)$ is a Dirichlet series for which $\sigma_a - \sigma_u = \sigma$. Similar examples prove Theorem VIII. It may be interesting however, to see that the method used to obtain best possible examples is flexible enough to give the whole range of possible values for the difference $\sigma_a - \sigma_u$.

A STUDY OF INDEFINITELY DIFFERENTIABLE AND QUASI-ANALYTIC FUNCTIONS. I.[1]

By W. J. Trjitzinsky.[2]

In the first few sections of this paper we shall study series of the form

$$(1) \qquad \sum x_i f(a_i x),$$

where $|f^{(p)}(x)| < h \, (p = 0, 1, \cdots)$ for all real values of x. Theorems and cases of interest will be established when, for all real values of x, the series (1) represents an indefinitely differentiable function with assigned initial values at $x = 0$.

Further, by the aid of the two generalized Cauchy formulas established by Borel,[3] we shall investigate some of the properties of Borel monogenic functions. These functions form a class of quasi-analytic functions of a complex variable.

Quasi-analytic functions are of interest since they possess an important property in common with analytic functions: the values of the function and of its derivatives at a point determine the function uniquely throughout the domain of definition. A formal Taylor's series of a quasi-analytic function is not necessarily convergent.

In particular, beginnings of a theory of functions $P(x, y)$ (x, y real)—defined in regions over which a Borel monogenic function may exist—will be made by establishing formulas analogous to Green's identities.

In the second part, to appear later,[4] several problems of representation of quasi-analytic functions are considered. The methods used for the latter purpose are, to an extent, of the type utilized by Carleman.[5]

Series $\sum x_i f(a_i x)$. Their Application in the Representation of Indefinitely Differentiable Functions.

1. General properties of the series. We introduce the following definition.

[1] Received October 23, 1930.

[2] National Research Fellow.

[3] É. Borel, Leçons sur les fonctions monogènes $\cdots$. Paris, 1917. This monograph will be referred to as (B_2).

[4] These Annals.

[5] T. Carleman, Les fonctions quasi analytiques. Paris, 1926. This monograph will be referred to as (C).

DEFINITION. *An entire function $f(x)$, such that the inequalities $|f(x)|$, $|f^{(p)}(x)| < h$, $p = 1, 2, \cdots$ (h independent of x and p) hold for all real values of x, will be said to be of class A.* Among functions of class A are $\sin x$, $\cos x$ and any of the Bessel's functions with an integral subscript.[6]

The following theorem will be proved.

THEOREM I. *Let $f(x)$ be of class A and the a_k be real and such that for a positive integer p the series $\sum_k |1/a_k^p|$ converges. In order that*

$$S(x) = \sum x_k f(a_k x) \tag{1}$$

and all the series $S^{(p)}(x)$, derived by differentiating (1) *term by term, should be uniformly convergent for all real values of x it is necessary and sufficient that*

$$\lim_{k\to\infty} x_k a_k^m = 0 \qquad (m = 0, 1, 2, \cdots). \tag{2}$$

Proof. Suppose $S^{(m)}(x) = \sum_k x_k a_k^m f^{(m)}(a_k x)$ converges uniformly on the axis of reals, $m = 0, 1, 2, \cdots$ ($S^{(0)} = S$); it follows in particular that $S^{(m)}(0) = f^{(m)}(0)\cdot\sum x_k a_k^m$ converges for $m = 0, 1, 2, \cdots$. Hence $\lim_{k\to\infty} x_k a_k^m = 0$, $m = 0, 1, 2, \cdots$. Conversely, if (2) holds, or what amounts to the same, if $\lim_{k\to\infty} x_k a_k^{m+p} = 0$ ($m = -p, -p+1, \cdots, 0, 1, 2, \cdots$), then $|x_k a_k^{m+p}| < h_m$, where h_m is independent of k and p. Since the a_k are real and $f(x)$ is of class A, $|f^{(m)}(a_k x)| < h$ ($m = 0, 1, 2, \cdots$) for all real values of x. Hence $|S^{(m)}(x)| \leqq \sum_k |x_k a_k^m f^{(m)}(a_k x)| < h \sum_k |x_k a_k^m|$. Making use of the inequalities $|x_k| < |h_m/a_k^{m+p}|$ we derive

$$\sum_k |x_k a_k^m f^{(m)}(a_k x)| < h\, h_m \sum_k |a_k^m/a_k^{m+p}| = h\, h_m \sum_k |1/a_k^p| \tag{3}$$
$$(m = 0, 1, 2, \cdots).$$

The series in the last member of (3) converges by hypothesis. The inequalities (3) prove uniformity of convergence for all real values of x of all the series for the $S^{(m)}(x)$. This completes the proof.

The following corollary is immediate.

COROLLARY. *Let $\sum_k |1/a_k^p|$ be convergent (the a_k real) and let the x_k be such that there exist finite constants h_m independent of k, for which $|x_k a_k^m| < h_m$ ($m = 0, 1, 2, \cdots$). For the derivatives of the function $S(x)$, defined by the series $\sum_k x_k f(a_k x)$ ($f(x)$ of class A, $|f|$, $|f^{(m)}| < h$), the following inequalities hold for all real values of x*

[6] For Bessel's functions we have $J_m(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(m\theta - x\sin\theta)\, d\theta$ and consequently for all real values of x $|J_m(x)|$, $|J_m^{(p)}(x)| \leqq 1$.

$$|S^{(m)}(x)| < h\,h_m \sum_k |1/a_k^p| \qquad (m = 0, 1, 2, \cdots). \tag{4}$$

Consider the particular case when the $a_k = k$ and the law of decrease of the $|x_k|$ is expressed by the inequalities

$$|x_k| < c \cdot e^{-\psi(k)}.$$

The function $\psi(k)$ is assumed to be non-decreasing and such that $\lim_{k\to\infty} \psi(k) = \infty$; moreover, its derivative is such that with a sufficiently small positive α and for all positive values of x

$$x\,\psi^{(1)}(x)/\psi(x) > \alpha.$$

Then

$$|S^{(p)}(x)| < h \sum_k k^p |x_k| < c\,h \sum_k k^p\, e^{-\psi(k)}$$

$$= h \sum_k (k\, e^{-\psi(k)/2}) \cdot e^{-\psi(k)/2} < (h_1^p) \cdot (\text{max. in } k \text{ of } (k^p\, e^{-\psi(k)/2})).$$

The last member in these inequalities, with $\psi(x)$ satisfying the above assumptions, occurs in the discussion of trigonometric series by de la Vallée Poussin. He shows that

$$\text{max. in } k \text{ of } (k^p\, e^{-\psi(k)/2}) < \left[h^{(1)}\,\varphi\left(\frac{2p}{\alpha}\right)\right]^p,$$

where $\varphi(x)$ is the function inverse to $\psi(x)$ and $h^{(1)}$ is a suitable constant.[7] Hence the following is true.

With $f(x)$ of class A let the coefficients of the series

$$S(x) = \sum_k x_k f(kx)$$

satisfy the inequalities $|x_k| < c\, e^{-\psi(k)}$. Let $\psi(k)$ be a non-decreasing function which approaches ∞ with k and which has its derivative such that for a sufficiently small positive α the inequality $x\,\psi^{(1)}(x)/\psi(x) > \alpha$ holds for all positive values of x. Then for all real values of x

$$|S^{(p)}(x)| < \left[h_2\,\varphi\left(\frac{2p}{\alpha}\right)\right]^p \qquad (p = 0, 1, 2, \cdots),$$

where h_2 is a suitable constant and $\varphi(x)$ is the function inverse to $\psi(x)$.

2. **Representation of functions with assigned initial values at a point; with the $|a_{i+1}/a_i|$ not necessarily $> \lambda > 1$.** We shall call

[7] C. de la Vallée Poussin, The Rice Institute Pamphlet, vol. 12 (1925), no. 2, pp. 128–130. This Pamphlet will be referred to as (V).

the values $F(0)$, $F^{(p)}(0)$ initial values of $F(x)$ at $x=0$; the constants a_k will be supposed to be real, $|a_{k+1}|>|a_k|$, $|a_1|\geqq 1$, $|a_k|\to\infty$. We note that the set of equations with the A_i bounded as a set

$$\sum_{k=1}^{\infty} a_k^i y_k = A_i \qquad (i=1,2,\cdots) \tag{5}$$

has been solved by Borel.[8] If $\{y_k\}$ is a solution of (5), making the first members of (5) convergent, then the equations

$$\sum_{k=1}^{\infty} a_k^i x_k = A_{i+1} = F_i \qquad (i=0,1,2,\cdots) \tag{6}$$

are satisfied for the $x_k = a_k y_k$.

The following is the way a set of solutions of (5) is given by Borel. Let $\varphi(z)=\sum \varphi_n z^n$ be an entire function whose zeros are the a_k and are simple; and let $\theta(z)=\sum\theta_n z^n$ be another entire function, such that

$$\sum_{k=1}^{\infty} |a_k^i/\varphi^{(1)}(a_k)\,\theta(a_k)| \tag{7}$$

converges for $i=1,2,\cdots$. With

$$\varphi(z)\,\theta(z)/(z-a_k) = \sum_{n=0}^{\infty} c_n^{(k)} z^n, \tag{8}$$

the y_k are defined by

$$y_k = \left[\sum_{i=1}^{\infty} c_i^{(k)} A_i\right] \Big/ \varphi^{(1)}(a_k)\,\theta(a_k) \qquad (k=1,2,\cdots). \tag{9}$$

Let the $F^{(n)}(0)$ be an assigned set of constants. We shall seek to determine the coefficients x_k of the series

$$S(x) = \sum_{k=1}^{\infty} x_k f(a_k x) \tag{10}$$

so that $S^{(p)}(0) = F^{(p)}(0)\ (p=0,1,2,\cdots)$. It will be assumed that

$$\sum_n n^2\, |F^{(n)}(0)/f^{(n)}(0)|,\quad F^{(n)}(0)/f^{(n)}(0) = F_n = A_{n+1}, \tag{11}$$

converges. Suppose that (7) converges and that in (10) $x_k = a_k y_k$, where the y_k are defined by (9) and (8).

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[8] É. Borel, Sur quelques points de la Théorie des fonctions, Annales de l'École Norm. sup., 3e serie, t. XII, 1895, p. 44. This paper will be referred to as (B_1). Also see F. Riesz, Les systèmes d'équations linéaires à une infinité d'inconnues, Paris, 1913 (pp. 19–20). This monograph will be referred to as (R).

Then

$$(12)\quad \begin{aligned} |S^{(m)}(x)| &= \left|\sum_{k=1}^{\infty} x_k a_k^m f^{(m)}(a_k x)\right| < h\sum_{k=1}^{\infty} |x_k a_k^m| = h\sum_{k=1}^{\infty} |y_k a_k^{m+1}| \\ &= h\sum_k \left| a_k^{m+1}\left(\sum_{i=1}^{\infty} c_i^{(k)} A_i\right) \Big/ \varphi^{(1)}(a_k)\,\theta(a_k)\right| = G_m \quad (m = 0, 1, 2, \cdots). \end{aligned}$$

Now, since

$$\sum_n c_n^{(k)} z^n = \left(\sum_i \varphi_i z^i\right)\left(\sum_m \theta_m z\right) \Big/ -a_k\left(1 - \frac{z}{a_k}\right),$$

it follows that

$$(13)\quad c_n^{(k)} = -\frac{1}{a_k}[\delta_0 a_k^{-n} + \delta_1 a_k^{-n+1} + \cdots + \delta_{n-1} a_k^{-1} + \delta_n],$$

where $\delta_m = \varphi_0\,\theta_m + \varphi_1\,\theta_{m-1} + \cdots + \varphi_m\,\theta_0$ $(m = 0, 1, 2, \cdots)$. The functions $\varphi(z)$ and $\theta(z)$ being entire, there exists a constant h_1 independent of k such that $|\varphi_k|$ and $|\theta_k| < h_1$; consequently $|\delta_m| < (m+1)\,h_1^2$. This together with the inequalities $|a_k| \geqq 1$ $(k = 1, 2, \cdots)$ gives

$$(14)\quad \begin{aligned} |c_n^{(k)}| &< |\delta_0| + \cdots + |\delta_n| < h_1^2[1 + 2 + \cdots + (n+1)] \\ &= h_1^2(n+1)(n+2)/2 = l_n \qquad (n = 0, 1, 2, \cdots). \end{aligned}$$

From (12) it follows that

$$(15)\quad \begin{aligned} |S^{(m)}(x)| &< G_m < h\sum_k \left| a_k^{m+1}\left(\sum_{i=1}^{\infty} l_i\,|A_i|\right) \Big/ \varphi^{(1)}(a_k)\,\theta(a_k)\right| \\ &= h\left(\sum_{i=1}^{\infty} l_i\,|A_i|\right)\cdot \sum_k |a_k^{m+1}/\varphi^{(1)}(a_k)\,\theta(a_k)| \quad (m = 0, 1, 2, \cdots). \end{aligned}$$

The first series in the last member of (15),

$$(16)\quad \sum_{i=1}^{\infty} l_i\,|A_i| = \sum_{i=0}^{\infty} l_{i+1}\,|F^{(i)}(0)/f^{(i)}(0)| = h' \quad (h'\text{-independent of } m),$$

converges on account of the convergence of the series (11); the second series converges by hypothesis. The inequalities (15) are valid for all real values of x, and they justify differentiation term by term of (10) and of all the derived series. Furthermore, by (6) (where $A_{n+1} = F^{(n)}(0)/f^{(n)}(0)$) it follows that $S^{(n)}(0) = F^{(n)}(0)$ $(n = 0, 1, \cdots)$. In the above demonstration as well as throughout the rest of the paper, if for a value of n we have $f^{(n)}(0) = 0$, then the corresponding numbers $F^{(n)}(0)$, $F^{(n)}(0)/f^{(n)}(0)$ will be supposed to be equal to zero. The following theorem will now be stated.

THEOREM II. *The series*

$$(17)\quad S(x) = \sum_{k=1}^{\infty} x_k f(a_k x),\ |a_{k+1}| > |a_k|,\ |a_1| \geqq 1,\ |a_k| \to \infty \qquad (a_k \textit{ real}),$$

where $f(x)$ is of class A, is indefinitely differentiable, term by term, for all real values of x, and represents a function with assigned initial values $F^{(n)}(0)$ $(n=0,1,\cdots)$ at $x=0$ provided the following conditions hold. The series $\sum_n n^2\,|F^{(n)}(0)/f^{(n)}(0)|$ converges. The x_k are defined by the expressions

$$(18)\qquad x_k = a_k\left(\sum_{i=0}^{\infty} c_{i+1}^{(k)}\,F^{(i)}(0)/f^{(i)}(0)\right)\Big/\varphi^{(1)}(a_k)\;\theta(a_k)\qquad (k=1,2,\cdots)$$

where $\varphi(x)$ and $\theta(x)$ are suitable entire functions; $\varphi(x)$, with the a_k for simple and the only zeros, and $\theta(x)$, such that (7) converges.

$$\varphi(z)\,\theta(z)/(z-a_k) = \sum_{n=0}^{\infty} c_n^{(k)}\,z^n.$$

The derivatives of any representation (17), defined as above, satisfy the inequalities

$$(19)\qquad |S^{(m)}(x)| < G_m < h\,h'\sum_k |\,a_k^{m+1}/\varphi^{(1)}(a_k)\,\theta(a_k)| \qquad (m=0,1,2,\cdots)$$

for all real values of x.

The x_k of Theorem II can also be expressed in the form of a contour integral

$$(20)\qquad x_k = \frac{a_k}{2\pi i}\int_C \frac{\varphi(z)\,\theta(z)\,\psi(z)\,dz}{(z-a_k)\,\varphi^{(1)}(a_k)\,\theta(a_k)},$$

where $\psi(z)=\sum_{i=0}^{\infty} F_i/z^{i+2}$ $(F^{(i)}(0)/f^{(i)}(0)=F_i)$ and C is a circle with the origin for the center and a radius greater than the radius of convergence for $\psi(z)$. This can be verified by noting that in

$$\frac{\varphi(z)\,\theta(z)\,\psi(z)}{z-a_k} = (c_0^{(k)}+c_1^{(k)}z+\cdots)\cdot\left(\frac{F_0}{z^2}+\frac{F_1}{z^3}+\cdots\right)$$

the coefficient of $1/z$ is $[c_1^{(k)}F_0+c_2^{(k)}F_1+\cdots+c_{n+1}^{(k)}F_n+\cdots]$.

When $f(x)$ is even, Theorem II continues to hold. However a representation, in general different from that resulting from Theorem II, can be derived for this case as follows. We let the $F^{(2i)}(0)$ be an assigned set of numbers such that $\sum_i i^2\,|F^{(2i)}(0)|$ converges. Assume the b_k real; $|b_{k+1}|>|b_k|$, $|b_1|\geqq 1$, $|b_k|\to\infty$. The $a_k=b_k^2$ satisfy the same conditions. The equations

$$(21)\qquad \sum_{k=1}^{\infty} a_k^i\,x_k = F_i\quad (i=0,1,2,\cdots),\qquad F_i = F^{(2i)}(0)/f^{(2i)}(0),$$

are the same as (6) and are satisfied by (20) with the F_i denoting not $F^{(i)}(0)/f^{(i)}(0)$ but $F^{(2i)}(0)/f^{(2i)}(0)$ and the a_k replaced by the b_k^2. In the series $S(x)=\sum x_k f(b_k x)$ we take for the x_k the above set of values.

The following inequalities hold for all real values of x

$$|S^{(2m)}(x)| < h \sum_k |x_k a_k^m| = G_m, \qquad a_k = b_k^2 \qquad (m = 0, 1, 2, \cdots),$$

where G_m has the same meaning as in (12). Hence by (19)

$$|S^{(2m)}(x)| < h_1 h' \sum_k |a_k^{m+1} / \varphi^{(1)}(a_k)\, \theta(a_k)| = h_2 S_m. \tag{22}$$

On the other hand since

$$S^{(2i)}(0) = f^{(2i)}(0) \sum_k x_k b_k^{2i} = f^{(2i)}(0) \sum_k x_k a_k^i \qquad (i = 0, 1, 2, \cdots),$$

it follows from (21) that $S^{(2i)}(0) = F^{(2i)}(0)$. We state the following theorem.

THEOREM III. *The series*

$$S(x) = \sum_{k=1}^{\infty} x_k f(b_k x), \ |b_{k+1}| > |b_k|, \ |b_1| \geqq 1, \ |b_k| \to \infty, \ (b_k \text{ real}), \tag{23}$$

where $f(x)$ is an even function of class A, is indefinitely differentiable term by term for all real values of x, and represents a function with assigned initial values $F^{(2n)}(0)$ $(n = 0, 1, 2, \cdots)$ at $x = 0$, provided the following conditions hold. The series $\sum_n n^2 \, |F^{(2n)}(0)/f^{(2n)}(0)|$ converges. The x_k are defined by (20), *with the a_k replaced by the b_k^2 and the F_i denoting $F^{(2i)}(0)/f^{(2i)}(0)$. The derivatives of the function represented by* (23) *satisfy for all real values of x the inequalities $|S^{(2m)}(x)| < h_2 S_m$ (S_m defined by* (22)*).*

Remark. The validity of the results of this section is asserted only in those instances when the expressions (8) and (9), as given by Borel, actually furnish a solution of the equations (5).

3. Representations, for the case $a_i = i$, in connection with certain Fourier expansions. We shall prove the following theorem.

THEOREM IV. *Let $p(x)$ be an even function of class A; let the $F^{(2n)}(0)$ $(n = 0, 1, 2, \cdots)$ be an assigned set of constants such that*

$$\psi(x) = \sum_i \frac{(-1)^i F^{(2i)}(0)\, x^{2i}}{p^{(2i)}(0) \cdot (2\,i)!}$$

and all the derived series of $\psi(x)$ converge on the closed interval $(0, \pi)$. Then the series

$$b_0\, p(0)/2 + \sum_{k=1}^{\infty} b_k\, p(kx) \tag{24}$$

is indefinitely differentiable term by term for all real values of x and represents a function $S(x)$ for which $S^{(2n)}(0) = F^{(2n)}(0)$ $(n = 0, 1, 2, \cdots)$, provided

the b_k are the Fourier constants of the cosine expansion of a function $g(x)$, $g(x) = b_0/2 + \sum_{n=1}^{\infty} b_n \cos nx$, where

$$g(x) = \psi(x) + e^{-1/x^2} \cdot \theta(x). \tag{25}$$

The power series $\theta(x)$ is determined so that the series and all the derived series converge on the closed interval $(0, \pi)$ and so that $g^{(2i+1)}(\pi) = 0$ $(i = 0, 1, \cdots)$. A power series $\theta(x)$, satisfying the above conditions, exists and may be determined by Borel's method.

Proof. Letting $e^{-1/x^2} \cdot \theta(x) = \varphi(x)$, the conditions $g^{(2i+1)}(\pi) = 0$ $(i = 0, 1, \cdots)$ may be written in the form

$$\varphi^{(2i+1)}(\pi) = -\psi^{(2i+1)}(\pi) \qquad (i = 0, 1, 2, \cdots), \tag{26}$$

where the second members exist and can be found in succession in terms of the $F^{(2n)}(0)$ and the $p^{(2n)}(0)$. From (26) it results that

$$\theta^{(i)}(\pi) = d_i \qquad (i = 0, 1, 2, \cdots), \tag{27}$$

where the d_i are constants which may be determined in succession for $i = 0, 1, 2, \cdots$ from (26). (The d_{2i} $(i = 0, 1, \cdots)$ may be taken arbitrarily.) The problem of finding a power series $\theta(x) = \sum_{n=0}^{\infty} \theta_n x^n$ which together with all its derivatives converges on the closed interval $(0, \pi)$ and is such that $\theta^{(i)}(\pi) = G_i$ (where the G_i are any constants whatever) is referred to by F. Riesz[9] and has been solved by Borel[10]. We shall suppose now $\theta(x)$ has been determined so that (27) is satisfied. The Fourier expansion of $g(x)$,

$$g(x) = \psi(x) + e^{-1/x^2} \cdot \theta(x) = b_0/2 + \sum_n b_n \cos nx, \tag{28}$$

is indefinitely differentiable term by term. The various series derived by differentiating (28) term by term any number of times are the Fourier expansions of the corresponding derivatives of $g(x)$; they are correspondingly equal to these derivatives on the closed interval $(0, \pi)$[11]. In particular, for $x = 0$ we have

$$g(0) = b_0/2 + \sum_n b_n = \psi(0) = F(0)/p(0),$$

$$g^{(2i)}(0) = (-1)^i \sum_n b_n \cdot n^{2i} = \psi^{(2i)}(0) = (-1)^i F^{(2i)}(0)/p^{(2i)}(0) \tag{29}$$
$$(i = 1, 2, \cdots).$$

---

[9] (R; 19).

[10] (B_1; 38); also see É. Borel, Leçons sur les fonctions de variable réelles, pp. 70–73. Paris, 1928. This monograph will be referred to as (B_3).

[11] (B_1; 45).

To find a law of decrease for the b_n the method used by de la Vallée Poussin for the case of a Fourier development of a periodic function[12] can be employed. Noting that

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\cdot \cos nx \cdot dx \qquad (n = 0, 1, \cdots);$$

performing the integration by parts p times and utilizing the fact that $g^{(2i+1)}(\pi) = g^{(2i+1)}(0) = 0$, we have

$$b_n = \frac{2(-1)^p}{\pi n^p}\int_0^{\pi} g^{(p)}(x)\cdot q_p(nx)\cdot dx,$$

where $q_p(x)$ is $\cos x$ if p is even and $\sin x$ if p is odd. Hence

$$(30) \qquad |b_n| < \frac{2M_p}{n^p},\ |g^{(p)}(x)| < M_p \qquad (0 \leqq x \leqq \pi),$$

where p is any positive integer.

Letting $S_i = \sum_n b_n n^i$ and using (30) with $p = i+2$,

$$|S_i| \leqq \sum_n |b_n| n^i < 2\sum_n M_{i+2}\cdot n^i/n^{i+2},$$

so that

$$(31) \qquad \sum_n |b_n| n^i < h_1 M_{i+2}\ (i = 0, 1, \cdots), \quad h_1 = 2\sum_n 1/n^2.$$

With the b_n defined as above, consider the series

$$S(x) = b_0 p(0)/2 + \sum_{n=1}^{\infty} b_n p(nx).$$

It is indefinitely differentiable for all real values of x, since by (31)

$$|S^{(i)}(x)| \leqq \sum_n |b_n n^i p^{(i)}(nx)| < h\sum_n |b_n| n^i < h h_1 M_{i+2}.$$

In particular, for $x = 0$

$$(32) \quad S(0) = p(0)\cdot\left(\frac{b_0}{2}+\sum_n b_n\right), \quad S^{(2i)}(0) = p^{(2i)}(0)\cdot\sum_n b_n n^{2i} \ (i = 1, 2, \cdots).$$

By (29)

$$S(0) = p(0)\,\psi(0) = F(0),$$
$$S^{(2i)}(0) = p^{(2i)}(0)\,(-1)^i\,\psi^{(2i)}(0) = F^{(2i)}(0) \qquad (i = 1, 2, \cdots).$$

This completes the proof of the theorem.

Theorem IV may be generalized. With $q(x)$ an odd and $p(x)$ an even function, both of class A, and the a_n and the b_n suitable constants, a series

[12] (V; 124).

of the form $b_0 p(0)/2 + \sum_n [a_n q(nx) + b_n p(nx)]$ will represent a function with assigned initial values at $x = 0$. If for functions of class A in any of the representations of this kind trigonometric functions are substituted we have a problem solved by Borel.[13]

We shall now seek to determine the b_n and a_n in

$$b_0 p(0)/2 + \sum_n [a_n q(nx) + b_n p(nx)]$$

so that this series will represent an indefinitely differentiable function with zero initial values at $x = 0$. The equations

$$(33) \quad \sum_{n=1}^{\infty} n^{2i+1} a_n = 0, \quad \sum_{n=1}^{\infty} n^{2i} b_n = 0, \quad \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n = 0 \quad (i = 1, 2, \cdots)$$

are known to be satisfied by

$$(34) \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} L(x)\cdot\sin(nx)\cdot dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} L(x)\cdot\cos(nx)\cdot dx$$
$$(n = 0, 1, \cdots),$$

where

$$L(x) = e^{-1/x^2}\varphi(x) + e^{-1/x^2}\psi(x)$$

with $\varphi(x)$, $\psi(x)$ power series determined so that $L^{(i)}(\pi) = L^{(i)}(-\pi)$ $(i = 0, 1, \cdots)$.[14] The $|a_n|$ and $|b_n|$ satisfy the inequalities (30) with $M_p > |L^{(p)}(x)|$, $(-\pi \leqq x \leqq \pi)$. By (31)

$$(35) \qquad \sum_n |a_n| n^i \text{ and } \sum_n |b_n| n^i < h^{(1)} M_{i+2} \qquad (i = 0, 1, \cdots)$$

with $h^{(1)}$ a suitable constant independent of i and n. Hence

$$S(x) = \frac{b_0 p(0)}{2} + \sum_n [a_n q(nx) + b_n p(nx)]$$

is differentiable term by term any number of times; in fact, for all real values of x

$$(35a) \quad \begin{aligned} |S^{(i)}(x)| &\leqq \sum_n (|a_n n^i q^{(i)}(nx)| + |b_n n^i p^{(i)}(nx)|) \\ &< h\cdot\sum_n (|a_n| + |b_n|)\cdot n^i < 2hh' M_{i+2}. \end{aligned}$$

For $x = 0$, $S(0) = p(0)\cdot(b_0/2 + \sum_n b_n)$ and

$$S^{(i)}(0) = q^{(i)}(0)\sum_n n^i a_n + p^{(i)}(0)\sum_n n^i b_n \qquad (i = 1, 2, \cdots).$$

[13] (B_1), (B_3; 68–74).

[14] (B_1; 45).

Since $p^{(2i+1)}(0) = q^{(2i)}(0) = 0$, from (33) it follows that $S^{(i)}(0) = 0$ $(i = 0, 1, 2, \cdots)$. Hence the following theorem can be stated.

THEOREM V. *Let $p(x)$ be an even and $q(x)$ an odd function, both of class A. Let $L(x) = e^{-1/x^2}\varphi(x) + e^{-1/x^2}\psi(x)$ where $\varphi(x)$, $\psi(x)$ are power series and are with all their derivatives convergent on the closed interval $(-\pi, \pi)$. They are determined by Borel's method so that $L^{(i)}(\pi) = L^{(i)}(-\pi)$ $(i = 0, 1, 2, \cdots)$. The series*

$$
(36) \qquad \begin{gathered} S(x) = \frac{b_0 p(0)}{2} + \sum_{n=1}^{\infty} [a_n q(nx) + b_n p(nx)], \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} L(x)\cdot \sin(nx)\cdot dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} L(x)\cdot \cos(nx)\cdot dx \\ (n = 0, 1, \cdots) \end{gathered}
$$

is indefinitely differentiable for all real values of x and it represents a function with zero initial values at $x = 0$. On the axis of reals

$$
(37) \quad |S^{(i)}(x)| < h_2 M_{i+2} \, (M_i > |L^{(i)}(x)| \text{ for } -\pi \leqq x \leqq \pi; \quad i = 0, 1, \cdots)
$$

with h_2 a constant independent of i and x.

The a_n and b_n of Theorem V are arbitrary to the extent that one of the two power series involved in the expression for $L(x)$ can be taken arbitrary, provided this series and its derivatives converge on the closed interval $(-\pi, \pi)$. A greater multiplicity of expressions for the a_n and the b_n may be obtained by using a result of Poincaré which can be stated as follows. If the equations

$$
(38) \quad \sum_{k=1}^{\infty} c_k^i x_k = 0 \quad (i = 0, 1, 2, \cdots), \qquad |c_{k+1}| > |c_k|, \quad |c_k| \to \infty
$$

are satisfied for a set of numbers $x_k = y_k \, (k = 1, 2, \cdots)$ such that the series $\sum_k |c_k^i y_k|$ converges $(i = 0, 1, \cdots)$, then the $x_k = y_k \cdot r(c_k)$ also constitute a solution provided the following holds. The entire function $r(x) = \sum_i r_i x^i$ is such that the series $\sum_{i=1}^{\infty} |r_i| \cdot s_i$, where $s_i = \sum_k |c_k^i y_k|$, converges.[15]

With $c_k = k^2$ the equations (38) are satisfied for $x_k = y_k = b_k$, where the b_k are defined by (34). Moreover, with $x_k = b_k$ and $c_k = k^2$ the left members of (38) are absolutely convergent, and by (35)

$$
(39) \qquad s_i = \sum_k |c_k^i y_k| = \sum_k |k^{2i} b_k| < h^{(1)} M_{2i+2} \qquad (i = 0, 1, \cdots).
$$

[15] H. Poincaré, Sur les déterminants d'ordre infini, Bull. Soc. math. de France, t XIV, 1886, p. 77–90. This paper will be referred to as (P). Also see (R; 17–18).

Thus with $r(x) = \sum_i r_i x^i$, an entire function such that $\sum_i |r_i| M_{2i+2}$ converges, convergence of $\sum_i |r_i| s_i$ will be secured. With $r(x)$ so defined, the equations $\sum_n n^{2i} x_n = 0$ are satisfied for $x_n = b_n \cdot r(n^2)$ $(n = 1, 2, \cdots)$. The $|b_n|$ satisfy the inequalities (30) (with $M_p > |L^{(p)}(x)|$ $(-\pi \leqq x \leqq \pi)$). Utilizing these inequalities with a particular value of p, $p = i + 2$, we have

$$\text{(39a)} \quad \begin{gathered} \sum_n |n^i x_n| = \sum_n |n^i \cdot b_n \cdot r(n^2)| < 2 M_{i+2} \sum_n \frac{n^i |r(n^2)|}{n^{i+2}} = h^{(2)} M_{i+2} \\ \left(h^{(2)} = 2 \sum_n \left| \frac{r(n^2)}{n^2} \right|, \quad i = 0, 1, \cdots \right) \end{gathered}$$

provided $\sum_n |r(n^2)|/n^2$ converges.

Consider now the first set of equations (33). This set may be written in the form

$$\text{(39b)} \qquad \sum_{n=1}^{\infty} n^{2i} z_n = 0, \qquad z_n = n x'_n \qquad (i = 1, 2, \cdots).$$

Since the $x'_n = a_n$ (a_n defined by (34)) satisfy the inequalities $|a_n| < 2 M_p / n^p$, it follows that for the $z_n = n a_n$ we have

$$\text{(40)} \qquad |z_n| < 2 M_p / n^{p-1}, \; p \text{ any positive integer } (n = 1, 2, \cdots).$$

The equations (38) with $c_k = k^2$ are satisfied for $x_k = z_k = k a_k$ $(k = 1, 2, \cdots)$, where the a_k are defined by (34). These specifications along with (40), where $p = 2i + 3$, render the inequalities $|z_k| < 2 M_{2i+3}/k^{2i+2}$ $(k = 1, 2, \cdots)$. Hence

$$\text{(41)} \qquad s'_i = \sum_k |c_k^i y_k| = \sum_k |k^{2i} z_k| < h^{(1)} M_{2i+3} \qquad (i = 0, 1, \cdots).$$

Consequently with $r_1(x) = \sum_i r'_i x^i$, an entire function such that $\sum_i |r'_i| M_{2i+3}$ converges, the convergence of $\sum_i |r'_i| s'_i$ will be secured. By Poincaré's result, the equations $\sum_n n^{2i} z_n \equiv \sum_n n^{2i+1} x'_n = 0$ $(i = 1, 2, \cdots)$ will then be satisfied for $z_n = n a_n r_1(n^2)$ $(n = 1, 2, \cdots)$. That is, the first set of equations (33), if written in the form $\sum_n n^{2i+1} x'_n = 0$ $(i = 1, 2, \cdots)$, will be satisfied for $x'_n = a_n r_1(n^2)$ $(n = 1, 2, \cdots)$, the a_n being defined by (34). Using the inequalities $|a_n| < 2 M_p / n^p$, $p = i + 2$ $(n = 1, 2, \cdots)$, we shall have

$$\text{(41a)} \quad \begin{aligned} \sum_n |n^i x'_n| &= \sum_n |n^i a_n r_1(n^2)| < 2 M_{i+2} \sum_n \frac{n^i |r_1(n^2)|}{n^{i+2}} \\ &= h_1^{(2)} M_{i+2} \quad \left(h_1^{(2)} = 2 \sum_n |r_1(n^2)|/n^2; \quad i = 0, 1, \cdots \right), \end{aligned}$$

provided $\sum_n |r_1(n^2)|/n^2$ converges.

Let us now consider the series

$$S(x) = \frac{x_0 p(0)}{2} + \sum_{n=1}^{\infty} [x'_n q(nx) + x_n p(nx)], \tag{42}$$
$$x_n = b_n r(n^2), \qquad x'_n = a_n r_1(n^2).$$

It is observed that for all real values of x

$$|S^{(i)}(x)| < h \sum_{n=1}^{\infty} (|x'_n n^i| + |x_n n^i|) \qquad (i = 0, 1, \cdots), \tag{43}$$

where h is a suitable constant independent of i and x. By (39a) and (41a) we have $|S^{(i)}(x)| < 2hh^{(3)} M_{i+2}$ where $h^{(3)}$ is the greater of the two constants $h_1^{(2)}$ and $h^{(2)}$. These inequalities show that (42) is indefinitely differentiable term by term for all real values of x. For $x = 0$

$$S(0) = p(0)(x_0/2 + \sum x_n), \; S^{(2i)}(0) = p^{(2i)}(0) \sum_n x_n n^{2i} \quad (i = 1, 2, \cdots); \tag{44}$$
$$S^{(2i+1)}(0) = q^{(2i+1)}(0) \sum_n x'_n n^{2i+1} \qquad (i = 0, 1, \cdots).$$

Hence $S(0) = S^{(i)}(0) = 0$ $(i = 1, 2, \cdots)$, and the following theorem can be stated.

THEOREM VI. *Let $p(x)$, $q(x)$, $L(x)$ and the constants M_i be defined as in Theorem V. Let $r(x) = \sum_i r_i x^i$, $r_1(x) = \sum_i r'_i x^i$ be entire functions such that the series*

$$\sum |r_i| M_{2i+2}, \quad \sum |r'_i| M_{2i+3}, \quad \sum |r(i^2)|/i^2, \quad \sum |r_1(i^2)|/i^2$$

converge. Then the series

$$S(x) = \frac{x_0 p(0)}{2} + \sum_{n=1}^{\infty} [x'_n q(nx) + x_n p(nx)], \tag{45}$$
$$x'_n = r_1(n^2) \frac{1}{\pi} \int_{-\pi}^{\pi} L(x) \sin nx \, dx, \quad x_n = r(n^2) \frac{1}{\pi} \int_{-\pi}^{\pi} L(x) \cos nx \, dx,$$

is indefinitely differentiable for all real values of x and represents a function with zero initial values at $x = 0$. On the axis of reals

$$|S^{(i)}(x)| < h_3 M_{i+2} \qquad (i = 0, 1, \cdots) \tag{46}$$

where h_3 is a constant independent of i and x.

4. On some series of the form $\sum_{-\infty}^{+\infty} x_k f(a_k x)$ with $|a_k| \to \infty$, $|a_{-k}| \to 0$, as $k \to \infty$. Appell has solved the set of equations

(47) $$c(-1)^p + \sum_{n=-\infty}^{\infty} c_n \lambda^{np} = 0 \qquad (p = 0, 1, 2, \cdots), \quad \lambda > 1,$$

in connection with the study of elliptic functions. Poincaré gave a rigorous proof that the numbers

(47a) $$c_n = (-1)^n \Big/ \left[\lambda^{(n^2+n)/2} \left(1 + \frac{1}{\lambda^n}\right) \prod_{i=1}^{\infty} \left(1 - \frac{1}{\lambda^i}\right)^2 \right] \quad (n = 0, \pm 1, \pm 2, \cdots)$$
$$c = -1/2 \prod_{i=1}^{\infty} \left(1 + \frac{1}{\lambda^i}\right)^2$$ [16]

constitute a solution of (47).

From (47 a) it follows that

(47 b) $$|c_n| < h_1 \lambda^{(-n^2-|n|)/2} \qquad (n = 0, \pm 1, \pm 2, \cdots) \ |c| < h_1,$$

where h_1 is a suitable constant independent of n. Now the series

(48) $$S(x) = c f(-x) + \sum_{-\infty}^{\infty} c_n f(\lambda^n x)$$

is indefinitely differentiable for all real values of x, since for $(p = 0, 1, \cdots)$ and all real values of x

(48 a) $$|S^{(p)}(x)| \leqq |c(-1)^p f^{(p)}(-x)| + \sum_{n=-\infty}^{\infty} |c_n \lambda^{np} f^{(p)}(\lambda^n x)|$$
$$< h\left[|c| + \sum_{n=-\infty}^{\infty} |c_n \lambda^{np}|\right] < h h_1 \left[1 + \sum_{n=-\infty}^{\infty} \lambda^{(-n^2-|n|+2np)/2}\right] = S_p,$$

where the series in the last member converges $(p = 0, 1, \cdots)$. For $x = 0$

$$S^{(p)}(0) = c(-1)^p + \sum_{n=-\infty}^{\infty} c_n \lambda^{np} \qquad (p = 0, 1, 2, \cdots).$$

By (47) it follows that the $S^{(p)}(0) = 0$. Thus the following result can be stated.

Let $f(x)$ be of class A and $\lambda > 1$. The series

$$S(x) = c f(-x) + \sum_{-\infty}^{\infty} c_n f(\lambda^n x),$$

where

$$c = -1 \Big/ \left[2 \prod_{i=1}^{\infty} \left(1 + \frac{1}{\lambda^i}\right)^2\right],$$
$$c_n = (-1)^n \Big/ \left[\lambda^{(n^2+n)/2} \left(1 + \frac{1}{\lambda^n}\right) \prod_{i=1}^{\infty} \left(1 - \frac{1}{\lambda^i}\right)\right],$$

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[16] H. Poincaré, Remarques sur l'emploi de la méthode précédente, Bull. Soc. math. de France, t. XIII, 1885, p. 26. He uses $q = 1/\sqrt{\lambda}$.

is indefinitely differentiable for all real values of x; it represents a function with zero initial values at $x = 0$. On the axis of reals $|S^{(p)}(x)| < S_p$ $(p = 0, 1, \cdots)$, where the S_p are defined by (48 a).

We shall now consider a set of equations of the form

$$\sum_{n=-\infty}^{\infty} b_n \lambda^{np} = 0 \qquad (p = 0, 1, 2, \cdots) \tag{48b}$$

differing from the set (47) in having the term $c(-1)^p$ absent. According to Poincaré the equations $\sum_{-\infty}^{\infty} x_n a_n^p = 0$ $(p = 0, 1, \cdots)$ in which $|a_{n+1}| > |a_n|$, $\lim_{n\to\infty} |a_n| = \infty$ and $\lim_{n\to\infty} |a_{-n}| = 0$, are satisfied by $x_n = A_n (n = 0, \pm 1, \pm 2, \cdots)$, where A_n is the residue at $x = a_n$ of a certain function $1/F(x)$. The poles of $1/F(x)$, each of them simple, are the points $x = a_n$ (the origin being consequently an essentially singular point). Moreover, $F(x)$ should be such that

$$\lim_{n\to\pm\infty} \int_{C_n} [x^p/F(x)]\, dx = 0 \qquad (p = 0, 1, 2, \cdots), \tag{49}$$

where C_n is a circle with center at the origin and radius r_n, $|a_n| < r_n < |a_{n+1}|$. The $a_n = A_n$ make the left members of the equations convergent (not necessarily absolutely).[17] Having this result in view we form

$$\begin{aligned} F(x) &= (x-1)\prod_{n=1}^{\infty}\left(1-\frac{1}{x\lambda^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{\lambda^n}\right) \\ &= (x-1)\prod_{n=1}^{\infty}\left[1-\frac{1}{\lambda^n}\left(x+\frac{1}{x}\right)+\frac{1}{\lambda^{2n}}\right] \\ &= \lambda_1 \sqrt{x}\, \theta_1(\log x) = \lambda_1 \sqrt{x}\, \varphi(x), \end{aligned} \tag{50}$$

where $\theta_1(x)$ is a function denoted thus by Briot and Bouquet.[18]
Let

$$\begin{aligned} J'_{m,n} &= \int_{C_m} \frac{x^n\, dx}{\sqrt{x}\, \varphi(x)}, \qquad K'_{m,n} = \int_{C_m} \left| \frac{x^n\, dx}{\sqrt{x}\, \varphi(x)} \right|, \\ \Lambda_{m,n} &= \int_{C_m} \left| \frac{x^n\, dx}{\varphi(x)} \right|, \end{aligned} \tag{51}$$

where C_m is a circle of radius r_m, $r_{m+1}/\lambda = r_m$ $(m = 0, \pm 1, \pm 2, \cdots)$.[19] Then, since on C_m we have $|\sqrt{x}| = \sqrt{r_m}$,

[17] (P).

[18] See (P; 24).

[19] The same assumption is made concerning the r_m in (P; 24).

(51a) $|J'_{m,n}| < K'_{m,n} = \frac{1}{\sqrt{r_m}} \Lambda_{m,n} \quad (m = 0, \pm 1, \pm 2, \cdots; \ n = 0, 1, \cdots).$

Poincaré has shown that, with the r_n defined as in (51),

$$\Lambda_{m+1,n}/\Lambda_{m,n} = \delta \lambda^{2n+1}/r_m,$$

where δ is a constant depending on the function φ. Hence

$$\text{(51b)} \qquad \frac{K'_{m+1,n}}{K'_{m,n}} = \frac{\Lambda_{m+1,n}/\sqrt{r_{m+1}}}{\Lambda_{m,n}/\sqrt{r_m}} = \frac{\delta \lambda^{2n+1}}{\sqrt{r_m r_{m+1}}} = h_{m,n}.$$

Now, when $m \to \infty$, $r_m \to \infty$ so that $\lim_{m\to\infty} h_{m,n} = 0$ $(n = 0, 1, \cdots)$. Hence $\lim_{m\to\infty} K'_{m,n} = 0$ $(n = 0, 1, \cdots)$. On the other hand, when $m \to -\infty$, $r_m \to 0$ and $h_{m,n} \to \infty$ $(n = 0, 1, \cdots)$. Consequently $\lim_{m\to -\infty} K'_{m,n} = 0$ $(n = 0, 1, \cdots)$. Therefore, from (51a), it follows that $\lim_{m\to\infty} |J'_{m,n}| = 0$, $\lim_{m\to -\infty} |J'_{m,n}| = 0$ $(n = 0, 1, \cdots)$. The conditions (49) are thus satisfied when $F(x)$ is defined by (50). Hence the $b_n = A_n$ $(n = 0, \pm 1, \pm 2, \cdots)$, where A_n is the residue at $x = \lambda^n$ of the function $1/F(x)$ ($F(x)$ defined by (50)), constitute a solution of the equations (48c).

The constants c_n of Theorem VII are the residues of a function

$$\text{(52)} \qquad 1/\lambda_1 \cdot (x+1)\sqrt{x}\,\varphi(x) \equiv 1/(x+1) F(x) \qquad (F(x) \text{ given by (50)})$$

at the points $x = \lambda^n$.[20] The residues A_n of $1/F(x)$ can be determined in terms of the c_n as follows. In the neighborhood of $x = \lambda^n$

$$1/F(x) = \frac{A_n}{x - \lambda^n} + P_n(x); \qquad 1/(x+1)F(x) = \frac{c_n}{x - \lambda^n} + Q_n(x),$$

where $P_n(x)$ and $Q_n(x)$ are regular near $x = \lambda^n$. Hence

$$(x - \lambda^n)/F(x) = A_n + (x - \lambda^n) P_n(x) = c_n(x+1) + (x+1)(x - \lambda^n) Q_n(x),$$

so that (using (47b))

$$\text{(53)} \qquad A_n = c_n(\lambda^n + 1),$$

$$|A_n| < g\lambda^n |c_n| < g h_1 \lambda^{(-n^2 - |n| + 2n)/2} \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

The series

$$\text{(54)} \qquad S(x) = \sum_{-\infty}^{\infty} A_n f(\lambda^n x)$$

[20] (P).

is indefinitely differentiable for all real values of x, since for $p = 0, 1, \cdots$ and for all real values of x

$$(54\text{a}) \qquad \begin{aligned} |S^{(p)}(x)| &\leqq \sum_{n=-\infty}^{\infty} |A_n \lambda^{np} f^{(p)}(\lambda^n x)| < h \sum_{n=-\infty}^{\infty} |A_n \lambda^{np}| \\ &< h \sum_{n=-\infty}^{\infty} \lambda^{(-n^2-|n|+2n+2np)/2} = S_p', \end{aligned}$$

where the series in the last member converges ($p = 0, 1, \cdots$). For $x = 0$

$$S^{(p)}(0) = f^{(p)}(0) \sum_{n=-\infty}^{\infty} A_n \lambda^{np} \qquad (p = 0, 1, \cdots)$$

so that, by (48b), $S^{(p)}(0) = 0$. The following result can now be stated.

Let $f(x)$ be of class A *and $\lambda > 1$. The series*

$$S(x) = \sum_{-\infty}^{\infty} A_n f(\lambda^n x),$$

where

$$A_n = (-1)^n \Big/ \left[\lambda^{(n^2-n)/2} \prod_{i=1}^{\infty} \left(1 - \frac{1}{\lambda^i}\right)\right] \quad (n = 0, \pm 1, \pm 2, \cdots),$$

is indefinitely differentiable for all real values of x and represents a function with zero initial values at $x = 0$. On the axis of reals $|S^{(p)}(x)| < S_p'$ ($p = 0, 1, \cdots$), where the S_p' are defined by (54a).

Borel Monogenic Functions.

1. **Representations of Borel's monogenic functions.** Borel has generalized the notion of monogenic functions of a complex variable z. His monogenic functions have, in general, divergent power series expansions. At the same time they have the property of quasi-analyticity;[21] that is, the initial values of the function, at any point of the domain of definition, determine the function uniquely throughout the domain. It is known that quasi-analytic functions, representable by series of the form $\sum_n c_n/(z - a_n)$ where the a_n are everywhere dense in some portion of the plane, are a particular case of the generalized monogenic functions.[22] The general expression for these functions in terms of rational fractions, as far as I know, has not been derived. With the latter purpose in view it will be recalled that the domain of definition C (Cauchy domain) is defined as follows. Let the a_n form an enumerable everywhere dense set of points

[21] (B_2; 125, 144).

[22] See an example in (B_2; 144).

interior to a closed region R. About each point a_n, as center, describe a circle $S_n'^{(h)}$ of radius $r_n/2^h$; the numbers r_n being such that

$$(1) \qquad \sum_n r_n \text{ converges}, \quad \sum_{n+1}^{\infty} r_h < \frac{r_n}{4}, \quad \lim_{n\to\infty} n/\log\log\log\left(\frac{1}{r_n}\right) = 0.$$

There exist numbers $r_n^{(h)}$ such that

$$(1a) \qquad r_n/2^{h+1} < r_n^{(h)} < r_n/2^h \qquad (r, h = 1, 2, \cdots)$$

and such that, if $S_n^{(h)}$ denotes a circle of radius $r_n^{(h)}$, its center at a_n, then the circles $S_n^{(h)}$ $(n = 1, 2, \cdots)$ do not cut each other. The perfect set $C^{(h)}$ is defined as the set consisting of the points of R, the interiors of the circles $S_n^{(h)}$ $(n = 1, 2, \cdots)$ having been excluded. The point set C consists of points P, each one of which belongs to $C^{(h)}$ for all $h > h_0$, where h_0 depends on P. We have $C = R - G$, where G is a certain point set of measure zero and contains the points a_n; furthermore, G is of the power of the continuum. "Reduced domains" $\Gamma^{(p)}$ and Γ are defined by means of a set of numbers ϱ_n,

$$(2) \qquad 1/\varrho_n^2 < \log\log(1/r_n), \qquad \lim_i i^\alpha \varrho_i = 0$$

$$(\alpha \text{ any fixed number}; \; n = 1, 2, \cdots),$$

in the same way as the $C^{(p)}$ and C were defined with the aid of the r_n. We note that $\Gamma^{(p)}$ is contained in $C^{(p)}$ $(p = 1, 2, \cdots)$ and Γ in C; C contains also some points not belonging to Γ.

A function $f(z)$ is monogenic in a domain C if:

1° *It is continuous in C.*

2° *It possesses at every point P of C a unique and continuous derivative.*

Monogenuity may be similarly defined in $C^{(p)}$. In every case it is assumed that the increment ratio approaches the derivative uniformly.

With this definition Borel gives the following representation

$$(3) \qquad f(x) = \frac{1}{2\pi i}\int_K \frac{f(z)\,dz}{z-x} - \sum_n{}' \frac{1}{2\pi i}\int_{S_n^{(p)}} \frac{f(z)\,dz}{z-x},$$

where x is any point of Γ and p is sufficiently great so that x is also an interior point of a perfect subset $\Gamma^{(p)}$ of Γ. K is a simple closed curve all of whose points are interior to $C^{(p)}$; moreover, K contains x in its interior. The summation refers to all the circles $S_n^{(p)}$ interior to K and exterior to each other; and the integrals are taken in the direct sense.

When we say that K is interior to $C^{(p)}$ it is meant that K is interior to R and that no point of K belongs to the frontier of $C^{(p)}$. The frontier

of $C^{(p)}$ consists of the contour of R and of the circumferences of all those circles $S_k^{(p)}$ which are exterior to each other and are interior to the region R. In the sequel R, in general, will not be mentioned. It is also to be noted that the $L_n^{(p)}$ are the circles used in defining $\Gamma^{(p)}$. The center of $L_n^{(p)}$ is at a_n and its radius is $\varrho_n^{(p)}$,

$$\varrho_n/2^{p+1} < \varrho_n^{(p)} < \varrho_n/2^p.$$

We consider

$$(4) \qquad {}_p\varphi_n(x) = \frac{-1}{2\pi i}\int_{S_n^{(p)}} \frac{f(z)\,dz}{z-x}.$$

About the point a_n, as center, there are two concentric circles $S_n^{(p)}$, $L_n^{(p)}$ with radii $r_n^{(p)}$ and $\varrho_n^{(p)}$, respectively ($r_n^{(p)} < \varrho_n^{(p)}$). The point z is on $S_n^{(p)}$, and x is exterior to $L_n^{(p)}$. Hence

$$|z-x| > \varrho_n^{(p)} - r_n^{(p)}.$$

The function ${}_p\varphi_n(x)$ represents two different analytic functions exterior and interior to $S_n^{(p)}$. The point x is in $\Gamma^{(p)}$, that is, exterior to $S_n^{(p)}$. Accordingly, consider the analytic function, represented by (4) exterior to $S_n^{(p)}$. Expanding $1/(z-x)$ in powers of $1/(x-a_n)$

$$(4a) \quad {}_p\varphi_n(x) = \sum_{m=1}^{\infty} {}_nb_m^{(p)}/(x-a_n)^m, \qquad {}_nb_m^{(p)} = \frac{1}{2\pi i}\int_{S_n^{(p)}} f(z)\,(z-a_n)^{m-1}\,dz.$$

In (4a) $|z-a_n| = r_n^{(p)}$. Letting $|f(z)| < M$ for all z on C,

$$(4b) \qquad |{}_nb_m^{(p)}| < M(r_n^{(p)})^m < M(r_n/2^p)^m.$$

With the summation, with respect to n extended as in (3), the double sum

$$(5) \qquad \sum_n{}' \, {}_p\varphi_n(x) = \sum_n{}' \sum_{m=1}^{\infty} {}_nb_m^{(p)}/(x-a_n)^m$$

is seen to be absolutely convergent for all x in $\Gamma^{(p)}$. This can be shown by noting that, for x in $\Gamma^{(p)}$, $|x-a_n| \geqq \varrho_n^{(p)} > \varrho_n/2^{p+1}$ so that $|1/(x-a_n)|^m < (2^{p+1}/\varrho_n)^m$. This latter inequality, together with (4b), gives

$$(5a) \qquad \begin{aligned} \sum_n{}' \sum_{m=1}^{\infty} |{}_nb_m^{(p)}/(x-a_n)^m| &< M \sum_n{}' \sum_{m=1}^{\infty} \left(\frac{2r_n}{\varrho_n}\right)^m \\ &= M \sum_n{}' \frac{2r_n}{\varrho_n} \cdot \frac{1}{1-2r_n/\varrho_n} < \frac{2M}{1-\lambda} \sum_n{}' \frac{r_n}{\varrho_n}, \end{aligned}$$

where $2r_n/\varrho_n < \lambda < 1$ $(n = 1, 2, \cdots)$. The assumption $2r_n/\varrho_n < \lambda < 1$ entails no loss of generality. The series, in the last member, converges

since, by hypothesis, the $r_n \to 0$ much more rapidly than the ϱ_n. Concerning the integral around K, it can be said that it represents an analytic function of x (x interior to K). Thus with c, a suitable point interior to K,

$$(6) \quad \frac{1}{2\pi i}\int_K \frac{f(z)\,dz}{z-x} = \sum_{m=0}^{\infty} d_m^{(p)}(x-c)^m, \qquad d_m^{(p)} = \frac{1}{2\pi i}\int_K \frac{f(z)\,dz}{(z-c)^{m+1}}.$$

Hence the following theorem can be stated.

THEOREM I. *Let the r_n (as given by (1)) define, by means of the circles $S_n^{(p)}$, a sequence of domains $C^{(p)}$. These domains will be supposed to determine a Cauchy domain C in connection with a set of points $\{a_n\}$, which may be everywhere dense in the whole of a region R or in some portions of this region. Let the ϱ_n (as given by (2)) define reduced domains $\Gamma^{(p)}$ and Γ by means of the circles $L_n^{(p)}$. Let K denote a simple closed curve all of whose points are interior to $C^{(p)}$. Every Borel monogenic function $f(x)$, defined on C, is representable when x is in Γ and is interior to K by the double series*

$$(7) \qquad f(x) = \sum_{m=0}^{\infty} d_m^{(p)}(x-c)^m + \sum_{n=1}^{\infty}{}' \sum_{m=1}^{\infty} {}_nb_m^{(p)}/(x-a_n)^m$$

(c *a suitable point interior to* K).

In this representation the ${}_nb_m^{(p)}$ and $d_m^{(p)}$ are given by (4a) and (6). The integer p is any integer such that $p \geqq p_1$, where p_1 is the smallest integer such that x is interior to $\Gamma^{(p_1)}$. The series (7) is absolutely and uniformly convergent in $\Gamma^{(p)}$.

We have shown that, except for an additive analytic function, a Borel monogenic function is representable by a series of the form

$$(8) \qquad F(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{b_{n,m}}{(x-a_n)^m}.$$

Conversely, we should expect that it is possible to specify the law of decrease of the $|b_{n,m}|$ so that the series (8) represents a function Borel monogenic in some $C^{(p)}$. The a priori possibility of this has been indicated by Borel.[23]

Accordingly, we proceed to find such a law of decrease of the $|b_{n,m}|$.

By definition it is sufficient to have $F(x)$, $F^{(1)}(x)$ unique and continuous in $C^{(p)}$ and, given $\varepsilon(>0)$, have a number h so that

$$\left|\frac{F(x_1)-F(x)}{x_1-x} - F^{(1)}(x)\right| < \varepsilon,$$

[23] (B_2; 152).

whenever x_1, x are points of $C^{(p)}$ for which

$$|x_1 - x| < h_1.$$[24]

Letting

$$g_{n,m}(x_1, x) = \left| \frac{1}{(x_1 - x)} \left(\frac{1}{(x_1 - a_n)^m} - \frac{1}{(x - a_n)^m} \right) + \frac{m}{(x - a_n)^{m+1}} \right|,$$

it is seen that

$$\left| \frac{F(x_1) - F(x)}{x_1 - x} - F^{(1)}(x) \right| < \sum_{n,m} |b_{n,m}| \, g_{n,m}(x_1, x)$$

provided the second member converges.

Now

$$g_{n,m}(x_1, x) = \left| \frac{(x - a_n)\,[(x - a_n)^{m-1} + \cdots + (x_1 - a_n)^{m-1}] - m(x_1 - a_n)^m}{(x_1 - a_n)^m (x - a_n)^{m+1}} \right|$$

$$= |x - x_1| \left| \frac{e_0 (x_1 - a_n)^{m-1} + e_1 (x_1 - a_n)^{m-2}(x - x_1) + \cdots + e_{m-1}(x - x_1)^{m-1}}{(x_1 - a_n)^m (x - a_n)^{m+1}} \right|$$

where

$$\text{(8a)} \quad \begin{gathered} e_0 = C_1^m + C_1^{m-1} + \cdots + C_1^1, \quad e_1 = C_2^m + C_2^{m-1} + \cdots + C_2^2, \\ \cdots, e_{m-1} = C_m^m. \end{gathered}$$[25]

Without any loss of generality it may be assumed that

$$|x_1 - a_n| \leqq 1;$$

moreover, for x, x_1 in $C^{(p)}$,

$$|x_1 - a_n|, \; |x - a_n| \geqq r_n^{(p)}.$$

Hence, whenever x, x_1 are in $C^{(p)}$ and

$$|x - x_1| < h,$$

it follows that

$$g_{n,m}(x_1, x) < h \frac{g_1(m)}{(r_n^{(p)})^{2m+1}}$$

where

$$\text{(8b)} \qquad g_1(m) = e_0 + e_1 h_1 + \cdots + e_{m-1} h_1^{m-1} \qquad (h_1 > h).$$

Consequently

$$\left| \frac{F(x_1) - F(x)}{x_1 - x} - F^{(1)}(x) \right| < h \sum_{n,m} \frac{|b_{n,m}|\, g_1(m)}{(r_n^{(p)})^{2m+1}} = h S_1.$$

Here it will be necessarily assumed that the $|b_{n,m}|$ are such that the series for S_1 converges for some positive h_1.

[24] See (B_2; 135).

[25] The C_n^k are binomial coefficients.

With this assumption, given $\varepsilon(>0)$, a constant h $(h<\varepsilon/S_1)$ can be found so that, whenever x and x_1 are in $C^{(p)}$, the inequality $|x_1-x|<h$ will imply that

$$\left|\frac{F(x_1)-F(x)}{x_1-x}-F^{(1)}(x)\right|<\varepsilon.$$

Moreover, the convergence of the series for S_1 implies convergence of the series

$$\sum_{n,m}\frac{|b_{n,m}|}{(r_n^{(p)})^m}, \qquad \sum_{n,m}\frac{m\,|b_{n,m}|}{(r_n^{(p)})^{m+1}};$$

that is, existence and continuity in $C^{(p)}$ of $F(x)$ and of $F^{(1)}(x)$ $(F^{(1)}(x)=\sum -m\,b_{n,m}(x-a_n)^{-m-1})$ will be secured.

THEOREM II. *The function* $F(x)$ *of the form*

$$\sum_{n,m=1}^{\infty}\frac{b_{n,m}}{(x-a_n)^m} \tag{8}$$

is Borel monogenic in $C^{(p)}$ *if the series*

$$\sum_{n,m}\frac{|b_{n,m}|\,g_1(m)}{(r_n^{(p)})^{2m+1}} \tag{9}$$

converges for a positive h_1. *Here* $g_1(m)$ *is defined by means of* (8a) *and* (8b).

Suppose now that $a_n \neq 0$ $(n=1,2,\cdots)$. What are the conditions under which a given point, say $x=0$, will belong to $C^{(p)}$? This is equivalent to the requirement that $x=0$ should be on or exterior to all the $S_n^{(p)}$ which constitute the frontier of $C^{(p)}$. We see therefore that it is sufficient that, in addition to (1), the inequalities

$$(r_n^{(p)}<)\ \frac{r_n}{2^p}\leqq|a_n| \qquad (n=1,2,\cdots) \tag{9a}$$

should be satisfied.

On the other hand, a closed interval (a,b) will consist of interior points of $C^{(p)}$ under the following conditions. There are no a_n $(n=1,2,\cdots)$ on the interval $(a\leqq x\leqq b)$; in addition to (1) the inequalities

$$\begin{aligned} &(r_{n'}^{(p)}<)\ r_{n'}/2^p \leqq |\Im\, a_{n'}| && (a\leqq \Re\, a_{n'}\leqq b),\\ &(r_{n_1}^{(p)}<)\ r_{n_1}/2^p \leqq |a_{n_1}-a| && (\Re\, a_{n_1}<a),\\ &(r_{n_2}^{(p)}<)\ r_{n_2}/2^p \leqq |a_{n_2}-b| && (b<\Re\, a_{n_2}) \end{aligned} \tag{9b}$$

hold.

A similar discussion may be given with $C^{(p)}$ replaced by any reduced domain $\Gamma^{(p)}$. This eventually would amount to additional restrictions on the $r_n^{(p)}$.

Thus we see that *it is possible to specify the law of decrease of the* $|b_{n,m}|$ *so that, provided none of the* a_n *are on* $(a \leq x \leq b)$, *the series* (8) *represents a function* $F(x)$, *Borel monogenic in a region* $C^{(p)}$; *the region* $C^{(p)}$ *itself or any reduced domain containing in its interior the closed interval* (a, b). *A similar result may be stated with the interval* (a, b) *replaced by a single point.*

2. **Some properties of Borel monogenic functions.** A fundamental property of these functions is that there exists for them a generalized Cauchy integral (3). This circumstance makes the class of Borel monogenic functions especially susceptible to study. On the other hand, the problem of quasi-analytic functions of a complex variable, in its greatest possible generality, it seems, would have to be studied by other methods.[26] At the present time this more general problem will not be considered.

For brevity let

$$f(z) = \frac{1}{2\pi i}\int_K \frac{f(\zeta)\,d\zeta}{\zeta - z} - \sum_k{}' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f(\zeta)\,d\zeta}{\zeta - z} \tag{10}$$

$$\equiv B(f; K, C^{(p)}, \zeta; z).$$

Here $f(z)$ is Borel monogenic in $C^{(p)}$, K is a simple closed curve interior to $C^{(p)}$, and the summation is extended over those $S_k^{(p)}$ (forming the frontier of $C^{(p)}$) which are interior to K. The formula (10) is valid for z interior to K and interior to a reduced domain $\Gamma^{(p)}$ (the latter, as noted before, is formed by means of certain circles $L_k^{(p)}$). A fortiori (10) holds for z interior to K and in $\Gamma^{(p-1)}$.

An analogue will be derived of the theorem of Weierstrass on the analyticity of the limit of a sequence of analytic functions converging uniformly in a Weierstrassian domain. The known extensions of Weierstrass's theorem, the Stieltjes and Vitali-Porter's theorems, cannot be conveniently extended, by means of (10), to Borel monogenic functions.[27]

Let $C_1^{(p)}$ denote the region consisting of K and of the part of $C^{(p)}$ interior to K. Let Q be a simple closed curve interior to K and interior to $\Gamma^{(p-1)}$; and let $\Gamma_1^{(p-1)}$ denote the region consisting of Q and of the part of $\Gamma^{(p-1)}$ interior to Q.

The following theorem will be proved.

THEOREM III. *Let*

$$f_1(z),\ f_2(z),\ \cdots,\ f_n(z),\ \cdots \tag{11}$$

be a sequence of functions each Borel monogenic in $C_1^{(p)}$.

[26] Note, for instance, a theorem of Carleman in (C; 99, 100).

[27] Montel states, without proof, that the Vitali-Porter theorem, holds for quasi-analytic functions. See, P. Montel, Les familles normales ..., Paris, 1927, pp. 30. This monograph will be referred to as (M).

If this sequence converges uniformly on the frontier of $C_1^{(p)}$ *to* $f(z)$, *it converges uniformly in* $\Gamma_1^{(p-1)}$ *to a function* $f_0(z)$. *This function is indefinitely differentiable in* $\Gamma_1^{(p-1)}$. *Moreover, the sequences*

$$f_1^{(m)}(z),\ f_2^{(m)}(z),\ \cdots,\ f_n^{(m)}(z),\ \cdots \qquad (m = 1, 2, \cdots)$$

will converge uniformly in $\Gamma_1^{(p-1)}$ *to the* $f_0^{(m)}(z)$, *respectively.*

We have

$$f_n(z) = B(f_n;\ K,\ C^{(p)},\ \zeta;\ z)$$

for z in $\Gamma_1^{(p-1)}$. Since it is known that B is indefinitely differentiable, term by term,[28] we have

$$f_n^{(m)}(z) = B^{(m)}(f_n;\ K,\ C^{(p)},\ \zeta;\ z)$$

$$(12) \qquad = \frac{m!}{2\pi i}\int_K \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^{m+1}} - \sum_k{}' \frac{m!}{2\pi i}\int_{S_k^{(p)}} \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^{m+1}}$$
$$(m = 1, 2, \cdots).$$

Consider the function

$$f_0(z) = B(f;\ K,\ C^{(p)},\ \zeta;\ z)$$

which exists since, by hypothesis, for ζ on the frontier of $C_1^{(p)}$, the function $f(\zeta)$ exists and is necessarily bounded. Thus $|f(\zeta)| < M$; moreover, for z in $\Gamma_1^{(p-1)}$ and ζ on $S_k^{(p)}$

$$|1/(\zeta - z)| < 1/(\varrho_k^{(p)} - r_k^{(p)}), \qquad (\varrho_k^{(p-1)} > \varrho_k^{(p)}).$$

Hence, in virtue of the convergence of the series

$$(12a) \qquad \sum_k{}' \frac{r_k^{(p)}}{(\varrho_k^{(p)} - r_k^{(p)})^{m+1}} = G_m^{(p)} \qquad (m = 0, 1, 2, \cdots)$$

it follows that for z in $\Gamma_1^{(p-1)}$ the series $B(f;\ K,\ C^{(p)},\ \zeta;\ z)$ is indefinitely differentiable term by term; the series itself and all the derived series being absolutely convergent in $\Gamma_1^{(p-1)}$. Thus $f_0(z)$ is indefinitely differentiable in $\Gamma_1^{(p-1)}$.

Form

$$f_n(z) - f_0(z) = B(f_n - f;\ K,\ C^{(p)},\ \zeta;\ z),$$
$$f_n^{(m)}(z) - f_0^{(m)}(z) = B^{(m)}(f_n - f;\ K,\ C^{(p)},\ \zeta;\ z).$$

Now, by hypothesis, for ζ on the frontier of $C_1^{(p)}$

$$|f_n(\zeta) - f(\zeta)| < \epsilon \qquad (n \geqq N(\epsilon)).$$

[28] See (B_2; 140).

Hence, if $\delta(>0)$ denoted the least distance from a point on Q to a point on K and if the length of K is denoted by $2\pi l$, we have

$$|f_n^{(m)}(z)-f_n^{(m)}(z)|<\varepsilon\, m!\left[\frac{l}{\delta^{m+1}}\right]+\sum_k\left[\frac{r_k^{(p)}}{(\varrho_k^{(p)}-r_k^{(p)})^{m+1}}\right]=\varepsilon\, Q_m^{(p)}$$
$$(m=0,1,2,\cdots).$$

The numbers $Q_m^{(p)}$ $(m=0,1,2,\cdots)$ all exist since the series (12a) all converge. Consequently, the sequence $(f_n(z))$ converges uniformly in $\Gamma_1^{(p-1)}$ to $f_0(z)$ and, in the same region, the sequences $(f_n^{(m)}(z))$ $(m=1,2,\cdots)$ converge uniformly to $f^{(m)}(z)$ $(m=1,2,\cdots)$, respectively. Thus the theorem has been proved.

By means of (10) it is possible to derive some results concerning families of Borel monogenic functions. A theorem will be proved involving equicontinuity.

Suppose $f(z)$ is a bounded family of functions Borel monogenic in $C^{(p)}$. Thus, for z in $\Gamma_1^{(p-1)}$

$$f(z)=B(f;K,C^{(p)},\zeta;z).$$

If z' is another point of $\Gamma_1^{(p-1)}$ it follows that

$$f(z)-f(z')=\frac{1}{2\pi i}\int_K\frac{(z-z')f(\zeta)\,d\zeta}{(\zeta-z)(\zeta-z')}-\frac{1}{2\pi i}\sum_k{}'\int_{S_k^{(p)}}\frac{(z-z')f(\zeta)\,d\zeta}{(\zeta-z)(\zeta-z')}.$$

Whence, noting that $|\zeta-z|$, $|\zeta-z'|>\delta>0$, for ζ on K and z and z' in $\Gamma_1^{(p-1)}$, and observing that $|\zeta-z|$, $|\zeta-z'|>\varrho_k^{(p)}-r_k^{(p)}$ (>0) for ζ on $S_k^{(p)}$ and z and z' in $\Gamma_1^{(p-1)}$, we have

$$|f(z)-f(z')|<M\left[\frac{l}{\delta^2}+\sum_k{}'\frac{r_k^{(p)}}{(\varrho_k^{(p)}-r_k^{(p)})^2}\right]|z-z'|.$$

Here $2\pi l$ is the length of K.

Thus there exists a number $k^{(p)}$ such that the inequality

$$|f(z)-f(z')|\leqq k^{(p)}|z-z'|$$

is verified for all functions $f(z)$ of the family and for whatever points z, z' of $\Gamma_1^{(p-1)}$. Consequently, given ε, we have for all $f(z)$

$$|f(z)-f(z')|<\varepsilon$$

provided $|z-z'|<\delta$ $(=\varepsilon/k^{(p)})$.

THEOREM IV. *Let $f(z)$ form a family of Borel monogenic functions defined and bounded as a set in $C^{(p)}$.*

In every reduced set $\Gamma_1^{(p-1)}$ *of* $C^{(p)}$ *the functions of the family satisfy a condition of equi-continuity.*

Borel has established the formula

$$\int_K f(z)\,dz - \sum_k{}' \int_{S_k^{(p)}} f(z)\,dz = 0 \tag{13}$$

which is valid for functions $f(z)$ monogenic in $C^{(p)}$.[29]

The following definition will be given.

DEFINITION. *A function* $f(z)$ *will be called Borel meromorphic if there exists a polynomial* $P(z)$ *such that the product* $P(z)\,f(z)$ *is Borel monogenic.*

As an application of (13) a result will be derived concerning the zeros and poles of a Borel meromorphic function. Let $f(z)$ be defined in $C^{(p)}$ and have no zeros nor poles on the frontier of $C_1^{(p)}$. We shall denote the interior points (the points not belonging to the frontier) of a region G by $\bar{G}$. In $\bar{C}_1^{(p)}$ let z_1 be the only zero or pole, of multiplicity m. We have

$$f(z) = (z - z_1)^m f_1(z)$$

where m is a positive or negative integer and $f_1(z)$ is Borel monogenic in $C^{(p)}$; moreover,

$$|f_1(z)| > M > 0$$

for z in $C_1^{(p)}$. Thus $f_1^{(1)}(z)/f_1(z)$ is monogenic in $C_1^{(p)}$, and

$$\int_K \frac{f_1^{(1)}(z)\,dz}{f_1(z)} - \sum_k{}' \int_{S_k^{(p)}} \frac{f_1^{(1)}(z)\,dz}{f_1(z)} = 0. \tag{13 a}$$

Let us calculate the expression

$$N(f) = \frac{1}{2\pi i}\int_K \frac{f^{(1)}(z)\,dz}{f(z)} - \sum_k{}' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f^{(1)}(z)\,dz}{f(z)}. \tag{13 b}$$

We have

$$\begin{aligned} N(f) &= \frac{1}{2\pi i}\int_K \left(\frac{m}{z-z_1} + \frac{f_1^{(1)}(z)}{f_1(z)}\right) dz \\ &\quad - \sum_k{}' \frac{1}{2\pi i}\int_{S_k^{(p)}} \left(\frac{m}{z-z_1} + \frac{f_1^{(1)}(z)}{f_1(z)}\right) dz \\ &= m + \frac{1}{2\pi i}\int_K \frac{f_1^{(1)}(z)}{f_1(z)}\,dz - \sum_k{}' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f_1^{(1)}(z)}{f_1(z)}\,dz. \end{aligned}$$

Hence, by (13a),

$$N(f) = m.$$

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[29] (B_2; 135).

Thus, counting a critical point of multiplicity m, as m points, we have the theorem.

THEOREM V. *Let the zeros and poles of a Borel meromorphic function $f(z)$, defined in $C^{(p)}$, be isolated in $\bar{C}_1^{(p)}$; and let there be no zeros or poles on the frontier of $C_1^{(p)}$.*

Then

$$N(f) = \frac{1}{2\pi i}\int_K \frac{f^{(1)}(z)\,dz}{f(z)} - \sum_k{}' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f^{(1)}(z)\,dz}{f(z)} = \mathfrak{z} - p,$$

where $\mathfrak{z}$ denotes the number of zeros and p the number of poles in $\bar{C}_1^{(p)}$ (that is, interior to K and exterior to the $S_k^{(p)}$ which form the frontier of $C^{(p)}$ interior to K).

The following corollary will be proved.

COROLLARY. *Let $(f_n(z))$ be a sequence of functions Borel monogenic in $C^{(p)}$ $(f_n \not\equiv a)$. Assume that for z on the frontier of $C_1^{(p)}$ the sequences*

$$\begin{array}{llll} f_1(z), & f_2(z), & \cdots, & f_n(z), \quad \cdots \\ f_1^{(1)}(z), & f_2^{(1)}(z), & \cdots, & f_n^{(1)}(z), \quad \cdots \end{array}$$

converge uniformly to $f(z)$ and $f^{(1)}(z)$, respectively; the function $f(z)$ being Borel monogenic in $C^{(p)}$. Let the functions

$$f(z)-a, \quad f_n(z)-a \qquad (n \geqq n_1)$$

have no zeros on the frontier of $C_1^{(p)}$; let there be only isolated zeros in $\bar{C}_1^{(p)}$.

Then, for n sufficiently great, $(f_n(z)-a)$ will have in $\bar{C}_1^{(p)}$ as many zeros as $(f(z)-a)$.

It is sufficient to prove this corollary under the assumption that $(f(z)-a)$ has only one zero, z_1, in $\bar{C}_1^{(p)}$; let it be of multiplicity p. From Theorem V it would follow that

$$N(f-a) = p.$$

Here N is formed by means of (13b) and for K we may choose a circle with z_1 for center.[30]

On the other hand, if p_n denotes the number of zeros of $(f_n(z)-a)$ in $\bar{C}_1^{(p)}$ by Theorem V we have

$$p_n = N(f_n - a).$$

Let

$$g_n(\zeta) = \frac{f^{(1)}(\zeta)}{f(\zeta)-a} - \frac{f_n^{(1)}(\zeta)}{f_n(\zeta)-a}.$$

[30] It is possible to construct a circle K, of an arbitrarily small radius, which will consist of points of $\bar{C}^{(p)}$. See (B_2; 139).

By hypothesis

$$|g_n(\zeta)| < \varepsilon \qquad (n \geqq n(\varepsilon))$$

when ζ is on the frontier of $C_1^{(p)}$.

Thus

$$\begin{aligned} |p - p_n| &= |N(f-a) - N(f_n - a)| \\ &= \frac{1}{2\pi}\left| \int_K g_n(\zeta)\, d\zeta - \sum_k{}' \int_{S_k^{(p)}} g_n(\zeta)\, d\zeta \right| \\ &< \varepsilon\, [l + \sum_k{}' r_k^{(p)}] \qquad (n \geqq n(\varepsilon)). \end{aligned}$$

Hence there exists an integer n_1 such that

$$p_n = p \qquad (n \geqq n_1).$$

This completes the demonstration of the corollary.

By means of (10) and (13) we may get a result which is an extension of a theorem of Painlevé[31] on, what will be termed Borel monogenic continuation.

Let AB be a rectifiable arc joining two points A and B of K; K, as before, being a simple closed curve interior to $C^{(p)}$. Except for the end points the arc AB will be assumed to lie in $\bar{C}_1^{(p)}$. Let APB and BQA be the two arcs into which K is separated by the points A and B. These points will be so chosen that $APBQA$ will denote the contour K described in the positive sense.

Further, we let O_1 denote the open region whose boundary is $APBA$; on the other hand, O_2 will denote the open region whose boundary is $BQAB$.

The following theorem will be proved.

THEOREM VI. *Let $f_1(z)$ be Borel monogenic in the region consisting of $APBA$ and the part of $C^{(p)}$ interior to $APBA$; let $f_2(z)$ be Borel monogenic in the region consisting of $BQAB$ and the part of $C^{(p)}$ interior to $BQAB$. Moreover, let $f_1(z) = f_2(z)$ on AB.*

Then by means of the expression

$$(14) \qquad \begin{aligned} G(f_1, f_2; z) &= \frac{1}{2\pi i}\int_{APB} \frac{f_1(\zeta)\, d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{BQA} \frac{f_2(\zeta)\, d\zeta}{\zeta - z} \\ &\quad - \sum_k{}'' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f_1(\zeta)\, d\zeta}{\zeta - z} - \sum_k{}''' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f_2(\zeta)\, d\zeta}{\zeta - z}, \end{aligned}$$

Borel monogenic continuation of $f_1(z)$ into $f_2(z)$, and conversely, can be effected as follows.

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[31] (M; 46).

When z is an interior point of $\Gamma^{(p)}$, in O_1, then

$$G(f_1, f_2; z) = f_1(z);$$

when z is an interior point of $\Gamma^{(p)}$, in O_2, then

$$G(f_1, f_2; z) = f_2(z).$$

The first summation in (14) *is extended over the $S_k^{(p)}$ in O_1, the second summation is extended over the $S_k^{(p)}$ in O_2.*

In fact, for z an interior point of $\Gamma^{(p)}$ in O_1, we have

$$f_1(z) = \frac{1}{2\pi i}\int_{APB} \frac{f_1(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{BA} \frac{f_1(\zeta)\,d\zeta}{\zeta - z} - \sum_k{}'' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f_1(\zeta)\,d\zeta}{\zeta - z}.$$

Moreover,

$$0 = \frac{1}{2\pi i}\int_{BQA} \frac{f_2(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{AB} \frac{f_2(\zeta)\,d\zeta}{\zeta - z} - \sum_k{}''' \frac{1}{2\pi i}\int_{S_k^{(p)}} \frac{f_2(\zeta)\,d\zeta}{\zeta - z}.$$

Noting that

$$\int_{AB} \frac{f_2(\zeta)\,d\zeta}{\zeta - z} = -\int_{BA} \frac{f_1(\zeta)\,d\zeta}{\zeta - z},$$

it follows that

$$f_1(z) = G(f_1, f_2; z),$$

where $G(f_1, f_2; z)$ is given by (14).

Repeating the above reasoning when z is an interior point of $\Gamma^{(p)}$ in O_1, we obtain

$$f_2(z) = G(f_1, f_2; z);$$

from which the theorem follows.

The purpose of this section had been to indicate the possibility of studying Borel monogenic functions by the aid of the formulas (10) and (13), just in the same way as analytic functions are studied by means of Cauchy's formulas (which are particular cases of (10) and (13)). The extension is not immediate since from the preceding discussion it can be seen that, in the "Borel monogenic theory", when a subset of a given region is taken, this subset cannot be arbitrary but must satisfy certain conditions securing convergence of certain series.

3. **Quasi-harmonic functions.** The following definition will be introduced.

DEFINITION I. *Every function* $h(x, y)$ $(x, y$ *real) such that, for some function* $f(z)$ $(z = x + iy)$ *Borel monogenic in* E, *we have*

$$\Re f(z) = h(x, y) \qquad (or\ \Im f(z) = h(x, y)) \tag{15}$$

will be said to belong to the quasi-harmonic class or the class H, *in* E.

For simplicity, though that is not necessary, we shall take $E = C^{(p)}$. Let $p(x, y)$ be of class H in $C^{(p)}$; then there exists a function $q(x, y)$, of the same class, so that

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \qquad \frac{\partial p}{\partial y} = \frac{-\partial q}{\partial x} \tag{15a}$$

for (x, y) in $C^{(p)}$.[32] Evidently, in $C^{(p)}$ we have

$$\Delta p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p = 0.$$

Furthermore, we note also that $\partial p / \partial x$, $\partial p / \partial y$ are unique and continuous in $C^{(p)}$, and the partial derivatives of all orders exist in $C^{(p)}$. The same is true of the function $q(x, y)$.

Borel has shown that

$$\iint_K -\frac{\partial p}{\partial y}\, dx\, dy = \int_{(K)} p\, dx - \sum_n{}' \int_{(S_n^{(p)})} p\, dx.\text{[33]} \tag{16}$$

Here the $(S_n^{(p)})$ and (K) denote contours; K denotes the area bounded by (K). In the double integral we let $\partial p / \partial y$ denote zero interior to the $(S_n^{(p)})$. Also

$$\iint_K \frac{\partial p}{\partial x}\, dx\, dy = \int_{(K)} p\, dy - \sum_n{}' \int_{(S_n^{(p)})} p\, dy. \tag{16a}$$

It will be convenient to introduce a class H'.

DEFINITION II. *A function* $h(x, y)$, *not necessarily such that* $\Delta h = 0$ (*in* $C^{(p)}$), *will be said to belong to the class* H' *provided that* h, $\partial h / \partial x$, $\partial h / \partial y$ *exist and are continuous in* $C^{(p)}$, *and provided formulas* (16) *and* (16a) *hold.*

Every function of class H is also of class H'. At present we leave undecided whether the class H' is more extensive than the class H. In the sequel, whenever necessary, we assume the existence of the partial derivatives.

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[32] (B_2; 155).

[33] (B_2; 155).

With p and q belonging to H', as a consequence of (16) and (16a) it follows that

$$\text{(16b)}\qquad \int\!\!\int_K \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) dx\,dy = \int_{(K)} (p\,dy - q\,dx) - \sum_n{}' \int_{(S_n^{(p)})} (p\,dy - q\,dx).$$

Imitating a known method of deriving Green's first identity, we let

$$p = v\frac{\partial u}{\partial x}, \qquad q = v\frac{\partial u}{\partial y}.$$

It follows then that

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\Delta u.$$

Thus, by (16b),

$$\text{(A)}\qquad \begin{aligned} &\int\!\!\int_K v\Delta u\,dx\,dy + \int\!\!\int_K \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right) dx\,dy \\ &= \int_{(K)} v\left(\frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx\right) - \sum_k{}' \int_{(S_k^{(p)})} v\left(\frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx\right). \end{aligned}$$

This formula is valid in $C^{(p)}$, provided that the functions

$$v\frac{\partial u}{\partial x}, \qquad v\frac{\partial u}{\partial y}$$

belong to the class H' (in $C^{(p)}$).

Interchanging u and v in (A) and subtracting the resulting identity from (A), we obtain

$$\text{(B)}\qquad \begin{aligned} &-\int\!\!\int_K (u\Delta v - v\Delta u)\,dx\,dy \\ &= \int_{(K)} \left[\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) dy - \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dx\right] \\ &\quad - \sum_n{}' \int_{(S_n^{(p)})} \left[\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) dy - \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dx\right]. \end{aligned}$$

(B) is a generalization of Green's second identity; it is valid in $C^{(p)}$, provided that $v\,\partial u/\partial x$, $v\,\partial u/\partial y$, $u\,\partial v/\partial x$, $u\,\partial v/\partial y$ belong to the class H' in $C^{(p)}$.

Letting $\partial/\partial u$ denote the normal derivative taken in a suitable sense, (B) may be written in the form

$$\text{(B}')\qquad \begin{aligned}&\iint_K (u\,\Delta v - v\,\Delta u)\,dx\,dy\\ &= \int_{(K)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds - \sum_k{}' \int_{(S_k^{(p)})} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds.\end{aligned}$$

This formula is valid under the conditions securing validity of (B).

As before, we let $\bar{G}$ denote the set of points of G which are not frontier points of G. Let $\delta(=\sigma+it)$ be in $\bar{\Gamma}^{(p)}$. There exists a sequence of concentric circles $\gamma_{s(q)}$ $(q \geqq q_1)$, each with center at δ, such that the circle $\gamma_{s(q)}$ belongs to $\bar{C}^{(p)}$ and its radius $s(q)$ satisfies the inequalities

$$\frac{1}{2^{q+1}} < s(q) < \frac{1}{2^q} \qquad (q = q_1, q_1+1, \cdots).$$[34]

Now let $g(z, \delta)$ be the Green's function corresponding to the region bounded by (K). Thus,

$$h(z, \delta) = g(z, \delta) + \log|z-\delta| \qquad (z = x+iy)$$

is harmonic for z interior to (K), and $g(z, \delta) = 0$ for z on (K). Let

$$v(x, y) = g(z, \delta)\,(=g), \quad u(x, y) = u(z)\,(=u),$$

and let u be of class H in $C^{(p)}$. Apply (B′) to the closed region K_1 consisting of the part of $C^{(p)}$ which lies between $\gamma_{s(q)}$ and (K) and of the contours $\gamma_{s(q)}$, (K). We have then

$$\begin{aligned}\iint_{K_1} (u\,\Delta g - g\,\Delta u)\,dx\,dy &= \int_{(K)} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds\\ - \int_{\gamma_{s(q)}} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds &- \sum_k{}'' \int_{(S_k^{(p)})} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds.\end{aligned}$$

Here the summation refers only to those of the $S_k^{(p)}$ which are exterior to each other and lie between $\gamma_{s(q)}$ and (K). Since for z in K_1

$$\Delta g = 0, \quad \Delta u = 0,$$

and since for z on (K) $g = 0$, it follows that

$$\text{(17)}\qquad \begin{aligned}\int_{\gamma_{s(q)}} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds &= \int_{(K)} u\frac{\partial g}{\partial n}\,ds\\ &- \sum_k{}'' \int_{(S_k^{(p)})} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds.\end{aligned}$$

[34] (B_2; 139).

When z is on $S_k^{(p)}$ we have, δ being exterior to $L_k^{(p)}$,

$$(1 \geqq) \, |z-\delta| > \varrho_k^{(p)} - r_k^{(p)}.$$

Hence

$$\left| \log |z-\delta| \right| < \log \frac{1}{\varrho_k^{(p)} - r_k^{(p)}},$$

$$\left| \frac{\partial \log |z-\delta|}{\partial n} \right| < \frac{h'}{(\varrho_k^{(p)} - r_k^{(p)})^2}.$$

Consequently

$$\left| \int_{(S_k^{(p)})} \left(u \frac{\partial g}{\partial n} - g \frac{\partial u}{\partial n} \right) ds \right|$$

$$(17\text{a}) \quad = \left| \int_{(S_k^{(p)})} \left[\left(u \frac{\partial h}{\partial n} - h \frac{\partial u}{\partial n} \right) + \left(\log |z-\delta| \frac{\partial u}{\partial n} - u \frac{\partial \log |z-\delta|}{\partial n} \right) \right] ds \right|$$

$$< r_k^{(p)} M \left(1 + \log \frac{1}{\varrho_k^{(p)} - r_k^{(p)}} + \frac{1}{(\varrho_k^{(p)} - r_k^{(p)})^2} \right) = q_k$$

where M is a sufficiently great constant independent of k.

It can be shown that the series

$$(17\text{b}) \qquad \sum_k q_k$$

converges.

Noting that, with z on $\gamma_{s(q)}$, $\log |z-\delta| = \log s(q)$, we write

$$\int_{\gamma_{s(q)}} \left(u \frac{\partial g}{\partial n} - g \frac{\partial u}{\partial n} \right) ds = M_q(\delta) + N_q(\delta),$$

$$M_q(\delta) = \int_{\gamma_{s(q)}} \left(u \frac{\partial h}{\partial n} - h \frac{\partial u}{\partial n} \right) ds + \log s(q) \int_{\gamma_{s(q)}} \frac{\partial u}{\partial n} ds,$$

$$N_q(\delta) = -\int_{\gamma_{s(q)}} u \frac{\partial \log |z-\delta|}{\partial n} ds.$$

It is observed that

$$\lim_{q \to \infty} M_q(\delta) = 0.$$

Now

$$N_q(\delta) = -\int_{\gamma_{s(q)}} u \frac{ds}{|z-\delta|} = -\frac{1}{s(q)} \int_{\gamma_{s(q)}} u \, ds$$
$$= -2\pi \cdot u(\delta + s(q) e^{i\theta(q)}),$$

where $\theta(q)$ is real. The numbers $N_q(\delta)$ $(q \geqq q_1)$ all exist since $u(x, y)$ is defined on every $\gamma_{s(q)}$.

Thus

$$(17\,\mathrm{c})\quad \lim_{q\to\infty}\int_{\gamma_{s(q)}}\left(u\frac{\partial g}{\partial n}-g\frac{\partial u}{\partial n}\right)ds = -\lim_{q\to\infty} 2\pi\cdot u(\delta+s(q)\,e^{i\theta(q)})$$
$$= -2\pi\, u(\delta) = -2\pi\, u(\sigma,\, t).$$

When in (17) q is replaced by a greater integer q' the summation involved in (17) will be extended over an increased number of circles. In fact, there will be added, in general, an infinity of terms corresponding to the $S_k^{(p)}$ between $\gamma_{s(q)}$ and $\gamma_{s(q_1)}$. As k approaches infinity, in the limit the summation will be extended over all the $S_k^{(p)}$ interior to (K). The existence of the limit follows from inequalities (17a) and from (17b). Making use of (17c) we obtain the relation

$$(\mathrm{C})\qquad 2\pi\, u(\delta) = -\int_{(K)} u\frac{\partial g}{\partial n}\,ds + \sum_k{}' \int_{(S_k^{(p)})}\left(u\frac{\partial g}{\partial n}-g\frac{\partial u}{\partial n}\right)ds.$$

We can state the following theorem.

THEOREM VII. *Let*

$$u(x,\, y)\quad (=u(z);\ z = x+iy)$$

be a function of class H (in $C^{(p)}$). Let $g(z,\,\delta)$ $(\delta=\sigma+i\,t)$ denote the Green's function belonging to the region K bounded by a simple closed curve (K) ((K) interior to $C^{(p)}$).

Then the formula (C) *will hold for all δ interior to (K) and interior to $\Gamma^{(p)}$.*

Of special interest among the applications of (A) we have, assuming that u belongs to H and $v=1$,

$$(18)\qquad \int_{(K)}\frac{\partial u}{\partial n}\,ds - \sum_k{}'\int_{(S_k^{(p)})}\frac{\partial u}{\partial n}\,ds = 0;$$

and, when $v=u$ and belongs to H,

$$(18\,\mathrm{a})\qquad \iint_K\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dx\,dy$$
$$= \int_{(K)} u\left(\frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx\right) - \sum_k{}'\int_{(S_k^{(p)})} u\left(\frac{\partial u}{\partial x}\,dy-\frac{\partial u}{\partial y}\,dx\right).$$

The relation (18) means that the integral of the normal derivative of a function of class H, extended over the contour (K) and the frontier of the part of $C^{(p)}$ interior to (K), is zero.

Some consequences analogous to those of the ordinary theory can be derived from (18a). For instance, we have the following theorem.

THEOREM VIII. *If u is of class H and if the normal derivative of u vanishes on (K) and on the $(S_k^{(p)})$, which form the frontier of the part of $C^{(p)}$ interior to (K), then u is a constant in $C^{(p)}$.*

A function of class H is determined, except for an additive constant, by the values of its normal derivatives on (K) and the $(S_k^{(p)})$.

Among the consequences of (B′) we have, when u and v belong to H (in $C^{(p)}$),

$$\int_{(K)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds-\sum_k{}'\int_{(S_k^{(p)})}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds=0. \tag{19}$$

Another identity may be obtained as follows. Let $\delta(=\sigma+it)$ be in $\overline{\Gamma^{(p)}}$, $v=\log r$ ($r=|z-\delta|=\sqrt{(x-\sigma)^2+(y-t)^2}$); assume that u is of class H in $C^{(p)}$, and let the $\gamma_{s(q)}$ be formed as before. Apply (B′) to the region K_1 (between (K) and $\gamma_{s(q)}$). Since $\Delta\log r=0$ it follows that

$$\begin{aligned}0=&\int_{(K)}\left(u\frac{\partial\log r}{\partial n}-\log r\frac{\partial u}{\partial n}\right)ds\\&-\int_{\gamma_{s(q)}}\left(u\frac{\partial\log r}{\partial n}-\log r\frac{\partial u}{\partial n}\right)ds-\sum_k{}''\int_{(S_k^{(p)})}\left(u\frac{\partial\log r}{\partial n}-\log r\frac{\partial u}{\partial n}\right)ds.\end{aligned} \tag{20}$$

Here the summation is extended over all the $(S_k^{(p)})$, limiting $C^{(p)}$, which lie between $\gamma_{s(q)}$ and (K).

As shown before,

$$\lim_q\int_{\gamma_{s(q)}}u\frac{\partial\log r}{\partial n}ds=2\pi u(\delta),$$

$$\lim_q\int_{\gamma_{s(q)}}\log r\frac{\partial u}{\partial n}ds=0.$$

Moreover, for z on $(S_k^{(p)})$ we have

$$|\log r|<\log\frac{1}{\varrho_k^{(p)}-r_k^{(p)}},\qquad\left|\frac{\partial\log r}{\partial n}\right|<\frac{h}{(\varrho_k^{(p)}-r_k^{(p)})^2}.$$

Since the series

$$\sum r_k^{(p)}\log\frac{1}{\varrho_k^{(p)}-r_k^{(p)}},\qquad\sum\frac{r_k^{(p)}}{(\varrho_k^{(p)}-r_k^{(p)})^2}$$

converge it follows that in the limit, as q approaches infinity, the second member of (20) approaches a limit. Hence

$$\begin{aligned}2\pi u(\delta)=&\int_{(K)}\left(u\frac{\partial\log r}{\partial n}-\log r\frac{\partial u}{\partial n}\right)ds\\&-\sum_k{}'\int_{(S_k^{(p)})}\left(u\frac{\partial\log r}{\partial n}-\log r\frac{\partial u}{\partial n}\right)ds\end{aligned}\qquad(r=|z-\delta|), \tag{D}$$

where the summation is extended over all the $(S_k^{(p)})$, limiting $C^{(p)}$, which lie interior to (K).

The following theorem can now be stated.

THEOREM IX. *Let u be of class H (in $C^{(p)}$). The identity* (D) *will hold for $\delta(=\sigma+it)$ interior to (K) and interior to $\Gamma^{(p)}$.*

Apply now (D) to a Borel monogenic function

$$f(\delta) = p(\sigma, t) + i q(\sigma, t).$$

Since

$$\frac{\partial p}{\partial n} + i \frac{\partial q}{\partial n} = \frac{\partial}{\partial n} f(z) = f^{(1)}(z)$$

it follows that

$$(\mathrm{D}') \quad \begin{aligned} f(\delta) = {} & \frac{1}{2\pi} \int_{(K)} \left(f(z) \frac{\partial \log r}{\partial n} - f^{(1)}(z) \log r \right) |dz| \\ & - \frac{1}{2\pi} \sum_k{}' \int_{(S_k^{(p)})} \left(f(z) \frac{\partial \log r}{\partial n} - f^{(1)}(z) \log r \right) |dz| \end{aligned} \qquad (r = |z - \delta|).$$

THEOREM X. *Let $f(z)$ be Borel monogenic in $C^{(p)}$. For $\delta(=\sigma+it)$ interior to (K) and interior to $\Gamma^{(p)}$ the identity* (D′) *holds. This formula expresses $f(\delta)$ in $\bar{\Gamma}^{(p)}$ (interior to (K)) in terms of the values of the function and of its first derivative on (K) and on the frontier of the part of $C^{(p)}$ interior to (K).*

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A STUDY OF INDEFINITELY DIFFERENTIABLE AND QUASI-ANALYTIC FUNCTIONS. II.[1]

BY W. J. TRJITZINSKY.[2]

SOME PROBLEMS OF REPRESENTATION OF QUASI-ANALYTIC FUNCTIONS.

1. Representations based on minimizing a certain definite integral. An indefinitely differentiable function $f(x)$, such that

$$(1)\qquad |f^{(i)}(x)| < Rk^i A_i \qquad (i = 1, 2, \cdots;\ 0 \leq x \leq a)$$

where

$$(1\text{a})\qquad \sum_i 1 \Big/ \sqrt[i]{A_i}$$

diverges, is quasi-analytic on $(0, a)$. (Theorem of Denjoy.)

In order that a function $F(x)$ satisfying (1) should be quasi-analytic it is necessary and sufficient that the integral

$$(1\text{b})\qquad \int_0^\infty \log\left(\sum_{i=0}^{\infty} r^{2i}/A_i^2\right)\frac{dr}{r^2}$$

should diverge. (Carleman's theorem.)[3]

We shall also note that a quasi-analytic function satisfying (1) is said to belong to the class C_A of quasi-analytic functions on the interval $(0, a)$.

In connection with minimizing a certain definite integral Carleman established the existence of a double sequence of functions $\omega_{n,i}(x)$ determined by and depending only on the $A_i (i = 1, 2, \cdots)$. By the aid of these functions the following representations of a quasi-analytic function with assigned initial values may be given. If there is a function $f(x)$ of class C_A having the constants $f^{(m)}(0) = c_m m!$ for initial values at $x = 0$, then

$$(2)\qquad f(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} c_i\, \omega_{n,i}(x);$$

$$(2\text{a})\qquad f(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \omega_{n,i}\, c_i\, x^i, \qquad \omega_{n,i} = \omega_{n,i}(1).[4]$$

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[1] Received October 23, 1930.—In this part of the paper use is made of definitions, notation and references of the first part, these Annals, preceding paper.

[2] National Research Fellow.

[3] (C; 61).

[4] (2), (2a) are found in (C; 68, 70) and (C; 72), respectively.

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Using Carleman's method I derived another representation of the form

$$(3) \qquad f(x) = \lim_{n \to \infty} \sum_{i=0}^{n-1} \omega_{n,i}\, \bar{c}_i(x),$$

where

$$(3\,\mathrm{a}) \qquad \bar{c}_i(x) = \sum_{r=0}^{r_i} \frac{(i-r)!\, c_{i-r}}{r!\,(i-2r)!} (x - p(x))^{i-2r} p^r(x) \qquad \left(r_i = \frac{i}{2} \text{ or } \frac{i-1}{2}\right),$$

with $p(x)$ a suitable analytic function and $c_i i! = f^{(i)}(0)$ $(i = 0, 1, \cdots)$.[5] Let

$$(4) \qquad \bar{\varphi}_k(x, t) = xt + t(t-1)\, p_k(x) \qquad (p_k(x) \text{ analytic}),$$

$$(4\,\mathrm{a}) \qquad F_k(t) = f(\bar{\varphi}_k(x, t)) \qquad (0 \leqq x \leqq a).$$

The development (3) was derived by applying (2a) to the function $F(t) = f(xt + t(t-1)\,p(x))$, considered as a function of t, and by letting subsequently $t = 1$. In (3a) $i!\,\bar{c}_i(x) = F^{(i)}(0)$. Since (2a) is applicable to functions of class C_A, $p(x)$ had to be such that $F(t)$, as function of t, would belong to the class C_A (on some closed interval $(0, b)$ containing $t = 1$). In general, the term a "suitable analytic function" will mean a function such that $F_k(t)$, defined by (4a) as function of t, belongs to the class C_A $(0 \leqq t \leqq b,\ b \geqq 1;\ 0 \leqq x \leqq a)$. With these preliminaries in view, consider the function $F_1(t) = f(\bar{\varphi}_1(x, t))$ formed with $p_1(x)$ a suitable analytic function. Instead of applying the development (2a) we shall apply to $F_1(t)$, as function of t, the development (3). Thus

$$(5) \qquad \begin{aligned} F_1(t) &= \lim_{n \to \infty} \sum_{i=0}^{n-1} \omega_{n,i}\, \bar{c}_i(t), \\ \bar{c}_i(t) &= \sum_{r=0}^{r_i} \frac{(i-r)!\, {}_1\bar{c}_{i-r}(x)}{r!\,(i-2r)!} (t - p(t))^{i-2r} p^r(t) \qquad \left(r_i = \frac{i}{2} \text{ or } \frac{i-1}{2}\right), \end{aligned}$$

where the ${}_1\bar{c}_i(x) \cdot i! = {}_1F^{(i)}(0)$ are formed by replacing in (3a) $p(x)$ by $p_1(x)$; thus

$$(5\,\mathrm{a}) \qquad {}_1\bar{c}_i(x) = \sum_{r=0}^{r_i} \frac{(i-r)!\, c_{i-r}}{r!\,(i-2r)!} (x - p_1(x))^{i-2r} p_1^r(x),$$

with $c_i i! = f^{(i)}(0)$ $(i = 0, 1, \cdots)$. Inserting $t = 1$ in (5), the first member becomes $f(x)$. Hence the following theorem results.

THEOREM I. *Given a set of constants $A_0, A_1, \cdots, A_i, \cdots$ defining a class C_A of quasi-analytic functions, there exists a set of numbers $\omega_{n,i}$ depending on the A_i (and independent of the c_i, $p_1(x)$, $p(x)$) such that, if there exists a function of class C_A for which $f^{(i)}(0) = c_i i!$ $(i = 0, 1, \cdots)$, the following representation holds*

[5] W. J. Trjitzinsky, On quasi-analytic functions. These Annals, vol. 30, October, 1929, pp. 526–546.

$$f(x) = \lim_{n \to \infty} \sum_{i=0}^{n-1} \omega_{n,i}\, \bar{c}_i(1),$$

(6)

$$\bar{c}_i(1) = \sum_{r=0}^{r_i} \frac{(i-r)!\,(1-p(1))^{i-2r}\, p^r(1)}{r!\,(i-2r)!}\; {}_1\bar{c}_{i-r}(x),$$

where

(6a) $${}_1\bar{c}_i(x) = \sum_{i=0}^{r_i} \frac{(i-r)!\, c_{i-r}}{r!\,(i-2r)!}(x-p_1(x))^{i-2r} p_1^r(x) \qquad \left(r_i = \frac{i}{2} \text{ or } \frac{i-1}{2}\right).$$

$p(x)$, $p_1(x)$ *are suitable analytic functions and the* $\omega_{n,i}$ *are the constants involved in* (2a).

Suppose that $p(1) = 1$, then in (6) $\bar{c}_i(1) \equiv 0$ for i odd and $\bar{c}_i(1) = {}_1\bar{c}_{i/2}(x)$ for i even. Consequently

(7) $$f(x) = \lim_{n \to \infty} \sum_{i=0}^{{}_1r_n} \omega_{n,2i} \cdot {}_1\bar{c}_i(x) \qquad \left({}_1r_n = \frac{n-1}{2} \text{ or } \frac{n-2}{2}\right).$$

To $F_2(t) = f(xt + t(t-1)\,p_2(x))$, considered as function of t, we apply the development (7). This gives

$$F_2(t) = \lim_{n \to \infty} \sum_{i=0}^{{}_1r_n} \omega_{n,2i} \cdot {}_1\bar{c}_i(t);$$

whence

$$F_2(1) = f(x) = \lim_{n \to \infty} \sum_{i=0}^{{}_1r_n} \omega_{n,2i} \cdot {}_1\bar{c}_i(x).$$

Suppose that the function $p_1(x)$ involved in the ${}_1\bar{c}_i(x)$ is such that $p_1(1) = 1$. Then, as before, it follows that ${}_1\bar{c}_i(1) = 0$ for i odd and ${}_1\bar{c}_i(1) = {}_2\bar{c}_{i/2}(x)$ for i even (${}_2\bar{c}_m(x)$ will be given by (6a) with $p_1(x)$ replaced by $p_2(x)$). Thus we have

(7a) $$f(x) = \lim_{n \to \infty} \sum_{i=0}^{{}_2r_n} \omega_{n,4i} \cdot {}_2\bar{c}_i(x) \qquad \left({}_2r_n = \frac{{}_1r_n}{2} \text{ or } \frac{{}_1r_n - 1}{2}\right).$$

Assuming $p_k(x) \equiv p(x)$ $(k = 1, 2, \cdots, (s-1))$, $p(1) = 1$ and $p(x)$ a suitable analytic function, the s-fold repetition of the process which rendered the developments (7), (7a) will result in the following representation

(8) $$f(x) = \lim_{n \to \infty} \sum_{i=0}^{{}_sr_n} \omega_{n,2^s i} \cdot {}_s\bar{c}_i(x) \qquad \left({}_sr_n = \frac{{}_{s-1}r_n}{2} \text{ or } \frac{{}_{s-1}r_n - 1}{2}\right).$$

The subscript s in the ${}_s\bar{c}_i(x)$ may be deleted if $p_s(x) \equiv p(x)$. It may be deleted otherwise, but with the understanding that $p_s(1) = p(1)$ is not necessarily equal to unity. The development (8) may be written in the form

$$f(x) = \lim_{m \to \infty} \sum_{i=0}^{{}_sr_e} \omega_{e,2^s i} \cdot \bar{c}_i(x), \quad e = n_m \;(n_{m+1} > n_m;\; m = 1, 2, \cdots).$$

Letting $n_m = m 2^s + 1$ and using the relations given for the ${}_s r_n$ by (8), we get ${}_s r_e = m$. Hence the following theorem is proved.

THEOREM II. *Given a set of constants* $A_0, A_1, \cdots, A_i, \cdots$ *defining a class* C_A *of quasi-analytic functions, let the* $\omega_{n,i}$ *be the sequence of constants of Theorem* I. *If there exists a quasi-analytic function* $f(x)$ *of class* C_A *for which* $f^{(i)}(0) = c_i\, i!$ $(i = 0, 1, \cdots)$, *such a function admits the representation*

$$(9) \qquad f(x) = \lim_{m \to \infty} \sum_{i=0}^{m} \omega_{e, 2^s i} \cdot \bar{c}_i(x), \qquad e = m\, 2^s + 1.$$

In (9) s *is any finite positive integer; and the* $\bar{c}_i(x)$ *are given by* (3 a) *with* $p(x)$ *a suitable analytic function. The representation holds provided that, for an analytic function* $p_1(x)$ $(p_1(1) = 1)$, $f(xt + t(t-1) p_1(x))$ *is of class* C_A *in* t *for* $0 \leqq t \leqq 1$ *and for* x *on a suitable interval.*

In (9) let $p(x) \equiv 0$. Then, from (3a), it follows that $\bar{c}_i(x) = c_i x^i$ $(i = 0, 1, \cdots)$; thus the following corollary results.

COROLLARY. *Suppose the set of constants* $A_0, A_1, \cdots, A_i, \cdots$ *defines a class* C_A *of quasi-analytic functions. If there exists a function* $f(x)$ *of class* C_A, *to which Theorem* II *is applicable and for which* $f^{(i)}(0) = c_i\, i!$ $(i = 0, 1, \cdots)$, *such a function admits the representation*

$$(10) \qquad f(x) = \lim_{m \to \infty} \sum_{i=0}^{m} \omega_{e, 2^s i}\, c_i x^i, \qquad e = n_m = m\, 2^s + 1.$$

In this representation s *is any finite positive integer and the* $\omega_{n,i}$ *are the constants involved in Theorem* I.

The representations (2a), (3) and (6) generally depend on the limiting properties of the constants in each row of the double array

$$\begin{array}{l}
\omega_{1,0}, \ \omega_{2,0}, \ \cdots, \ \omega_{n,0}, \ \cdots, \\
\omega_{1,1}, \ \omega_{2,1}, \ \cdots, \ \omega_{n,1}, \ \cdots, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\omega_{1,i}, \ \omega_{2,i}, \ \cdots, \ \omega_{n,i}, \ \cdots, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

The representations (9), (10) depend on the limiting properties of the constants just in certain subsequences of the rows of (10).

In general, a great multiplicity of representations results by applying to $F(t) = f(xt + t(t-1)p(x))$, as a function of t, any of the various representations already known. As remarked elsewhere, it is possible to derive representations for $f(x)$ from the consideration of a function $F(t) = f(xt + t(t-1)p(x, t))$, where $p(x, t)$ is an analytic function of x and t such that $F(t)$, as function of t, is of class C_A for $0 \leqq t \leqq b$ $(b \geqq 1, 0 \leqq x \leqq a)$. These representations, with $p(x, t)$ not restricted any further, would be unduly complicated. We shall now consider the case

when $p(x, t)$ is independent of x. Thus, suppose that $p(t)$ is an analytic function such that

$$F(t) = f(xt + t(t-1)p(t)) \qquad (0 \leqq x \leqq a), \tag{11}$$

as a function of t, is of class C_A for $0 \leqq t \leqq b$ $(b \geqq 1)$. Any of the developments of this section can be applied to $F(t)$; but we shall restrict ourselves to the application of (2a). For the latter purpose it will be necessary to compute the $F^{(n)}(0)$ $(n = 0, 1, \cdots)$. Now,

$$F^{(n)}(t) = \frac{\partial^n}{\partial t^n} f(u), \qquad u = xt + t(t-1)p(t) = xt + q(t).$$

Hence

$$F^{(n)}(t) = \frac{\partial^n}{\partial t^n} f(u) = \sum_{i_1, i_2, \cdots, i_e} \frac{n! f^{(p)}(u)}{i_1!\, i_2! \cdots i_e!} \left(\frac{u_1}{1!}\right)^{i_1} \left(\frac{u_2}{2!}\right)^{i_2} \cdots \left(\frac{u_e}{e!}\right)^{i_e}, \tag{12}$$
$$(p = i_1 + \cdots + i_e),$$

where

$$u_1 = \frac{\partial u}{\partial t} = x + q^{(1)}(t), \qquad u_i = \frac{\partial^i u}{\partial t^i} = q^{(i)}(t) \qquad (i = 2, 3, \cdots)$$

and the summation is extended over the positive integral values $i_1, i_2, \cdots, i_e$ such that $i_1 + 2i_2 + 3i_3 + \cdots + e i^e = n$.[6] Substituting $t = 0$

$$\begin{aligned} F^{(n)}(0) &= \sum_{i, i_2, \cdots, i_e} \frac{n! f^{(p)}(0)}{i!\, i_2! \cdots i_e!} \left(\frac{x + q^{(1)}(0)}{1!}\right)^{i} \left(\frac{q^{(2)}(0)}{2!}\right)^{i_2} \cdots \left(\frac{q^{(e)}(0)}{e!}\right)^{i_e} \\ &= n!\,[c_{n,0} + c_{n,1}(x + q^{(1)}(0))/1! + \cdots + c_{n,n}(x + q^{(1)}(0))^n/n!], \\ & \qquad (p = i + i_2 + \cdots + i_e), \end{aligned} \tag{12a}$$

where

$$\begin{aligned} c_{n,i} &= \sum_{i_2, i_3, \cdots, i_e} \frac{c_p\, p!}{i_2!\, i_3! \cdots i_e!} \left(\frac{q^{(2)}(0)}{2!}\right)^{i_2} \left(\frac{q^{(3)}(0)}{3!}\right)^{i_3} \cdots \left(\frac{q^{(e)}(0)}{e!}\right)^{i_e}, \\ c_p\, p! &= f^{(p)}(0), \qquad\qquad (p = i + i_2 + \cdots + i_e). \end{aligned} \tag{12b}$$

The summation in (12b) is extended over the positive integral values $i_2, i_3, \cdots, i_e$, such that $2i_2 + 3i_3 + \cdots + e i_e = n - i$. Applying (12a) we get

$$F(t) = \lim_n \sum_{i=0}^{n-1} \omega_{n,i}\, F^{(i)}(0)\, t^i/i!.$$

Hence

$$F(1) = f(x) = \lim_n \sum_{i=0}^{n-1} \omega_{n,i}\, F^{(i)}(0)/i!.$$

THEOREM III. *Given a set of constants* $A_0, A_1, \cdots, A_i, \cdots$ *defining a class* C_A *of quasi-analytic functions; if there exists a function* $f(x)$ *of*

[6] (12) is a direct application of a formula given by Faa de Bruno, Quarterly Journal of Mathematics, vol. I, p. 359.

class C_A, for which $f^{(i)}(0) = c_i\, i!$ $(i = 0, 1, \cdots)$, such a function admits the representation

$$(13)\quad f(x) = \lim_{n} \sum_{i=0}^{n-1} \omega_{n,i}\, P_i(x), \quad P_i(x) = c_{i,0} + c_{i,1}\frac{\bar{x}}{1!} + \cdots + c_{i,i}\frac{\bar{x}^i}{i!}.$$

Here $\bar{x} = x + q^{(1)}(0)$. The $c_{n,i}$ are given by (12b). *The function $q(t)$ involved in* (12b) *is defined by $q(t) = t(t-1)p(t)$ where $p(t)$ is any analytic function such that $f(xt + q(t))$, as function of t, is of class C_A for $0 \leq t \leq b$ $(b \geq 1)$, $0 \leq x \leq a$.*

REMARK. The existence of representations of this section is seen to depend on the existence of a function $p(x, t)$ such that the function of t $F(t) = f(xt + t(t-1)\, p(x, t))$ is of the same class as $f(x)$. Suppose that, with a given $p(x, t)$, $F(t)$ is of class C_B, different from the class C_A to which $f(x)$ belongs. Then, if C_L is a quasi-analytic class containing both C_A and C_B, as subclasses, we may derive a new representation for $f(x)$ by applying to $F(t)$, as function of t, a known expansion of functions of class C_L (the $\omega_{m,i}$ defined by the L_n). In the development, so derived, we let $t = 1$ $(F(1) = f(x))$.

2. **Necessary and sufficient conditions for quasi-analyticity.** The class of quasi-analytic functions (any particular class) possesses the property that a function of such a class is identically zero throughout its domain of definition if its initial values at a point are all zero. Carleman has proved a theorem establishing necessary and sufficient conditions which should be satisfied by the α_n in order that the function which is represented by the series $a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$, $|a_n| < g\alpha_n$ $(n = 0, 1, \cdots)$ should be quasi-analytic.[7] We raise the question: what are the necessary and sufficient conditions in order that the function, represented by the series

$$(14)\qquad F(x) = a_0 + \sum_{n=1}^{\infty} a_n\, p(nx), \quad |a_n| < g\alpha_n, \quad p(0) = 1,$$

($p(x)$ being an even function of class A), should be quasi-analytic? It will be necessary to distinguish between any even function of class A and those belonging to a subclass A' of A. The subclass A' will be defined as follows.

DEFINITION. *An even function $p(x)$ of class A will be said to belong to the subclass A', if the relation $\sum_{n=0}^{\infty} a_n\, p(nx) \equiv 0$, for all real values of x, implies that $a_n = 0$ $(n = 0, 1, \cdots)$.*

An example of a function of the subclass A' is $p(x) = \cos x$.

[7] (C; 95).

The reasoning which we are about to employ for the solution of the problem in question follows closely the method used by Carleman in proving his theorem concerning the cosine series.[8]

Let the γ_i $(i = 1, 2, \cdots)$ be a given set of positive constants. Under the conditions

$$(15) \qquad a_0 = 1, \qquad \sum_{i=0}^{\infty} a_i i^{2p} = 0 \qquad (p = 0, 1, \cdots, n),$$

the expression

$$(16) \qquad J(a) = \sum_{i=1}^{\infty} \gamma_i a_i^2$$

possesses a minimum ϱ_n. It is known that the sequence of the ϱ_n $(n = 1, 2, \cdots)$ is or is not bounded according as the Stieltjes problem of moments

$$\int_0^{\infty} x^p \, d\Psi(x) = c_p = \sum_{i=1}^{\infty} i^{2p}/\gamma_i \qquad (p = 0, 1, 2, \cdots)$$

is indeterminate or determinate, and conversely.[9] Let us denote by $a_i^{(n)}$ $(i = 0, 1, 2, \cdots)$, $a_0^{(n)} = 1$, the set of constants rendering $J(a)$ minimum under conditions (15). We have then $\varrho_n = J(a^{(n)})$. Let $F_n(x)$ be the function corresponding to the set $a_i^{(n)}$ $(i = 0, 1, \cdots)$ and defined as follows

$$(17) \quad F_n(x) = a_0^{(n)} + a_1^{(n)} p(x) + a_2^{(n)} p(2x) + \cdots + a_i^{(n)} p(ix) + \cdots.$$

From conditions (15) it follows that $F_n^{(2p)}(0) = 0$ $(p = 0, 1, \cdots, n)$. We can say for brevity that $F_n(x)$ minimizes $J(a)$ by rendering $J(a) = J(a^{(n)}) = \varrho_n$. Suppose $J(a^{(n)}) = \varrho_n < S^2$, $S > 0$ $(n = 0, 1, \cdots)$, with S independent of n. This assumption is equivalent to the set of inequalities $\sum_{i=1}^{\infty} \gamma_i (a_i^{(n)})^2 < S^2$ $(n = 0, 1, \cdots)$, from which it follows that $\gamma_i (a_i^{(n)})^2 < S^2$. Letting $\gamma_i = 1/\alpha_i^2$, where the α_i are the constants of (14), it follows that

$$(17\text{a}) \qquad |a_i^{(n)}| < S\alpha_i \qquad (i, n = 0, 1, \cdots).$$

With the assumption that the series $\sum_{i=0}^{\infty} \alpha_i i^m$ $(m = 0, 1, \cdots)$ converge, the following inequalities are satisfied for all real values of x

$$(17\text{b}) \qquad \begin{aligned} |F_n(x)| &\leqq h \sum_{i=0}^{\infty} |a_i^{(n)}| < hS \sum_{i=0}^{\infty} \alpha_i = A_0, \\ |F_n^{(m)}(x)| &\leqq h \sum_{i=1}^{\infty} |a_i^{(n)} i^m| < hS \sum_{i=1}^{\infty} \alpha_i i^m = A_m \qquad (m = 0, 1, \cdots). \end{aligned}$$

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[8] (C; 92, 95).

[9] See (C; 93). When $\Psi(x)$ is a solution of the problem of moments, this problem is said to be determinate provided that every other solution differs from $\Psi(x)$ by a constant at every point of continuity of $\Psi(x)$.

The sequences of functions

$$\text{(18a)} \qquad F_1(x), \quad F_2(x), \quad \cdots, \quad F_n(x), \quad \cdots$$

$$\text{(18b)} \qquad F_1^{(m)}(x), \quad F_2^{(m)}(x), \quad \cdots, \quad F_n^{(m)}(x), \cdots \qquad (m = 1, 2, \cdots)$$

are each bounded and equi-continuous; the equi-continuity of the sequences (18b) being inferred in succession from that of (18a) by means of the relations

$$F_n^{(i)}(x) = \int_0^x F_n^{(i+1)}(y)\, dy \qquad (i \leqq n-1).$$[10]

Consequently it is possible to find a sequence of indices $n_1, n_2, \cdots, n_i, \cdots$ such that the sequences

$$\text{(19)} \qquad \begin{array}{llll} F_{n_1}(x), & F_{n_2}(x), & \cdots, & F_{n_i}(x), \quad \cdots \\ F_{n_1}^{(m)}(x), & F_{n_2}^{(m)}(x), & \cdots, & F_{n_i}^{(m)}(x), \quad \cdots \qquad (m = 1, 2, \cdots) \end{array}$$

are all uniformly convergent, the function $F(x)$, defined by

$$\text{(19a)} \qquad F(x) = \lim_i F_{n_i}(x),$$

being indefinitely differentiable (for all real values of x). The derivatives of $F(x)$ are given by

$$\text{(19b)} \qquad \lim_i F_{n_i}^{(m)}(x) = F^{(m)}(x) \qquad (m = 1, 2, \cdots).$$

Since $F_{n_i}(x) = a_0^{(n_i)} + a_1^{(n_i)} p(x) + a_2^{(n_i)} p(2x) + \cdots$, and in virtue of the convergence conditions, it follows from (19a) that

$$\text{(20)} \qquad F(x) = a_0 + a_1 p(x) + a_2 p(2x) + \cdots + a_k p(kx) + \cdots,$$

where $a_k = \lim_i a_k^{(n_i)}$ and (by (17a)) $|a_k| \leqq S\alpha_k$ $(k = 0, 1, \cdots)$. Moreover, $F(0) = F^{(m)}(0) = 0$ $(m = 1, 2, \cdots)$. With $p(x)$ belonging to the subclass A', we have a series (20) not all of whose coefficients are zero.[11] Therefore the function $F(x)$, represented by (20), cannot be identically zero (on the axis of reals); that is, it is not quasi-analytic. Hence, if the ϱ_n form a bounded set (the problem of moments with $\gamma_i = 1/\alpha_i^2$ being indeterminate) then the class of functions, defined by (14) and formed with $p(x)$ of subclass A', is not quasi-analytic.

Let us consider the other alternative. Suppose that the problem of moments $\int_0^\infty x^n\, d\Psi(x) = \sum_{i=1}^\infty \alpha_i^2\, i^{2n}$ is determinate, then the same will be

[10] Compare with (C; 65).

[11] At least $a_0 \neq 0$, since $a_0 = \lim_i a_0^{(n_i)} = 1$ $(a_0^{(n)} = 1;\ n = 0, 1, \cdots)$.

true of the problem $\int_0^\infty x^n\, d\Psi(x) = \sum_{i=1}^{\infty} i^2 \alpha_i^2\, i^{2n}$ $(n = 0, 1, \cdots)$.[12] This, however, would imply that the ϱ_n' $(n = 0, 1, \cdots)$, where ϱ_n' is the minimum of

(21) $$_1J(a) = \sum_{i=1}^{\infty} a_i^2 / i^2 \alpha_i^2$$

under the conditions

(21a) $$_1a_0 = 1, \qquad \sum_{i=0}^{\infty} {}_1a_i\, i^{2p} = 0 \qquad (p = 0, 1, \cdots, n),$$

approach infinity with n. Let the set of constants, satisfying (21a) and rendering (21) a minimum, be denoted by $_1a_i^{(n)}$ $(i = 0, 1, \cdots)$, $_1a_0^{(n)} = 1$. Then

(22) $$\varrho_n' = {}_1J({}_1a^{(n)}) = \sum_{i=1}^{\infty} ({}_1a_i^{(n)})^2 / i^2 \alpha_i^2 \qquad (n = 0, 1, \cdots).$$

We form

(22a) $$_1F_n(x) = \sum_{i=0}^{\infty} {}_1a_i^{(n)}\, p(ix), \qquad p(0) = 1, \qquad {}_1a_0^{(n)} = 1.$$

By (21a), $_1F_n^{(2p)}(0) = 0$ $(p = 0, 1, \cdots, n)$. The coefficients of (22a) minimize (21). The conditions $_1a_0^{(n)} = 1$, $_1a_0 = 1$ or $a_0 = 1$ can be replaced by a condition of the form $\sum_{i=1}^{N} a_i \beta_i = 1$,[13] provided not all the a_i are zero. With this in view we shall examine whether or not there exists a function $F(x) = \sum_{i=0}^{\infty} a_i\, p(ix)$, $p(0) = 1$, satisfying the following conditions: $1°$ $F^{(2p)}(0) = 0$ $(p = 0, 1, \cdots)$; $2°$ $|a_i| < S\alpha_i$ $(i = 0, 1, \cdots)$; $3°$ $F(x) \not\equiv 0$.

On account of (3°) not all the a_i are zero. If $a_0 \neq 0$, we define M by $Ma_0 = 1$. The function

(23) $$_1F(x) = MF(x) = \sum_{i=0}^{\infty} {}_1a_i\, p(ix) \qquad ({}_1a_i = Ma_i)$$

will have $|{}_1a_i| < |M|\, S\alpha_i$ $(i = 0, 1, \cdots)$, $_1F^{(2p)}(0) = 0$ $(p = 0, 1, \cdots)$ and $_1a_0 = 1$. With the $a_i = {}_1a_i$ the expression $_1J(a)$ of (21) becomes

(23a) $$_1J({}_1a) = \sum_{i=1}^{\infty} {}_1a_i^2 / i^2 \alpha_i^2 < M^2 S^2 \sum 1/i^2 = g_1.$$

Consequently the ϱ_n', as given by (22) and derived under the condition $_1a_0^{(n)} = 1$, are such that $\varrho_n' < g_1$ (since $\varrho_n' \leqq {}_1J({}_1a)$). This contradicts the assumption that $\lim_n \varrho_n' = \infty$. Hence, $_1F(x)$ and therefore $F(x)$ are identically zero if conditions (1°), (2°) are satisfied.

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[12] (C; 94).
[13] (C; 94).

If $a_0 = 0$, let i_1 be the smallest subscript such that $a_{i_1} \neq 0$. Define the ϱ'_n as before, except that the condition ${}_1a_0^{(n)} = 1$ will now be replaced by $\sum_{i=1}^{i_1} {}_1a_i^{(n)} \beta_i = 1$, $\beta_{i_1} \neq 0$. Then for a constant η

$$\eta \sum_{i=1}^{i_1} a_i \beta_i = \eta\, {}_1a_{i_1} \cdot \beta_{i_1} = 1.$$

The function

$$(24) \qquad {}_1F(x) = \eta F(x) = \sum_{i=i_1}^{\infty} {}_1a_i\, p(ix) \qquad ({}_1a_i = \eta a_i)$$

will have $|{}_1a_i| < |\eta| S \alpha_i$ $(i = i_1, i_1+1, \cdots)$, ${}_1F^{(2p)}(0) = 0$ $(p = 0, 1, \cdots)$ and $\sum_{i=1}^{i_1} {}_1a_i \beta_i = 1$. The ${}_1a_i$ of ${}_1F(x)$ (as given by (24)) render ${}_1J(a)$ of (21) equal to ${}_1J({}_1a) = \sum_i {}_1a_i^2 / i^2 \alpha_i^2 < \eta^2 S^2 \sum 1/i^2 = g_2$. Hence, if the ϱ'_n are defined by (22) with the conditions ${}_1a_0^{(n)} = 1$ replaced by $\sum_{i=0}^{i_1} {}_1a_i^{(n)} \beta_i = 1$ $(\beta_{i_1} \neq 0)$, then $\varrho'_n < g_2$ $(n = 0, 1, \cdots)$. A contradiction arises again and therefore ${}_1F(x)$ as well as $F(x)$ are identically zero. Thus a function satisfying conditions (1°) and (2°) has to be identically zero for all real values of x.

THEOREM IV. *Let $F(x)$ be defined for all real values of x by the series*

$$(25) \quad F(x) = a_0 + a_1 p(x) + a_2 p(2x) + \cdots + a_n p(nx) + \cdots, \quad p(0) = 1,$$

where $p(x)$ is an even function of the subclass A' and the $|a_n|$ satisfy a law of decrease $|a_n| < A \alpha_n$ $(n = 0, 1, \cdots)$. Let the series $\sum_i i^p \alpha_i$ $(p = 0, 1, \cdots)$ be convergent. $F(x)$, as defined by (25), is indefinitely differentiable. In order that $F(x)$ should be determined by its initial values $F^{(2p)}(0)$ at $x = 0$ it is necessary and sufficient that the problem of moments

$$\int_0^{\infty} x^n \, d\psi(x) = \sum_{i=1}^{\infty} \alpha_i^2 i^{2n} \qquad (n = 0, 1, \cdots)$$

should be determinate. The sufficient part continues to hold when $p(x)$ is any even function of class A $(p(0) = 1)$.[14]

3. Application of the Stieltjes continued fractions. The problem of representation of quasi-analytic functions in terms of its initial values is very difficult, and therefore it seems of interest to mention first a case of a very simple representation. Unfortunately, this representation will depend on the possibility, as yet not established, of having at the same time:

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[14] A theorem of this type is given by Carleman for the case $p(x) = \cos x$ in (C; 95).

1°. A set of values, $f^{(n)}(0)$ $(n = 0, 1, \cdots)$, which are the initial values of a function $f(x)$ belonging to a quasi-analytic class $C_M\left(\sum_n 1 \middle/ \sqrt[n]{M_n}\right.$ divergent, $\left. M_n \geqq 1;\ 0 \leqq x \leqq a\right)$.

2°. A function $p(x)$, of class A, such that $|p^{(n)}(0)| \geqq k^n > 0$ and

$$c_n = f^{(n)}(0)/p^{(n)}(0) > 0 \qquad (n = 0, 1, 2, \cdots).$$

Granted (1°) *and* (2°), *it follows that*

$$\begin{aligned} f(x) &= \int_0^\infty p(tx)\, d\Psi(t), \\ f^{(m)}(x) &= \int_0^\infty t^m p^{(m)}(tx)\, d\Psi(t) \qquad (m = 1, 2, \cdots) \end{aligned} \tag{26}$$

where $\Psi(t)$ *is a uniquely determined function.*

In fact, we have

$$\sqrt[2n]{c_n} \leqq \sqrt[2n]{M_n}/\sqrt{k} \leqq \sqrt[n]{M_n}/\sqrt{k}.$$

Hence, from the divergence of $\sum 1/\sqrt[n]{M_n}$ divergence of

$$\sum_n \frac{1}{\sqrt[2n]{c_n}}$$

can be inferred. Consequently, by a result of Carleman,[15] the problem of moments

$$\int_0^\infty t^n\, d\Psi(t) = c_n \qquad (n = 0, 1, \cdots) \tag{26a}$$

is determinate. Thus we have (26a) satisfied for a non-decreasing function $\Psi(t)$ which is unique except for an additive constant. The integral (26) will be indefinitely differentiable for all real values of x. In fact,

$$\begin{aligned} |f^{(m)}(x)| \leqq h\int_0^\infty t^m\, d\Psi(t) &= h c_m = h f^{(m)}(0)/p^{(m)}(0) \\ &\leqq \frac{h M_m}{k^m} \qquad (|p^{(m)}(u)| \leqq h;\ m = 0, 1, \cdots). \end{aligned}$$

Moreover, as seen from the above, the function represented by this integral will be of class C_M. It will possess the assigned initial values, as can be seen differentiating (26) and using the fact that $\Psi(t)$ satisfies (26a).

In the following better, though more complicated, representations in terms of initial values will be derived. They constitute an extension of a representation given by Carleman;[16] the latter uses $\cos x$ in place of our more general function $p(x)$.

[15] (C; 81).

[16] (C; 95, 96).

DEFINITION I. *An even function* $p(x)$, *of class* A, *will be said to belong to the subclass* B *if there exists an odd entire function* $g(x)$ *so that* $q(ix) = p(x) + ig(x)$ *possesses the property that*

$$\lim_{r\to\infty} M(r) = 0. \tag{27}$$

The function $M(r)$ *is the maximum in* θ *of*

$$|q(tre^{i\theta})| \qquad \left(\frac{\pi}{2} + \varepsilon \leqq \theta \leqq \frac{3\pi}{2} - \varepsilon;\ t > 0\right).$$

Here ε *approaches zero when* r *approaches infinity.*

An even function $p(x)$ which belongs to the subclasses A' and B will be said to belong to the subclass B'. (B' is, of course, a subclass of B.)

LEMMA. *If* $p(x)$ *is a function of the subclass* B *then*

$$p(t\sqrt{u}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{zq(tz)\,dz}{u+z^2} \qquad (a, u, t > 0). \tag{27a}$$

To prove this lemma we form a contour

$$C_r = L_r + R_r.$$

Here L_r is a circular arc for which

$$|z| = r, \qquad \frac{3\pi}{2} - \varepsilon \leqq \arg z \leqq \frac{\pi}{2} + \varepsilon;$$

and R_r denotes the two segments of straight lines joining $z = a$ with $z = re^{i(3\pi/2-\varepsilon)}$ and $z = re^{i(\pi/2+\varepsilon)}$, respectively. When r is sufficiently great, so that the points $\pm i\sqrt{u}$ are interior to C_r, it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_r}\frac{zq(tz)dz}{u+z^2} &= \frac{1}{2}[q(it\sqrt{u}) + q(-it\sqrt{u})]\\ &= \frac{1}{2}[(p(t\sqrt{u}) + ig(t\sqrt{u})) + (p(t\sqrt{u}) - ig(t\sqrt{u}))]\\ &= p(t\sqrt{u}).\end{aligned}$$

We note that

$$\left|\int_{L_r}\frac{zq(tz)dz}{u+z^2}\right| = \left|\int_{\frac{\pi}{2}+\varepsilon}^{\frac{3\pi}{2}-\varepsilon}\frac{q(tre^{i\theta})\,d\theta}{1+ue^{-2i\theta}/r^2}\right| < \frac{\pi}{1-u/r^2}M(r) \quad (\sqrt{u} < r),$$

so that

$$\lim_{r\to\infty}\int_{L_r}\frac{zq(tz)\,dz}{u+z^2} = 0.$$

On the other hand,

$$\lim_{r}\int_{R_r}\frac{zq(tz)\,dz}{u+z^2} = \int_{a-i\infty}^{a+i\infty}\frac{zq(tz)\,dz}{u+z^2}.$$

Consequently

$$2\pi i\, p(t\sqrt{u}) = \lim_r \int_{L_r} + \lim_r \int_{R_r} = \int_{a-i\infty}^{a+i\infty} \frac{z q(tz)\, dz}{u+z^2}.$$

The function used by Carleman, $p(x) = \cos x$, is of subclass B ($q(x) = e^x$); as matter of fact, it belongs to the more special subclass B'. While existence of functions different from $\cos x$ and still of subclass B' is left undecided, we shall show that there actually exist functions of the subclass B which are different from $\cos x$.

In fact, *every function of the form*

$$p(x) = \int_a^b \cos(x\overline{\varphi}(\varphi))\, d\varphi \tag{28}$$

is of subclass B, provided

$$0 < \lambda \leqq \overline{\varphi}(\varphi) \leqq 1 \qquad (a \leqq \varphi \leqq b).$$

The function $g(x)$, corresponding to $p(x)$, will be defined by

$$g(x) = \int_a^b \sin(x\overline{\varphi}(\varphi))\, d\varphi. \tag{28a}$$

We have

$$p^{(2n)}(x) = \int_a^b (-1)^n \cos(x\overline{\varphi}(\varphi))\, \overline{\varphi}^{2n}(\varphi)\, d\varphi,$$

$$p^{(2n+1)}(x) = \int_a^b (-1)^{n+1} \sin(x\overline{\varphi}(\varphi))\, \overline{\varphi}^{2n+1}(\varphi)\, d\varphi,$$

so that for all real values of x

$$|p^{(n)}(x)| \leqq (b-a) \qquad (n = 0, 1, \cdots).$$

Thus, $p(x)$ is an even function of class A. Using (28a),

$$q(ix) = p(x) + ig(x) = \int_a^b e^{ix\overline{\varphi}(\varphi)}\, d\varphi,$$

$$q(u) = \int_a^b e^{u\overline{\varphi}(\varphi)}\, d\varphi.$$

It can be shown that

$$|q(tre^{i\theta})| \leqq (b-a)\, e^{\lambda tr\cos\theta}$$

$$\leqq (b-a)\, e^{-\lambda tr\sin\epsilon} \qquad \left(\frac{\pi}{2} + \epsilon \leqq \theta \leqq \frac{3\pi}{2} - \epsilon\right).$$

Hence the choice $\epsilon = \log r/r$ will insure that

$$\lim_{r\to\infty} M(r) = 0.$$

We shall prove the following theorem.

THEOREM V. *Let $p(x)$ be of class B and let the α_n be such that the problem of moments*

$$\int_0^\infty x^n\, d\Psi(x) = \sum_{m=0}^\infty \alpha_m^2\, m^{2n} \qquad (n = 0, 1, \cdots) \tag{29}$$

is determinate and that the series

$$\sum_n \frac{1}{\sqrt[2n]{A_n}} \qquad \left(A_n = \sum_{m=0}^{\infty} \alpha_m m^{2n}\right) \tag{29a}$$

diverges.

Let the set of constants $f^{(n)}(0)$ $(n = 0, 1, \cdots)$ *be the initial values, at* $x = 0$, *belonging to the class of functions of the form*

$$f(x) = \sum_0^{\infty} a_n p(nx) \qquad (p(0) = 1;\ |a_n| < A\alpha_n).$$

The function $f(x)$ *(which will be quasi-analytic) is expressible in terms of the* $f^{(n)}(0)$ *as follows.*

$$\begin{aligned} f(x) &= AF(x) - \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} z\,q(xz)\,\Omega(z^2)\,dz, \\ F(x) &= \sum \alpha_n p(nx) \qquad\qquad (a > 0). \end{aligned} \tag{29b}$$

Here the continued fraction $\Omega(z)$ *is the development of the series*

$$\sum_{p=0}^{\infty} \frac{(-1)^p (AF^{(2p)}(0) - f^{(2p)}(0))}{p^{(2p)}(0)\, z^{p+1}}; \tag{29c}$$

and $q(ix) = p(x) + ig(x)$, *the function* $g(x)$ *being the one associated with* $p(x)$ *by means of Definition* I.

If $p(x)$ *is of subclass* B', *and the* α_n *are such that* $\sum a_n p(nx)$ *represents a quasi-analytic function, and the series* (29a) *diverges then* $f(x)$ *is representable by* (29b).

Suppose first that $p(x)$ is of subclass B. In the condition (29a) it is implied that the series

$$A_n = \sum_{m=0}^{\infty} \alpha_m m^{2n} \qquad (n = 0, 1, \cdots)$$

all converge.

Hence the series

$$f(x) = \sum a_n p(nx) \qquad (p(0) = 1;\ |a_n| < A\alpha_n) \tag{30}$$

and the series obtained by differentiating (30), term by term, any number of times are absolutely convergent for all real x. Since the problem (29) is determinate, by Theorem IV it follows that the series (30) and the various series of the derivatives represent a quasi-analytic function and its derivatives, respectively.

We have

$$AF^{(2p)}(x) - f^{(2p)}(x) = \sum_m (A\alpha_m - a_m)\, p^{(2p)}(mx)\, m^{2p}.$$

Hence

$$\frac{AF^{(2p)}(0) - f^{(2p)}(0)}{p^{(2p)}(0)} = \sum_{m=0}^{\infty} (A\alpha_m - a_m)\, m^{2p} > 0$$

since $|a_m| < A\alpha_m$ and the a_m are real.

The problem of moments

$$\int_0^\infty x^p \, dw(x) = \frac{A F^{(2p)}(0) - f^{(2p)}(0)}{p^{(2p)}(0)} = C_p \quad (>0)$$

has a solution which by substitution can be verified to be such that:

1° $w(x)$ is constant in $(m^2, (m+1)^2)$;

2° $w(x)$ has discontinuities

$$w(m^2+0) - w(m^2-0) = A\alpha_m - a_m \qquad (m = 0, 1, \cdots).$$

Observing that in virtue of the inequalities

$$A\alpha_n - a_n < 2A\alpha_n$$

it follows that

$$C_p < 2AA_p \qquad (p = 0, 1, \cdots),$$

we infer divergence of the series

$$\sum 1/\sqrt[2p]{C_p}$$

from that of the series (29a).

Hence, by a theorem of Carleman,[17] it follows that the problem of moments (29) is determinate. The function $w(x)$ is therefore the only solution.

By the theory of continued fractions

(30b)
$$\sum_{p=0}^{\infty} \frac{(-1)^p C_p}{z^{p+1}}$$

is developable into a continued fraction

$$\Omega(z) = \cfrac{1}{k_1 z + \cfrac{1}{k_2 + \cfrac{1}{k_3 z + \ddots}}}$$

(30c)
$$= \int_0^\infty \frac{dw(u)}{u+z} \qquad (k_m > 0;\ m = 1, 2, \cdots)$$

which is convergent except for negative values of z.[18]

We observe that, with $w(x)$ defined by (1°) and (2°),

$$\int_0^\infty p(\sqrt{u}\, t)\, dw(u) = \sum_{m=0}^{\infty} (A\alpha_m - a_m)\, p(mt).$$

Thus, this infinite integral represents $AF(t) - f(t)$. Since $p(x)$ belongs to B the lemma is applicable so that, using (27a),

[17] (C; 81).
[18] (C; 96).

$$AF(t)-f(t)=\int_0^\infty\left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{z\,q(tz)\,dz}{u+z^2}\right)dw(u).$$

By (30c) this becomes

$$AF(t)-f(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}z\,q(tz)\,\Omega(z^2)\,dz.$$

Since, with $p(x)$ belonging to B', both the necessary and sufficient conditions of Theorem IV hold, in this case it is not necessary to assume that the problem of moments is determinate. This problem will be determinate, by Theorem IV, in virtue of the assumption of quasi-analyticity of the function given by the series $\sum a_n\,p(nx)$. With the problem (29) determinate, the reasoning would be as for the case when $p(x)$ belongs to the subclass B; and the same representation (29b) will be derived.

The above method can be applied for the purpose of deriving representations in terms of initial values of functions, quasi-analytic for all real x, of the form

(31) $$f(x)=\sum_{n=1}^{\infty}\frac{a_n}{n^2x-i}\qquad(|a_n|<A\,\alpha_n;\ n=1,2,\cdots).$$ [19]

Since the class of quasi-analytic functions defined by (31) is very special we shall omit the proof.

Noting that

$$\frac{1}{t-i/u}=\frac{1}{2\pi i}\int_{-ia-\infty}^{-ia+\infty}\frac{z\,e^{2\pi i(u+z)}\,dz}{(tz+i)(u+z)}\qquad(a>0;\ 0<t<1/a)$$

and assuming that

$$\sum\frac{1}{\sqrt[2p]{A_p}}\qquad\left(A_p=\sum_{n=1}^{\infty}\alpha_n\,n^{2p}\right)$$

diverges, we observe that a quasi-analytic function $f(t)$, of the form (31), *is representable in terms of its initial values at $t=0$ as follows.*

(31a) $$f(t)=AF(t)-\frac{1}{2\pi i}\int_{-ia-\infty}^{-ia+\infty}\frac{z\,e^{2\pi iz}\,\Omega(z)\,dz}{tz+i}\qquad(0<t<1/a)$$

where the continued fraction $\Omega(z)$ is the development of the series

(31b) $$\sum_{p=0}^{\infty}\frac{(-i)^{p+1}(AF^{(p)}(0)-f^{(p)}(0))}{p!\,z^{p+1}},$$

and

(31c) $$F(t)=\sum_n\frac{\alpha_n}{n^2t-i}.$$

[19] The law of decrease of the $|a_n|$ can be so assigned that (31) should represent a quasi-analytic function. See Part I (Borel monogenic functions; § 1).

4. The method of linear inequalities. In this section we shall use the method due to de la Vallée Poussin and applied by him to trigonometric series.[20]

Let $K(\alpha, p)$ denote the class of functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n p(nx) \qquad (p(0) = 1; \ |a_n| < A\alpha_n) \tag{32}$$

where $p(x)$ is even and of class A. We recall that sufficient conditions for quasi-analyticity of $K(\alpha, p)$ are known when $p(x)$ is of class A, and both necessary and sufficient conditions for quasi-analyticity of $K(\alpha, p)$ are known when $p(x)$ is of subclass A' (§ 2).

We wish to solve the following problem.

If a set of values $f^{(2n)}(0) = C_{2n}$ belongs to a quasi-analytic class $K(\alpha, p)$ (*that is, if there exists a function $f(x)$ of class $K(\alpha, p)$ with these initial values at $x = 0$*), *determine the coefficients a_n of the series* (32).

By hypothesis there exist a_r $(r = 0, 1, \cdots)$ such that

$$C_{2i} = \sum_{r=0}^{\infty} p^{(2i)}(0)\, r^{2i} a_r \qquad (i = 0, 1, \cdots) \tag{32a}$$

We assign two positive numbers ε and n; ε being arbitrarily small and n—an integer arbitrarily great.

The series (32) being convergent, there exists a number p, depending on ε, n and the α_r, such that

$$\left| C_{2i} - \sum_{r=0}^{r=p} p^{(2i)}(0)\, r^{2i} a_r \right| \left(\leqq hA \sum_{r=p+1}^{\infty} r^{2i} \alpha_r \right) < \varepsilon \tag{32b}$$

$$(i = 0, 1, 2, \cdots, (n-1)).$$

The set of inequalities (32b) together with the inequalities

$$|a_r| < A\alpha_r \qquad (r = 0, 1, \cdots, p)$$

is satisfied by a particular system of solutions

$$a_0', \ a_1', \ \cdots, \ a_p'.$$

The function

$$\varphi(x) = a_0' + a_1' p(x) + \cdots + a_p' p(px) \tag{32c}$$

is an approximate solution of the problem.

Take a sequence of values

$$(\varepsilon_n, p_n) \qquad (n = 1, 2, \cdots; \ \varepsilon_n \to 0, \ p_n \to \infty).$$

Let $\varphi_n(x)$ be the function corresponding to ε_n, p_n. We shall show that a subsequence $(\varphi_{m_n}(x))$ exists such that

$$\lim_{n} \varphi_{m_n}(x) = \varphi(x). \tag{33}$$

[20] (V; 157, 162).

It is observed that, since the $a_0^1, a_1^1, \cdots$ satisfy the inequalities

$$|a'_m| < A\alpha_m,$$

each of the sets $(a'_0), (a'_1), \cdots$ is bounded, and consequently each set will have at least one limiting point. Choosing from the sequence (φ_n) a sequence for which $\lim a'_0$ exists, from this sequence we choose another sequence for which $\lim a'_1$ exists. Continuing this process, we derive a subsequence (φ_{m_n}) for which the limits

$$\lim a'_m = a^*_m \qquad (m = 0, 1, \cdots)$$

all exist. Noting that the $|a^*_m|$ satisfy inequalities of the same form as the $|a'_m|$, we have

$$\lim_n \varphi_{m_n}(x) = \varphi(x) = \sum_m a^*_m p(mx) \tag{33a}$$

where the series as well as the series derived by differentiating (33a), term by term, any number of times are all absolutely convergent for all real x. The function $\varphi(x)$ is of class $K(\alpha, p)$; moreover, it has the assigned initial values at $x = 0$. Hence

$$\varphi(x) = f(x).$$

If $p(x)$ is of subclass A' we have

$$\lim_n \varphi_n(x) = f(x).$$

Otherwise, say, the set (a'_0) would have at least two limiting points and there would be two series of the form (32) representing the same quasi-analytic function. The difference of these two series would be identically zero without all the coefficients of the series being zero. This would contradict the assumption that $p(x)$ is of class A'.

Let $K(\alpha; \beta)$ denote the class of functions of the form

$$f(z) = \bar{\varphi}(z) + \sum_{n,m=1}^{\infty} \frac{a_{n,m}}{(z-\gamma_n)^m}, \quad \bar{\varphi}(x) = \sum_0^{\infty} \frac{b_r z^r}{r!}, \tag{34}$$
$$|a_{n,m}| < A\alpha_{n,m}, \quad |b_r| < A\beta_r \qquad (n, m \geqq 1;\ r \geqq 0)$$

where the γ_n may be everywhere dense in some portions of the complex plane and $\gamma_n \neq 0$ $(n \geqq 1)$. The importance of this class is due to the fact, according to Theorem I (Part I; Borel monogenic functions), that every Borel monogenic function is representable by a series of the form (34) (except that $\varphi(x)$ may be in powers of $(x-d)$ with $d \neq 0$, but this is not essential). Moreover, taking account of (Part I, Borel monogenic

functions; § 1), it follows that conditions on the α_n, β_n can be found such that the following is true:

1°. The series (34) and all the series obtained by differentiating (34), term by term, are absolutely convergent in a region containing $z = 0$.

2°. The series (34) represents a function quasi-analytic in a region containing $z = 0$.

Let $K_0(\alpha; \beta)$ denote the class of functions belonging to $K(\alpha; \beta)$ and such that the conditions (1°), (2°) *hold.*

We shall solve the following problem.

Given that a set of values $f^{(n)}(0) = C_n$ belongs to class $K_0(\alpha; \beta)$, determine the coefficients b_n, $a_{n,m}$ of the series (34).

We have

$$f^{(i)}(0) = C_i = b_i + \sum_{n,m} \frac{(-1)^m m \cdots (m+i-1)}{\gamma_n^{m+i}} a_{n,m} \qquad (i = 0, 1, \cdots).$$

As before, we assign two numbers ε and n. The series involved here being all absolutely convergent, there exists a number (depending only on ε, n and the class) such that

$$(34\text{a}) \qquad C_i - \left[b_i + \sum_{r,m=1}^{r,m=p} \frac{(-1)^m m \cdots (m+i-1)}{\gamma_n^{m+i}} a_{r,m}\right] < \varepsilon$$
$$(i = 0, 1, \cdots, (n-1)).$$

Moreover,

$$(34\text{b}) \qquad |a_{r,m}| < A\alpha_{r,m}, \qquad |b_i| < A\beta_i$$
$$(1 \leqq r,\ m \leqq p; \quad 0 \leqq i \leqq n-1).$$

Thus, we have $p^2 + 2n$ inequalities (34a), (34b) involving $p^2 + n$ unknowns.

Let the numbers

$$a'_{r,m}\ (1 \leqq r,\ m \leqq p), \qquad b'_i\ (0 \leqq i \leqq n-1)$$

constitute a particular system of solutions.

We form the function

$$(34\text{c}) \qquad \varphi(z) = b'_0 + \frac{b'_1 z}{1!} + \cdots + \frac{b'_{n-1} z^{n-1}}{(n-1)!} + \sum_{r,m=1}^{r,m=p} \frac{a'_{r,m}}{(z-\gamma_r)^m}.$$

With the sequence (ε_n, p_n), formed as before, there is associated a sequence $(\varphi_n(z))$ of the form (34c).

Each of the sets (b'_i), as well as each of the sets $(a'_{r,m})$, is bounded; hence using the method of selecting sequences, applied before, we get a subsequence $(\varphi_{n'})$ for which the limits

$$(35) \qquad \lim b'_i = b^*_i$$

all exist. From this sequence by a similar process we select a sequence $(\varphi_{n''})$ for which not only the limits (35) but also the limits

$$\lim a'_{r,m} = a^*_{r,m} \tag{35a}$$

all exist.

Noting that the $|b^*_i|$ and the $|a^*_{r,m}|$ satisfy inequalities of the same form as the $|b'_i|$ and the $|a'_{r,m}|$, respectively, we have

$$\lim_{n''} \varphi_{n''}(z) = \varphi(z) = \sum_i \frac{b^*_i z^i}{i!} + \sum_{r,m=1}^{\infty} \frac{a^*_{r,m}}{(z-\gamma_r)^m} \tag{35b}$$

where $\varphi(z)$ is of class $K_0(\alpha;\beta)$. On the other hand,

$$\varphi^{(i)}(0) = C_i.$$

Consequently

$$\varphi(x) = f(x),$$

so that (35b) is a solution of the problem.

In connection with the above method it is impossible to find effectively necessary and sufficient conditions to be satisfied by the set (C_i) in order that there should exist a function of an assigned class. It is necessary and sufficient that each of the successive systems of linear inequalities should be compatible, however, as de la Vallée Poussin remarked[21] in his discussion of the cosine series, if we have encountered no incompatibility up to a certain step there is no certainty whether an incompatibility will or will not be encountered at a later stage.

However, at least theoretically, necessary and sufficient conditions on the initial values C_i may be found, in particular for the cosine series, applying a method involving least squares. While this method has been applied by de la Vallée Poussin for constructing a function, represented by a cosine series, in terms of its initial values, he has not pushed the method further to determine necessary and sufficient conditions on the C_i (in so far as they relate to a trigonometric series).[22]

This deficiency will be remedied in the following section not only for representations of quasi-analytic functions in the form of a trigonometric series, but also for representations by means of certain other series. The method for determining necessary and sufficient conditions on the initial values will be somewhat of the type used by Carleman in connection with a representation based on minimizing a certain definite integral.[23] In our

[21] (V; 162).
[22] (V; 160, 162).
[23] (C; 65, 73).

discussion the definite integral is replaced by a quadratic form. On the other hand quasi-analytic classes we define by the law of decrease of the moduli of the coefficients of certain series.

5. **The method of least squares.** In this section we shall modify the definitions used in the preceding section. Thus

DEFINITION I. *We let* $C(\alpha, p)$ *denote the class of functions of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n p(nx) \qquad (p(0) = 1;\ |a_n| < (A\alpha_n)^n) \tag{36}$$

where $p(x)$ *is even and of class A, the function* $p(x)$ *and the* a_n *being real.*

The following problem is proposed.

If a set of real numbers $f^{(2n)}(0) = C_{2n}$ *belongs to a quasi-analytic class* $C(\alpha, p)$, *determine the coefficients* a_n *of the series* (36).

It will be assumed that the quasi-analytic class $C(\alpha, p)$ is such that there exists another quasi-analytic class $C(\alpha^*, p)$ so that the series

$$\sum_n \left(\frac{A\alpha_n}{\alpha_n^*}\right)^{2n}, \qquad \sum_n \alpha_n^* n^{2i} \qquad (i = 0, 1, \cdots) \tag{36a}$$

converge.

Since the C_{2i} belong to a quasi-analytic class $C(\alpha, p)$ there exists a function

$$F(x) = \sum_{n=0}^{\infty} a_n p(nx), \qquad |a_n| < (A\alpha_n)^n, \tag{36b}$$

$$F^{(2n)}(0) = C_{2n} \qquad (n = 0, 1, \cdots). \tag{36c}$$

We have

$$C_{2i} = \sum_{n=0}^{\infty} p^{(2i)}(0)\, n^{2i} a_n \qquad (i = 0, 1, \cdots)$$

where the series all converge (absolutely).

Given ε_k, n_k we may chose p_k so that

$$\begin{aligned} \left| C_{2i} - \sum_{n=0}^{p_k-1} p^{(2i)}(0)\, n^{2i} a_n \right| &= \left| \sum_{n=p_k}^{\infty} p^{(2i)}(0)\, n^{2i} a_n \right| \\ &\leqq h \sum_{n=p_k}^{\infty} n^{2i} (A\alpha_n)^n < \sqrt{\frac{\varepsilon_k}{n_k}} \qquad (0 \leqq i \leqq n_k - 1). \end{aligned} \tag{37}$$

Consider the following positive quadratic form

$$E_i^*(y) = \sum_{r=0}^{n_i-1} \left[C_{2r} - \sum_{n=0}^{p_i-1} p^{(2r)}(0)\, n^r y_n \right]^2 + \varepsilon_i \sum_{n=0}^{p_i-1} \frac{y_n^2}{\alpha_n^{*2n}}. \tag{37a}$$

Let the

$$y_n = {}_i a_n \qquad (0 \leqq n \leqq p_i - 1)$$

minimize $E_i^*(y)$. Determination of the a_n will involve solving a set of linear equations. For the minimum we have

$$E_i^*({}_ia) \leqq E_i^*(a).$$

Now the a_n belong to the function $F(x)$ which, by hypothesis, is a solution and is therefore of class $C(\alpha, p)$.

Thus, using the inequalities

$$|a_n| < (A\,\alpha_n)^n$$

and (37), we get

$$E_i^*({}_ia) < n_i\left(\frac{\varepsilon_i}{n_i}\right) + \varepsilon_i \sum_{n=0}^{\infty}\left(\frac{A\,\alpha_n}{\alpha_n^*}\right)^{2n} \tag{38}$$

$$< G\cdot\varepsilon_i \qquad (i = 1, 2, \cdots)$$

since the series (36a) converges.

From (38) it follows that

$$\left| C_{2r} - \sum_{n=0}^{p_i-1} p^{(2r)}(0)\, n^{2r}\, {}_ia_n \right| < \sqrt{G}\,\varepsilon_i^{1/2}, \tag{38a}$$

$$|{}_ia_n| < \sqrt{G}\,\alpha_n^{*n} \qquad (0 \leqq n \leqq p_i - 1). \tag{38b}$$

We form a sequence

$$(\varepsilon_i, n_i) \qquad (i = 1, 2, \cdots;\ \varepsilon_i \to 0,\ n_i \to \infty)$$

and let

$$\varphi_i(x) = \sum_{n=0}^{p_i-1} {}_ia_n\, p(nx) \tag{38c}$$

be the function corresponding to ε_i, n_i. By (38b) the set $({}_ia_n)$, formed by letting $i = 1, 2, \cdots$, is bounded. Hence, using the process of § 4, we show that there exists a subsequence $(\varphi_{i'})$ such that the limits

$$\lim_{i'} {}_{i'}a_n = a_n^* \qquad (n = 0, 1, \cdots)$$

all exist.

Since the $|a_n^*|$ satisfy inequalities of the form (38b) and since the series (36a) converge, we have

$$\lim_{i'} \varphi_{i'}(x) = \sum_{n=0}^{\infty} a_n^*\, p(nx) = \varphi(x) \tag{39}$$

where $\varphi(x)$ is of class $C(\alpha^*, p)$; moreover, by (38a), we have

$$\varphi^{(2n)}(0) = C_{2n}.\text{[24]}$$

Letting

$$G(x) = F(x) - \varphi(x) = \sum_n (a_n - a_n^*)\, p(nx),$$

[24] The reasoning here is of the kind used by de la Vallée Poussin in (V; 161).

we note that $G^{(2n)}(0) = 0$. On the other hand,

$$|a_n - a_n^*| \leqq (A\alpha_n)^n + \sqrt{G}\,\alpha_n^{*n},$$

so that $G(x)$ is of class $C(\alpha^*, p)$, that is, quasi-analytic. Consequently $G(x) \equiv 0$ and

$$\varphi(x) = F(x).$$

Thus $\varphi(x)$, as given by (39), satisfies the problem. We have the sequence $(\varphi_i(x))$ itself convergent; thus

$$\lim \varphi_i(x) = F(x). \tag{39a}$$

In fact, if the sequence

$$\varphi_1, \cdots, \varphi_i, \cdots$$

does not converge, there exists a subsequence $(\varphi_{i''})$ such that

$$\lim \varphi_{i''}(x) = h(x)$$

where $h(x)$ is different from $\varphi(x)$ $\big(\varphi(x)$ being the limit of the sequence $(\varphi_i(x))\big)$. Now, by the method which we have used to show that $\varphi(x) = F(x)$, it can be shown that $h(x) = F(x)$. We have then $h(x) = \varphi(x)$. Consequently it is impossible for the sequence $(\varphi_i(x))$ to have two different limiting functions. Hence (39a) follows.[25] The possibility of two different series of the form (39) representing the solution is not excluded when $p(x)$ is of class A; this possibility is excluded when $p(x)$ is of subclass A'.

Noting that $E_i^*({}_ia)$ is a function of the C_{2k}, α_r, ε_i $(0 \leqq k \leqq n_i - 1;$ $0 \leqq n \leqq p_i - 1)$, we let

$$E_i^*({}_ia) = L_i(C, \alpha^*; \varepsilon_i). \tag{40}$$

From the preceding it is seen that *in order that the C_{2k} should be the initial values belonging to a quasi-analytic class $C(\alpha, p)$ it is necessary that the C_{2k} should satisfy the inequalities*

$$\frac{1}{\varepsilon^i} L_i(C, \alpha^*; \varepsilon_i) < G \qquad (i = 1, 2, \cdots) \tag{40a}$$

Assume that the C_{2i} belong to a quasi-analytic class $C(\alpha, p)$. Let $F(x)$ be the function of this class

$$F(x) = \sum_0^\infty a_n\, p(nx), \qquad |a_n| < (A\alpha_n)^n, \qquad F^{(n)}(0) = C_n. \tag{41}$$

Form $E_i(y)$, replacing in (37a) the α_n^* by the α_n. Let p_i be defined as before (by means of the α_n). The minimum of $E_i(y)$ can be obtained

[25] This reasoning is of the type used by Carleman in (C; 70, 71).

replacing in (40) the α_n^* by the α_n. Letting the ${}_i a_n'$ $(0 \leqq n \leqq p_i - 1)$ denote the minimizing constants, this minimum will be given by

$$E_i({}_i a') = L_i(C, \alpha; \varepsilon_i).$$

Now, with the a_n satisfying (41), we have

$$\text{(41 a)} \qquad \begin{aligned} L_i(C, \alpha; \varepsilon_i) \leqq E_i(a) &= \sum_{r=0}^{n_i-1} \left[C_{2r} - \sum_{n=0}^{p_i-1} p^{(2r)}(0)\, n^{2r} a_n \right]^2 + \varepsilon_i \sum_{n=0}^{p_i-1} \frac{a_n^2}{\alpha_n^{2n}} \\ &< n_i \left(\frac{\varepsilon_i}{n_i}\right) + \varepsilon_i \sum_{n=0}^{\infty} \left(\frac{A\alpha_n}{\alpha_n}\right)^{2n} = \left(1 + \sum_0^\infty A^{2n}\right) \varepsilon_i = G' \varepsilon_i \end{aligned}$$

provided $A < 1$.

On the other hand, sufficiency of the inequalities (41 a) can be proved easily. Accordingly, we shall state the following theorem.

THEOREM VI. *Let $A < 1$. In order that an assigned set of real initial values C_{2k} should belong to a quasi-analytic class $C(\alpha, p)$ it is necessary and sufficient that the sequence*

$$\frac{1}{\varepsilon_i} L_i(C, \alpha; \varepsilon_i) \qquad (i = 1, 2, \cdots)$$

should be bounded. ($A < 1$ *for the necessary part, else* $A = 1$).

We shall consider now the analogous problem for quasi-analytic functions representable in terms of rational fractions.

It will be assumed that *the C_n, b_n, $a_{n,m}$, γ_n ($\gamma_n \neq 0$) are all real.*

Let $C(\alpha; \beta)$ denote the class of functions of the form

$$\text{(42)} \qquad \begin{aligned} &f(z) = \bar{\varphi}(z) + \sum_{n,m=1}^{\infty} \frac{a_{n,m}}{(z-\gamma_n)^m}, \quad \bar{\varphi}(z) = \sum_0^\infty \frac{b_r z^r}{r!}, \\ &|a_{n,m}| < (A\alpha_{n,m})^{n+m}, \quad |b_r| < (A\beta_r)^r \qquad (n, m \geqq 1;\ r \geqq 0). \end{aligned}$$

DEFINITION II. *Let $C_0(\alpha; \beta)$ denote the class of functions belonging to $C(\alpha; \beta)$; and let $C_0(\alpha; \beta)$ be such that the conditions* (1°), (2°) (§ 4), *as applied to* (42), *hold.*

We assume, moreover, that the quasi-analytic class $C_0(\alpha; \beta)$ is such that there exists another quasi-analytic class $C_0(\alpha^*; \beta^*)$ so that the series

$$\text{(42 a)} \qquad \sum_{n,m} \left(\frac{A\alpha_{n,m}}{\alpha_{n,m}^*}\right)^{2(m+n)}, \qquad \sum_r \left(\frac{A\beta_r}{\beta_r^*}\right)^{2r}$$

converge.

We shall solve the following problem.

If a set of real numbers $f_{(0)}^{(i)} = C_i$ belongs to a quasi-analytic class $C_0(\alpha; \beta)$, determine the coefficients $a_{n,m}$, b_r of the series (42).

Let

$$F(z) = \sum_{0}^{\infty} \frac{b_i z^i}{i!} + \sum_{n,m=1}^{\infty} \frac{a_{n,m}}{(z-\gamma_n)^m}, \tag{42b}$$

$$|a_{n,m}| < (A\alpha_{n,m})^{n+m}, \quad |b_r| < (A\beta_r)^r \qquad (n, m \geqq 1;\ r \geqq 0),$$

be the function for which $F^{(i)}_{(0)} = C_i$ $(i = 0, 1, \cdots)$.

Choose p_k so that

$$\begin{aligned} &\left| C_i - \left[b_i + \sum_{n,m=1}^{n,m=p_k} \frac{(-1)^m m \cdots (m+i-1)}{\gamma_n^{m+i}} a_{n,m} \right] \right| \\ &\quad \left(\leqq \sum_{n,m=p_k+1}^{\infty} \frac{m \cdots (m+i-1)}{|\gamma_n|^{m+i}} |a_{n,m}| \right. \\ &\quad \left. < \sum_{n,m=p_k+1} \frac{m \cdots (m+i-1)}{|\gamma_n|^{m+i}} (A\alpha_{n,m})^{n+m} \right) \\ &\quad < \sqrt{\frac{\varepsilon_k}{n_k}} \qquad\qquad (0 \leqq i \leqq n_k - 1). \end{aligned} \tag{42c}$$

This is possible since the double series

$$\sum_{n,m}^{\infty} \frac{m \cdots (m+i-1)}{|\gamma_n|^{m+i}} (A\alpha_{n,m})^{n+m} \qquad (i = 0, 1, 2, \cdots)$$

all converge.

Consider the following positive quadratic form

$$\begin{aligned} F_i^*(x; y) = &\sum_{r=0}^{n_i-1} \left[C_r - \left(x_r + \sum_{n,m=1}^{p_i} \frac{(-1)^m m \cdots (m+r-1)}{\gamma_n^{m+r}} y_{n,m} \right) \right]^2 \\ &+ \varepsilon_i \sum_{n,m=1}^{p_i} \frac{y_{n,m}^2}{\alpha^{*2n+2m}_{n,m}} + \varepsilon_i \sum_{r=0}^{n_i-1} \frac{x_r^2}{\beta_r^{*2r}}. \end{aligned} \tag{43}$$

Solving a set of linear equations we obtain the constants

$$x_r = {}_ib_r, \quad y_{n,m} = {}_ia_{n,m},$$

which minimize (43). Let $F_i({}_ib; {}_ia)$ be the minimum.

Using (42b) and (42c), we have

$$\begin{aligned} &F_i^*({}_ib; {}_ia) \leqq F_i^*(b; a) \\ &< n_i \left(\frac{\varepsilon_i}{n_i} \right) + \varepsilon_i \sum_{n,m} \left(\frac{A\alpha_{n,m}}{\alpha^*_{n,m}} \right)^{2n+2m} + \varepsilon_i \sum_r \left(\frac{A\beta_r}{\beta_r^*} \right)^{2r} \end{aligned} \tag{43a}$$

(in (43a) the b and the a belong to the function $F(z)$ and therefore satisfy inequalities (42b)). In virtue of the convergence of the series (42a), we have

$$F_i^*({}_ib; {}_ia) < G\varepsilon_i. \tag{43b}$$

Consequently

$$\left| C_r - \left({}_ib_r + \sum_{n,m=1}^{p_i} \frac{(-1)^m m \cdots (m+r-i)}{\gamma_n^{m+r}} \, {}_ia_{n,m} \right) \right| < \sqrt{G}\,\varepsilon_i^{1/2},$$

(43c)
$$|{}_ia_{n,m}| < \sqrt{G}\,(\alpha^*_{n,m})^{n+m}, \qquad |{}_ib_r| < \sqrt{G}\,(\beta^*_r)^r$$
$$(1 \leqq n,\, m \leqq p_i;\; 0 \leqq r \leqq n_i - 1).$$

We form a sequence

$$(\varepsilon_i,\, n_i) \qquad (i = 1, 2, \cdots;\; \varepsilon_i \to 0,\; n_i \to \infty).$$

Let

(44)
$$\varphi_i(z) = \sum_{r=0}^{n_i-1} \frac{{}_ib_r\, z^r}{r!} + \sum_{n,m=1}^{p_i} \frac{{}_ia_{n,m}}{(z-\gamma_n)^m}$$

be the function corresponding to ε_i, n_i. By a process of selecting subsequences we get a subsequence $(\varphi_{i''})$ for which

$$\lim {}_{i''}b_r = b^*_r, \qquad \lim {}_{i''}a_{n,m} = a^*_{n,m}.$$

We observe that the $|b^*|$ and the $|a^*|$ satisfy inequalities of the same form as the $|{}_ib|$ and the $|{}_ia|$. We have

(44a)
$$\lim \varphi_{i''}(z) = \varphi(z) = \sum_r \frac{b^*_r\, z^r}{r!} + \sum_{n,m} \frac{a^*_{n,m}}{(z-\gamma_n)^m}$$

where $\varphi(z)$ is of class $C_0(\alpha^*;\, \beta^*)$. Now the series (44a) is differentiable, term by term, any number of times, the resulting series being all convergent (absolutely) in a region containing $z = 0$. Hence, by (43c), it follows that

$$\varphi^{(r)}(0) = C_r \qquad (r \geqq 0).$$

Form

(44b)
$$G(z) = F(z) - \varphi(z) = \sum_r \frac{b_r - b^*_r}{r!} z^r + \sum_{n,m} \frac{a_{n,m} - a^*_{n,m}}{(z-\gamma_n)^m}.$$

Here

$$|b_r - b^*_r| < (A\beta_r)^r + \sqrt{G}\,\beta^{*r}_r,$$
$$|a_{n,m} - a^*_{n,m}| < (A\,\alpha_{n,m})^{n+m} + \sqrt{G}\,(\alpha^*_{n,m})^{n+m}.$$

Hence the function $G(z)$ is of class $C_0(\alpha^*;\, \beta^*)$; moreover, $G^{(n)}(0) = 0\,(n \geqq 0)$. Consequently $G(x) \equiv 0$, and

$$\varphi(z) = F(z).$$

Thus $\varphi(z)$, as defined by (44a), is a solution of the problem. Using the reasoning employed before, we show that

$$\lim \varphi_i(z) = \varphi(z) \;(= F(z)).$$

We note that the assumption that the C_i are initial values belonging to $C_0(\alpha;\beta)$ implies that the sequence of the minima,

$$\begin{aligned} &F_i^*({}_ib;{}_ia) = G_i(C;\gamma,\alpha^*,\beta^*;\varepsilon_i) \\ &(C_k,\beta_k^*\ (0 \leqq k \leqq n_i-1);\ \alpha_{n,m}^*,\gamma_n\ (1 \leqq n,\ m \leqq p_i)), \end{aligned} \tag{44c}$$

satisfies the inequalities

$$\frac{1}{\varepsilon_i} G_i(C;\gamma,\alpha^*,\beta^*;\varepsilon_i) < G \qquad (i = 1, 2, \cdots). \tag{45}$$

Thus, *in order that the C_k should be initial values belonging to a quasi-analytic class $C_0(\alpha;\beta)$ it is necessary that the C_k should satisfy the inequalities* (45).

Assume that the C_k belong to a quasi-analytic class $C_0(\alpha;\beta)$. Let $F(x)$ be the function of this class, given by (42b). Form $F_i(x;y)$ (in (43) replace the α^*, β^* by the α, β). Let p_i be defined as before. Let the numbers $x = {}_ib'$, $y = {}_ia'$ render $F_i(x;y)$ a minimum. We have

$$F_i({}_ib';{}_ia') = G_i(C;\gamma,\alpha,\beta;\varepsilon_i)$$

where the function G_i is the same as in (44c). Since the $a_{n,m}$ and the b_r satisfy (42b), it follows that

$$\begin{aligned} &G_i(C;\gamma,\alpha,\beta;\varepsilon_i) \leqq F_i(b;a) \\ &= \sum_{r=0}^{n_i-1}\left\{C_r - \left[b_r + \sum_{n,m=1}^{p_i} \frac{(-1)^m m\cdots(m+r-1)}{\gamma_n^{m+r}} a_{n,m}\right]\right\}^2 \\ &+ \varepsilon_i \sum_{n,m=1}^{p_i} \frac{a_{n,m}^2}{\alpha_{n,m}^{2n+2m}} + \varepsilon_i \sum_{r=0}^{n_i-1} \frac{b_r^2}{\beta_r^{2r}} \\ &< n_i\left(\frac{\varepsilon_i}{n_i}\right) + \varepsilon_i \sum_{n,m=1}^{p_i}\left(\frac{A\alpha_{n,m}}{\alpha_{n,m}}\right)^{2n+2m} + \varepsilon_i \sum_{r=0}^{n_i-1}\left(\frac{A\beta_r}{\beta_r}\right)^{2r} \\ &< G'\varepsilon_i \end{aligned} \tag{45a}$$

provided $A<1$. Thus we see that the inequalities (45a) are necessary; sufficiency of these inequalities can be proved without difficulty. Accordingly, we state the following theorem.

THEOREM VII. *Let $A<1$. In order that an assigned set of initial values (real) should belong to a quasi-analytic class $C_0(\alpha;\beta)$ it is necessary and sufficient that the sequence*

$$\frac{1}{\varepsilon_i} G_i(C;\gamma,\alpha,\beta;\varepsilon_i) \qquad (i = 1, 2, \cdots)$$

be bounded. ($A<1$ *for the necessary part, else* $A = 1$).

BROWN UNIVERSITY.

THE LAPLACE DIFFERENTIAL EQUATION OF INFINITE ORDER.[1]

By H. T. Davis.

1. **Introduction.** The object of the present study is the Laplace differential equation of infinite order,

$$(1) \qquad \sum_{n=0}^{\infty} (a_{n0} + a_{n1} x + a_{n2} x^2 + \cdots + a_{np} x^p)\, u^{(n)}(x) = f(x),$$

where p is a positive integer and not all the quantities a_{np} are zero.

It will be immediately observed that the general theory of equation (1) formally unifies the theories of the following essentially different types of linear functional equations in which the $p_i(x)$ are polynomials of degree not greater than p:

$$\text{(a)} \qquad u(x) + \int_x^{\infty} \sum_{i=1}^{m} p_i(x)\, \varphi_i(t-x)\, u(t)\, dt = f(x),$$

where we assume that the $\varphi_i(x)$ behave at infinity in such a manner that $\int_0^{\infty} \varphi_i(s)\, s^n\, ds$ exists for all values of n;

$$\text{(b)} \qquad u(x) + \int_a^{b} \sum_{i=1}^{m} p_i(x)\, q_i(t)\, u(x+ct)\, dt = f(x);$$

(c) $p_m(x)\, u(x+m) + p_{m-1}(x)\, u(x+m-1) + \cdots + p_0(x)\, u(x) = f(x)$;

(d) the Laplace differential equation of finite order.

The formal equivalence of these types with equation (1) is exhibited by means of the following Taylor's expansion:

$$u(T) = u(x) + (T-x)\, u'(x) + (T-x)^2\, u''(x)/2! + \cdots.$$

If we replace T by t and substitute in (a) we obtain an equation,

$$(2) \qquad P_0(x)\, u(x) + P_1(x)\, u'(x) + \cdots + P_n(x)\, u^{(n)}(x) + \cdots = f(x),$$

the coefficients of which are,

$$P_0(x) = 1 + \sum_{i=1}^{m} p_i(x) \int_0^{\infty} \varphi_i(s)\, ds, \quad P_n(x) = \sum_{i=1}^{m} p_i(x) \int_0^{\infty} \varphi_i(s)\, s^n\, ds/n!,$$
$$n > 0.$$

Similarly for (b) we let $T = x + ct$ and obtain equation (2) with the coefficients:

[1] Received May 19 and November 25, 1930. Presented to the American Mathematical Society, November 30, 1928.

$$P_0(x) = 1+\int_a^b \sum_{i=1}^m p_i(x)\, q_i(t)\, dt, \quad P_n(x) = \int_a^b \sum_{i=1}^m p_i(x)\, q_i(t)\, (ct)^n\, dt/n!, \quad n>0.$$

Equation (c) is transformed into the desired type by replacing T by $x+r$, $r = 0, 1, 2, \cdots, m$. We thus obtain the coefficients:

$$P_0(x) = \sum_{i=0}^m p_i(x), \quad P_n(x) = \sum_{i=1}^m i^n\, p_i(x)/n!, \qquad n>0.$$

The Laplace equation of finite order is derived from (1) by assuming that $a_{ni} = 0$ for all values of n greater than a fixed n'.

In this paper we shall be concerned only incidentally with the case for which $p = 0$, since this equation already has an extensive literature of its own.[2] This case, however, is essentially included in the statements of Theorems 4 and 6. The theory of what is now generally referred to as the Heaviside operational calculus, devised by Oliver Heaviside to simplify the study of electrical currents, in complicated net works, is also embraced by proper specialization.[3]

Historically the first discussion of equation (1) for $p>0$ was made by T. Lalesco[4] in 1908 who, incidental to a consideration of the problem of the inversion of Volterra integrals, applied to the homogeneous case the Laplace transformation,

$$u(x) = \int_L e^{xt}\, v(t)\, dt,$$

where the path is a conveniently chosen one depending upon the coefficients of the equation. This method was later developed in more detail by E. Hilb.[5]

O. Perron, returning to the problem in 1921,[6] derived a limitation upon

[2] See especially: C. Bourlet, Annales de l'École Normale Supérieure, 3d ser., vol. 14 (1897), pp. 133–190; T. J. Bromwich, Proc. London Math. Soc., vol. 15 (1916), pp. 401–445; H. T. Davis, Amer. Journal of Math., vol. 52 (1930), pp. 97–108; H. von Koch, Arkiv för Mat., Astro., och Fysik, vol. 15 (1921), No. 26, pp. 1–16; S. Pincherle, Memorie della R. Accademia di Bologna, ser. 4, vol. 9 (1886), pp. 45–71; Acta Mathematica, vol. 48 (1926), pp. 279–304; J. F. Ritt, Trans. Amer. Math. Soc., vol. 18 (1917), pp. 27–49; F. Schürer, Leipziger Berichte, vol. 70 (1918), pp. 185–246; I. M. Sheffer, Annals of Math., vol. 30 (1929), pp. 250–264; Trans. Amer. Math. Soc., vol. 31 (1929), pp. 250–264; G. Valiron, Annales de l'École Normale Supérieure, 3d ser., vol. 46 (1929), pp. 25–53; N. Wiener, Mathematische Annalen, vol. 95 (1926), pp. 557–584.

[3] For this see the author's paper: loc. cit., p. 107.

[4] Journal de Mathématique, vol. 73 (1908), pp. 125–202.

[5] Mathematische Annalen, vol. 82 (1920–21), pp. 1–39, vol. 84 (1921), pp. 16–30, vol. 84, pp. 43–52.

[6] Mathematische Annalen, vol. 84 (1921), pp. 31–42. See section 6 of this paper.

the number of solutions of equation (1) by means of an application of known results concerning the solution of the linear system,

$$\sum_{n=0}^{\infty}(a_n + b_{mn})\,x_{m+n} = c_m, \qquad m = 0, 1, 2, \cdots.$$

At the same time E. Hilb investigated the non-homogeneous equation by a method of considerable novelty. He replaced (1) by an infinite number of equations in an infinite number of unknowns through unlimited differentiation of the equation. The system thus obtained was found to come under the Hilbert-Schmidt theory of linear equations in an infinite number of unknowns as applied to Laurent forms and a solution of the system was thus explicitly obtained.[7] A distinguishing feature of Hilb's investigation was the attainment of conditions assuring the uniqueness of a solution obtained by means of an operator expansible as a power series in z.

I. M. Sheffer in two papers published in 1929[8] considered the details of solution for the cases $p = 0$ and $p = 1$. In the first of these, writing the equation in the form, $\{A_0(z) + xA_1(z)\} \rightarrow u(x) = f(x)$, $z = d/dx$, he discussed the cases (a) $A_1(z) = z - a$, and (b) $A_1(z) = (z - a)(z - b)$, and further showed that if $A_1(z)$ has r zeros of multiplicities $p_1, p_2, \cdots, p_r$, then equation (1) can be replaced by an equation of finite degree m, where $m = p_1 + p_2 + \cdots + p_r$.

In the second paper Sheffer employed methods similar to those used in somewhat more general considerations by S. Pincherle (loc. cit.), reducing equation (1) by means of a Laplace transformation to a contour integral equation and expressing the resolvent kernel by means of a second contour integral. He further proved that if $f(x)$ is expansible in a series of Appell polynomials, then a solution $u(x)$ can be expressed simply in terms of the coefficients of the expansion.

All writers who have previously studied equation (1) have restricted their investigations to the class of unlimitedly differentiable functions $\{g(x)\}$ of bounded *grade* (Stufe). By this term we shall mean a value q defined as the following limit:

$$\limsup_{n=\infty} q_n = q,$$

where we abbreviate, $q_n = |g^{(n)}(x)|^{1/n}$.

As the author has shown elsewhere, however, important equations are excluded from consideration by this limitation. For example the solution

[7] See E. Hellinger and O. Toeplitz, Grundlagen für eine Theorie der unendlichen Matrizen. Mathematische Annalen, vol. 69 (1910), pp. 289–330.

[8] (1) Annals of Mathematics, vol. 30 (1929), pp. 345–372; (2) Trans. Amer. Math. Soc., vol. 31 (1929), pp. 261–280.

of such equations as $u(x+1)-u(x)=1/x$ and $u(x+1)-xu(x)=0$, that is to say, the differential equations $(e^z-1)\rightarrow u(x)=1/x$, and $(e^z-x)\rightarrow u(x)=0$, $z=d/dx$, are important functions of unbounded grade. The exclusion of these equations from the applications of a general theory would appear to be an unfortunate blemish upon it. It is to be admitted that difficulties are thus introduced which appear usually in the form of divergent series, but in many cases these solutions can be rescued by means of Borel's method of summability[9] or can be identified as asymptotic series in the sense of Poincaré.[10]

The first object of the present paper is to discuss the theory of the Laplace equation by means of the calculus of operators. In this development we shall exhibit the efficacy of the Pincherle-Bourlet method of symbolic operators in which the *generatrix equation* instead of an infinite system of equations plays the dominant rôle.[11] By this means the formal solution of equation (1) can be reduced to three useful forms.

The second object of the paper is to discuss the validity of the solutions and the domain of functions to which the operators apply. A first step in generalization has been taken through the admission to the permissible domain of those functions which possess an infinite number of derivatives in the neighborhood of a point, but for which q_n may approach infinity as $O(n^a)$, $0\leqq a\leqq 1$. In this manner we adjoin to the domain considered by previous writers all entire functions $(0\leqq a<1)$ and all analytic functions with poles $(a=1)$. The difficulties admitted by this extension have not been entirely resolved, however, since they have been discovered to be inherent in the nature of asymptotic and summable series, the theory of which is still obscure in many points.

2. **The generatrix equation.** We shall survey briefly that part of the Pincherle-Bourlet theory which is necessary for our purposes.

Let us consider an operator developable in a power series:

$$F(x,z)\equiv a_0(x)+a_1(x)z+\cdots+a_n(x)z^n+\cdots, \tag{3}$$

where z^n is symbolic for the differential operator d^n/dx^n. The functions $a_i(x)$ we shall assume to be analytic in a common region R for every value of which the power series in z possesses a radius of convergence not smaller than $\varrho>0$.

By the symbol $F(x,z)\rightarrow f(x)$ we mean the function obtained when $F(x,d/dx)$ operates upon $f(x)$. We shall postpone consideration of the

[9] E. Borel, Leçons sur les séries divergentes, Paris (1901), chap. 4.

[10] H. Poincaré, Acta Mathematica, vol. 8 (1886), pp. 295–344.

[11] This method, although used also by Sheffer (loc. cit., p. 266), was known some years ago by the author. See Bull. Amer. Math. Soc., vol. 32 (1926), p. 221.

limitations to be imposed upon $f(x)$ to a later section and shall examine here only the formal aspects of the theory. Designating by $f_i(x)$ the function obtained by operating with $F(x, z)$ upon x^i, i an integer or zero, that is to say, $f_i(x) = F(x, z) \to x^i$, one easily demonstrates that the coefficients of (3) are expressible in terms of $f_i(x)$. We thus obtain:

$$(4) \qquad a_n(x) = \{f_n - nxf_{n-1} + n(n-1)x^2 f_{n-2}/2! - \cdots + (-1)^n x^n f_0\}/n!.$$

Let us consider a second operator,

$$(5) \qquad X(x, z) \equiv b_0(x) + b_1(x)z + b_2(x)z^2 + \cdots,$$

with coefficients analytic in R and convergent in z within a circle of radius $\varrho > 0$.

Bourlet designates the operation of $X(x, z)$ upon $F(x, z)$ by the symbol $[X \cdot F](x, z)$. It is occasionally useful to employ the symbol $X \to F$ for the same operation. This symbolic product in general defines a new operator which Bourlet shows to be formally equivalent to the series,

$$(6) \qquad \begin{aligned} X \to F = [X \cdot F](x, z) &= F(x, z)X(x, z) + (\partial F/\partial x)(\partial X/\partial z) \\ &+ (\partial^2 F/\partial x^2)(\partial^2 X/\partial z^2)/2! + \cdots + (\partial^n F/\partial x^n)(\partial^n X/\partial z^n)/n! + \cdots. \end{aligned}$$

If $F(x, z)$ is a polynomial in x or $X(x, z)$ a polynomial in z then series (6) is of finite order and $[X \cdot F]$ is again an analytic function in z for some domain R' of x. Since the former constitutes the circumstances of the present paper, it will be unnecessary to examine here the general case where (6) is an infinite series.

Let us consider the functional equation,

$$(7) \qquad F(x, z) \to u(x) = f(x).$$

It is obvious that the formal solution of (7) depends upon the discovery of a new operator, $X(x, z)$, which will solve the *generatrix equation*,

$$(8) \qquad [X \cdot F](x, z) = 1.$$

This operator we shall call the *resolvent generatrix*. It is clear that its rôle is analogous to that of the resolvent kernel in the solution of integral equations.

In general $X(x, z)$ will depend upon several arbitrary functions of x and may be written in the form,

$$X(x, z) = \sum_{i=1}^{s} C_i(x) X_i(x, z) + X_0(x, z),$$

where $X_i(x, z)$, $i = 1, 2, \cdots, s$, are s linearly independent solutions of the equation,

$$[X \cdot F](x, z) = 0.$$

Let $X(x, z)$ be expanded in a power series in z as in equation (5). Then $X(x, z) \rightarrow x^i$, i an integer or zero, is a solution of the equation,

$$F(x, z) \rightarrow u(x) = x^i.$$

Designating this solution by $u_i(x)$ we see from (4) that the coefficients of (5) and hence $X(x, z)$ itself are formally expansible in terms of a set of special solutions of (7), namely, $u_i(x)$, $i = 0, 1, 2, \cdots$. Since the coefficients of (5) are $b_r(x) = (\partial^r X/\partial z^r|_{z=0})/r!$, we obtain the identity:[12]

$$(9) \quad (\partial^r X/\partial z^r)|_{z=0} = u_r(x) - r x u_{r-1} + r(r-1)x^2 u_{r-2}/2! - \cdots \pm x^r u_0.$$

3. Calculation of the resolvent generatrix. Returning to equation (1) we define $F(x, z)$ to be the series,

$$F(x, z) = \sum_{n=0}^{\infty} (a_{n0} + a_{n1} x + a_{n2} x^2 + \cdots + a_{np} x^p) z^n.$$

The solution of equation (7) is reduced by (6) and (8) to the problem of finding a solution of the equation,

$$(10) \quad A_0(x,z)X(x,z) + A_1(x,z)\,\partial X/\partial z + A_2(x,z)\,\partial^2 X/\partial z^2 + \cdots + A_p(x,z)\,\partial^p X/\partial z^p = 1,$$

where we abbreviate,

$$A_r(x, z) = \sum_{n=0}^{\infty} \sum_{m=r}^{p} a_{nm}\, m!\, x^{m-r} z^n/(m-r)!\, r!.$$

In order to integrate (10) let us assume that $X(x, z)$ is a function of the form $X(x, z) = e^{-xz} X(z)$. Taking successive derivatives with respect to z we obtain,

$$\partial^n X(x, z)/\partial x^n = e^{-xz}\,[X^{(n)}(z) - n x X^{(n-1)}(z) + n(n-1)x^2 X^{(n-2)}(z)/2! - \cdots$$
$$\cdots + (-1)^n x^n X(z)], \qquad n = 1, 2, 3, \cdots.$$

When these values are substituted in (10) an interesting simplification takes place and the generatrix equation reduces to,

$$(11) \quad \{A_0(z) X(z) + A_1(z) X'(z) + A_2(z) X''(z) + \cdots + A_p(z) X^{(p)}(z)\}\, e^{-xz} = 1,$$

[12] Sheffer refers to the functions $u_r(x)$ as *Hurwitz polynomials* for the case where $F(x, z) = z A(z) + B(z) + x z B(z)$, $B(0) \neq 0$.

where the $A_i(z)$ are the functions,

$$A_i(z) = \sum_{n=0}^{\infty} a_{ni} z^n. \tag{12}$$

Making the convenient restriction that a_{0p} shall be different from zero,[13] the homogeneous equation,

$$A_0(z) X(z) + A_1(z) X'(z) + A_2(z) X''(z) + \cdots + A_p(z) X^{(p)}(z) = 0, \tag{13}$$

is seen to have $z = 0$ as a regular point. There will then exist p linearly independent solutions regular at the origin which we shall designate by $X_1(z)$, $X_2(z)$, $X_3(z)$, $\cdots$, $X_p(z)$.

Similarly the adjoint of equation (13), the *adjoint resolvent*,

$$A_0(z) Y(z) - \frac{d}{dz}[A_1(z) Y(z)] + \cdots + (-1)^p \frac{d^p}{dz^p}[A_p(z) Y(z)] = 0, \tag{14}$$

will have a set of p linearly independent solutions, $Y_1(z)$, $Y_2(z)$, $Y_3(z)$, $\cdots$, $Y_p(z)$, regular at the origin. As is well known[14] these solutions can be calculated in terms of the $X_i(z)$ by means of the formula,

$$Y_i(z) = (\partial \log W / \partial X_i^{(p-1)}) / A_p(z), \tag{15}$$

where W is the Wronskian, $W = |X_i^{(j-1)}|$, $i, j = 1, 2, \cdots, p$.

The solution of equation (11) may then be written in terms of the sets, $\{X_i(z)\}$, $\{Y_i(z)\}$, and we thus obtain as the formal value of the operator $X(x, z)$ the function,

$$X(x, z) = e^{-xz}\left\{\sum_{i=1}^{p} C_i(x) X_i(z) + \int_0^z e^{xt} W(z, t)\, dt\right\}, \tag{16}$$

where we make use of the abbreviation, $W(z, t) = X_1(z) Y_1(t) + X_2(z) Y_2(t) + \cdots + X_p(z) Y_p(t)$. The functions $C_i(x)$ thus far enter the solution arbitrarily, but we shall see that they are uniquely defined.

It is well known that $W(z, t)$ has the following properties:[15]

$$\begin{aligned} &\partial^{i+j} W(z, t)/\partial z^i \partial t^j |_{t=z} = 0, && i+j < p-1, \\ &\partial^{i+j} W(z, t)/\partial z^i \partial t^j |_{t=z} = (-1)^j / A_p(z), && i+j = p-1. \end{aligned} \tag{17}$$

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[13] When this is not the case Perron (loc. cit., p. 41) suggests the transformation, $u(x) = e^{ax} v(x)$. In the transformed equation $A_p(z)$ as defined by (12) becomes $A_p(z) = \sum_{n=0}^{\infty} a_{np}(z+a)^n$ and $A_p(0) = \sum_{n=0}^{\infty} a_{np} a^n$.

[14] For this and other relations between the adjoint functions see G. Darboux, La théorie des surfaces, Paris (1889), vol. 2, pp. 99–106.

[15] See Darboux, loc. cit., p. 103.

From these relations it is clear that we shall have,

$$\begin{aligned}\partial^r X(x,z)/\partial z^r|_{z=0} &= (\partial^r/\partial z^r)\{e^{-xz}[C_1(x)X_1(z)+\cdots+C_p(x)X_p(z)]\}_{z=0}\\ &= \left(\frac{d}{dz}-x\right)^r \to \{C_1(x)X_1(z)+\cdots+C_p(x)X_p(z)\}|_{z=0},\end{aligned}$$

for r less than or equal to $p-1$.

If this last equation be expanded and the coefficients of x compared with the coefficients of x in equation (9) we obtain the following system:

$$C_1(x)X_1^{(i)}(0)+\cdots+C_p(x)X_p^{(i)}(0) = u_i(x), \quad i = 0, 1, 2, \cdots, p-1.$$

If we further specialize the functions $X_i(z)$ so as to satisfy the conditions,

$$X_i^{(j-1)}(0) = \delta_{ij}, \tag{18}$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$, the arbitrary functions are identified with the special solutions $u_i(x)$, namely $C_i(x) = u_{i-1}(x)$.

We may formulate these conclusions in the following theorem:

THEOREM 1. *The resolvent generatrix* $X(x,z)$ *of equation* (1) *is expansible in the form*

$$X(x,z) = e^{-xz}\left[\sum_{i=1}^{p} u_{i-1}(x)X_i(z)+\int_0^z e^{xt}W(z,t)\,dt\right], \tag{19}$$

where the $u_i(x)$, $i = 0, 1, 2, \cdots, p-1$, *are solutions of equation* (1) *when* $f(x) = x^i$, $X_i(z)$ *are solutions of equation* (13) *subject to the defining conditions* (18) *and* $W(z,t)$ *is defined by* (17).

We proceed next to an explicit determination of the solutions $u_i(x)$. Designating by $H_j[Y](t)$ and $I_k[Y](t)$ the following functions:

$$H_j[Y](t) = A_j(t)Y(t)-[A_{j+1}(t)Y(t)]'+\cdots+(-1)^{p-j}[A_p(t)Y(t)]^{(p-j)}, \tag{20}$$

$$\begin{aligned}I_k[Y](t) = \int_0^t [&(t-s)^{k-1}A_0(s)Y(s)/(k-1)!-(t-s)^{k-2}A_1(s)Y(s)/(k-2)!\\ &+\cdots+(-1)^k A_{k-1}(s)Y(s)]\,ds,\end{aligned} \tag{21}$$

we can state the following theorem:

THEOREM 2. *The solutions* $u_i(x)$ *of theorem* 1 *are explicitly determined from the formula,*

$$u_{i-1}(x) = U(x)-\int_{l_i} e^{xt}Y_i(t)\,dt, \tag{22}$$

where $U(x)$ *is a solution of the homogeneous equation and* l_i *is a path of integration between zero and any other point in the complex plane for which* $H_j[Y_i](t)e^{xt}|_{l_i} = 0$, $j = i, i+1, \cdots, p$, and $I_k[Y_i](t)e^{xt}|_{l_i} = 0$, $k = 1, 2, \cdots, i-1$, *where the* $Y_i(t)$ *are the adjoints of* $X_i(t)$ *as defined by* (15) *and* (18).

Proof. We first observe that our definition of the solutions $X_i(t)$ at $t=0$, condition (18), has uniquely determined the values of the adjoints $Y_i(t)$ at the same point. Thus setting $t=z$ in equations (17) we have the bilinear relations,

$$\{\partial^i W(z, t)/\partial z^i\}|_{t=z} = 0, \qquad i<p-1;$$
$$\{\partial^i W(z, t)/\partial z^i\}|_{t=z} = 1/A_p(z), \qquad i=p-1.$$

Setting $z=0$ and noting the initial values of $X_i(z)$ we obtain, $Y_i(0)=0$, $i=1,2,\cdots,p-1$, $Y_p(0)=1/A_p(0)$. Similarly letting $j=1$ in equations (17) we get $Y_i'(0)=0$, $i=1, 2, \cdots, p-2$, $Y_{p-1}'(0)=-1/A_p(0)$; and in general,

$$(23) \quad Y_i^{(q)}(0) = 0, \; i=1,2,\cdots,p-q-1, \qquad Y_{p-q}^{(q)}(0) = (-1)^q/A_p(0).$$

From these values and those obtained by successive differentiation of equations (17) we obtain the following numerical results:

$$(24) \quad H_j[Y_i](0)=0, \;\; j=0,1,2,\cdots,i-1,i+1,\cdots,p, \; H_i[Y_i](0)=1.$$

Assuming proper convergence of the integral in (22) let us differentiate $v_{i-1}(x)=u_{i-1}(x)-U(x)$ n times. We thus obtain,

$$v_{i-1}(x) = -\int_{l_i} e^{xt}\, t^n\, Y_i(t)\, dt.$$

Substituting this function in the expression $F(x,z)\rightarrow u(x)$ we get,

$$(25) \quad F(x,z)\rightarrow v_{i-1}(x) = -\int_{l_i} e^{xt}\, Y_i(t)\{A_0(t)+A_1(t)\,x+\cdots+A_p(t)x^p\}\, dt.$$

Let us define now a new function,

$$U_k(A) = \int_0^t Y(s)\, A(s)\,(t-s)^{k-1}\, ds/(k-1)!,$$

which evidently has the property,

$$d^r U_k(A)/dt^r = U_{k-r}(A), \qquad k \geqq r.$$

Making use of this relationship and integrating by parts over any path L free from singularities we establish the identity,

$$\int_L Y(t)\, A(t)\, e^{xt}\, dt = \{U_1(A) - x\, U_2(A) + \cdots + (-1)^{k-1} x^k\, U_k(A)\}\, e^{xt}|_L$$
$$+ (-1)^k x^k \int_L e^{xt}\, U_k(A)\, dt.$$

Employing this identity and recalling the definition of $I_k(t)$ we obtain the following expansion:

$$\int_L e^{xt}\, Y(t)\,[A_0(t)+A_1(t)\,x+\cdots+A_{i-2}(t)\,x^{i-2}]\,dt$$

$$(26)\quad = \{I_1(t)-x I_2(t)+x^2 I_3(t)-\cdots+(-1)^{i-2}x^{i-2} I_{i-1}(t)\}\, e^{xt}|_L$$
$$+(-1)^{i-1}x^{i-1}\int_L e^{xt} I_{i-1}(t)\,dt.$$

A different integration by parts yields the following identity:

$$\int_L Y(t)\,A(t)\,e^{xt}\,dt$$

$$(27)\quad = \{YA/x-(YA)'/x^2+(YA)''/x^3-\cdots+(-1)^{k-1}(YA)^{(k-1)}/x^k\}\,e^{xt}|_L$$
$$+(-1)^k\int_L e^{xt}\,(YA)^{(k)}\,dt/x^k.$$

Making use of this expansion and recalling the definition (20) of $H_j(t)$ we obtain the following:

$$(28)\quad \int_L e^{xt}\, Y(t)\,[A_{i-1}(t)\,x^{i-1}+A_i(t)\,x^i+\cdots+A_p(t)\,x^p]\,dt$$
$$= \{x^{i-1}H_i(t)+x^i H_{i+1}(t)+\cdots+x^{p-1}H_p(t)\}\,e^{xt}|_L+x^{i-1}\int_L e^{xt} H_{i-1}(t)\,dt.$$

Combining (26) and (28), in which we now make the specialization $L=l_i$, we get for (25) the expression:

$$F(x,z)\to v_{i-1}(x)$$

$$(29)\quad = -\left\{\sum_{k=1}^{i-1}(-1)^{k-1}I_k[Y_i](t)\,x^{k-1}+\sum_{j=i}^{p}H_j[Y_i](t)\,x^{j-1}\right\}e^{xt}|_{l_i}$$
$$-x^{i-1}\int_{l_i}e^{xt}\{H_{i-1}[Y_i](t)+(-1)^{i-1}I_{i-1}[Y_i](t)\}\,dt.$$

We next establish the identity,

$$(30)\quad I_{i-1}[Y_i](t)+(-1)^{i-1}H_{i-1}[Y_i](t)=\int_0^t(t-s)^{i-2}H_0[Y_i](s)\,ds/(i-2)!.$$

To prove this consider the integral,

$$\int_0^t H_0[Y_i](s)\,ds = \{I_1[Y_i](t)-H_1[Y_i](t)\}|_0^t.$$

From equation (24) we see that $H_1[Y_2](0)=0$ and the identity is established for $i=2$. Similarly we have

$$\int_0^t(t-s)\,H_0[Y_i](s)\,ds = \{I_2[Y_i](t)+H_2[Y_i](t)\}|_0^t.$$

From this integration and the observation that $H_2[Y_3](0)=0$, the identity is established for $i=3$. The extension to the general case follows in an identical manner.

Returning now to equation (29) and recalling (30) we see that the integral of the right member vanishes identically since $H_0[Y_i](s)$ is identically zero. Furthermore we have $I_k[Y_i](0) = 0$ for every k and from (24) $H_i[Y_i](0) = 1$, $H_j[Y_i](0) = 0$, $j \neq i$. Hence, making the assumptions of Theorem 2 that there exists a path l_i from zero to some point a_i such that $\lim_{t=a_i} I_k[Y_i](t)e^{xt} = 0$, $k = 1, 2, \cdots, i-1$, $\lim_{t=a_i} H_j[Y_i](t)e^{xt} = 0$, $j = i, i+1, \cdots, p$, then it is clear that $u_{i-1}(x)$ defined by (22) is a solution of equation (1) in which $f(x) = x^{i-1}$. The negative sign is chosen for the integral of (22) since we have arbitrarily assumed that zero is the lower limit of integration.

COROLLARY. If the conditions of Theorem 1 are satisfied and if there exists a *unique limit* a for all the paths l_i satisfying the conditions of Theorem 2, then the resolvent generatrix of equation (1) takes the form,

$$X(x, z) = U(x)\, e^{-xz} X(z) + e^{-xz} \int_a^z X_i(z)\, Y_i(t)\, e^{xt}\, dt. \tag{31}$$

Since an arbitrary solution of the homogeneous equation is $C_i\, U(x)$, we get for the term of (19), independent of the integral,

$$U(x)\, e^{-xz} \sum_{i=1}^{p} C_i X_i = U(x)\, e^{-xz} X(z).$$

4. **Expansions of the resolvent.** It will now be desirable to state three useful forms which can be assumed by the resolvent generatrix. These expansions are contained in the following theorem:

THEOREM 3. *If there exists a point a in the complex plane for which we have,*

$$\lim_{t=a} H_j[Y_i](t)\, e^{xt} = 0, \qquad j = i, i+1, \cdots, p;$$
$$\lim_{t=a} I_k[Y_i](t)\, e^{xt} = 0, \qquad k = 1, 2, \cdots, i-1,$$

then the resolvent (16) *can be expanded in the following series:*

$$\text{(32) (A)} \qquad X(x, z) = U(x)\, e^{-xz} X(z) + \sum_{i=1}^{\infty} \psi_m(x)\, z^m/m!,$$

where we abbreviate,

$$\psi_m(x) = u_m(x) - m x\, u_{m-1}(x) + m(m-1)\, x^2\, u_{m-2}(x)/2! - \cdots \pm x^m u_0(x), \qquad m \leqq p-1,$$

$$\psi_m(x) = \sum_{i=1}^{p} \{u_{i-1}\}(x)\, [X_i]^m + m\{Y_i\}^0\, [X_i]^{m-1} + m(m-1)\, \{Y_i\}\, [X_i]^{m-2}/2! + \cdots + \{Y_i\}^{m-1}\, [X_i]^0, \qquad m > p-1,$$

in which we write,

$$u_i(x) = -\int_a^0 e^{xt}\, Y_i(t)\, dt,$$

$$[X_i]^m = X_i^{(m)}(0) - m x X_i^{(m-1)}(0) + m(m-1)x^2 X_i^{(m-2)}(0)/2! - \cdots \pm x^m X_i(0),$$
$$\{Y_i\}^m = Y_i^{(m)}(0) + m x Y_i^{(m-1)}(0) + m(m-1)x^2 Y_i^{(m-2)}(0)/2! + \cdots + x^m Y_i(0).$$

If the additional conditions are fulfilled,

$$\lim_{t=a} e^{xt} Y_i^{(m)}(t) = 0, \qquad m = 0, 1, 2, \cdots$$

then the resolvent has also the expansions,

$$\begin{aligned} X(x, z) &= U(x)\, e^{-xz} X(z) \\ (33)\ \ (\mathrm{B}) \qquad &+ \{(-1)^{p-1} e^{z(p-1)}/x(x+1)\cdots(x+p-1)\} \\ &\times \{w_{p-1}(z, z) - w_p(z, z)\, e^z/(x+p) \\ &\qquad + w_{p+1}(z, z)\, e^{2z}/(x+p)(x+p+1) - \cdots\}, \end{aligned}$$

where we abbreviate,

$$(34) \qquad w_n(z, t) = \left\{\frac{\partial}{\partial t} w_{n-1}(z, t)\right\} e^{-t},\ w_0(z, t) = W(z, t) = \sum_{i=1}^{p} X_i(z)\, Y_i(t).$$

$$\begin{aligned} (35)\ \ (\mathrm{C}) \qquad X(x, z) &= U(x)\, e^{-xz} X(z) + \{(-1)^{p-1}/x^p\} \\ &\times \{W_{p-1}(z, z) - W_p(z, z)/x + W_{p+1}(z, z)/x^2 - \cdots\}, \end{aligned}$$

in which we use the notation,

$$(36) \qquad W_n(z, t) = \frac{\partial}{\partial t} W_{n-1}(z, t), \quad W_0(z, t) = W(z, t) = \sum_{i=1}^{p} X_i(z)\, Y_i(t).$$

Proof. The derivation of the formulas contained in this theorem proceeds from the formal expansion of the operator (31). If $X_i(z)\, e^{-xz}$ and $\int_a^z e^{xt}\, Y_i(t)\, dt$ be expanded in power series in z and account taken of the defining conditions (18) and (23) we obtain (32). (A) is thus the power series solution of the generatrix equation (10).

Assuming next that a point exists such that $\lim_{t=a} e^{xt} Y_i^{(m)}(t) = 0$, we can perform the following integration by parts:

$$\begin{aligned} \int_a^z e^{xt}\, W(z, t)\, dt &= e^{xz}\, W(z, z)/x - \int_a^z e^{(x+1)t} \{\partial W(z, t)/\partial t\}\, e^{-t}\, dt/x \\ &= e^{xz}\, W(z, z)/x - e^{xz} e^z \{e^{-t}\, \partial W(z, t)/\partial t\}_{t=z}/x(x+1) \\ &\quad + \int_a^z e^{(x+2)t} \frac{\partial}{\partial t} \{[\partial W(z, t)/\partial t]\, e^{-t}\}\, e^{-t}\, dt/x(x+1), \\ &= e^{xz}\{w_0(z, z)/x - w_1(z, z)\, e^z/x(x+1) \\ &\qquad + w_2(z, z)\, e^{2z}/x(x+1)(x+2) - \cdots\}, \end{aligned}$$

where we use the abbreviations (34).

From (17) however, we observe that the first $p-2$ terms are identically zero and we thus obtain expression (33).

Finally the operator (C) is derived by means of a similar integration by parts and the employment of (17) to eliminate the first $p-2$ terms.

An interesting illustration is furnished by the classical equation,

$$(37) \qquad u(x+1)-xu(x) \equiv (e^z-x)\rightarrow u(x) = f(x).$$

Computing the values of $X(z)$ and $Y(z)$ from the equations, $X'(z)-e^z X(z)=0$, $Y'(z)+e^z Y(z)=0$, subject to the restrictions, $X(0)=1$, $X(z)\,Y(z)=-1$, we find: $X(z) = e^{e^z}/e$, $Y(z) = -e\cdot e^{-e^z}$.

Obviously the point a may be either $+\infty$ or $-\infty$. Choosing the latter, introducing new variables $u=e^t$, $p=e^z$, and recalling that the general solution of the homogeneous equation is $U(x)=\Pi(x)\,\Gamma(x)$, where $\Pi(x)$ is any function of unit period, we have,

$$X(x, z) = e^{-xz}\,X(z)\,\Pi(x)\,\Gamma(x)-p^{-x}\int_0^p u^{x-1}\,e^{p-u}\,du.$$

The integral is recognized as the incomplete gamma function which has the expansion,

$$\gamma(p)=\int_0^p u^{x-1}\,e^{p-u}\,du = p^x/x+p^{x+1}/x(x+1)+p^{x+2}/x(x+1)\,(x+2)+\cdots.$$

Noting the operational identities, $p^n\rightarrow f(x)=f(x+n)$ and $e^{-xz}X(z)\rightarrow f(x)$ $=$ a constant, the solution of equation (37) becomes,[16]

$$u(x)=\Pi(x)\Gamma(x)-\{f(x)/x+f(x+1)/x(x+1)+f(x+2)/x(x+1)(x+2)+\cdots\}.$$

Employing a second expansion of $\gamma(p)$ we obtain,

$$p^{-x}\,\gamma(p) = e^p\,p^{-x}\,\Gamma(x)-\{1/p+(x-1)/p^2+(x-1)\,(x-2)/p^3+\cdots\}.$$

Operating upon $f(x)$ with this new form of the operator, we get,

$$\begin{aligned} u(x) = \Pi(x)\,\Gamma(x)-\{f(0)+f(1)/1!+f(2)/2!+\cdots\}\,\Gamma(x) \\ +\{f(x-1)+(x-1)f(x-2)+(x-1)\,(x-2)f(x-3)+\cdots\}. \end{aligned}$$

The power series expansion of the operator is found from (32) to be

$$\begin{aligned} X(x,z)=\Pi(x)\Gamma(x)\,e^{-xz}X(z)-\int_0^1 u^{x-1}e^{1-u}du\{1+(1-x)z+(2-2x+x^2)z^2/2 \\ +(5-6x+3x^2-x^3)z^3/6+\cdots\}-\{z+(1-x)z^2/2+(3-2x+x^2)z^3/6+\cdots\}. \end{aligned}$$

5. Resumé of the properties of the grade (Stufe) of unlimitedly differentiable functions. Having attained in the preceding section the resolvent operator and the formal expansions to which it leads we now

[16] This is a well known result due to H. Mellin, Zur Theorie der linearen Differenzengleichungen erster Ordnung. Acta Mathematica, vol. 15 (1891), pp. 317–384.

propose to show how the convergence, or in the case of divergence, how the asymptotic and summable character of the formal solutions can be established by means of the properties of the grade (Stufe) of functions which possess differential coefficients of all orders over a range $a \leqq x \leqq b$ of the independent variable.

It will serve our purpose merely to recall here without proof how far an infinitely differentiable function may be characterized by its grade, the meaning of this term having been explained in section 1.

If $f(x)$ is an analytic function with a pole, branch point, or essential singularity in the finite plane then q is infinite. Moreover, $q_n = O(n)$. If $f(x)$ is an entire function of genus $p > 1$, then q is infinite and $q_n = o[n^{p/(p+1)}]$. If $f(x)$ is an entire function of genus 1, then q is finite and different from zero. If $f(x)$ is an entire function of genus zero then $q = 0$.[17]

It will also be convenient to have the following theorems relating to the grade of a function $f(x)$ where q is assumed to be finite:[18]

A. If $f(x)$ is a function of grade q, then $f^{(m)}(x)$ and $\int_a^m \cdots \int_a^m f(t)\, dt^m$ are of grade q.

B. If $f_1(x)$ and $f_2(x)$ are of grades q_1 and q_2 respectively where $q_1 \leqq q_2$, then $a_1 f_1(x) + a_2 f_2(x)$ will be at most of grade q_2.

C. If $f(x)$ is of grade q then $f(ax)$ is of grade $|a|q$.

D. If $f_1(x)$ is of grade q_1 and $f_2(x)$ is of grade q_2, then $f(x) = f_1(x) f_2(x)$ is at most of grade $q_1 + q_2$.

6. **The homogeneous equation.** Before we can discuss the grade of functions obtained by an application of the general operators of Theorem 3 it will be necessary to know the function-theoretic character of the solution of the homogeneous equation,

$$F(x, z) \to U(x) = 0.\tag{38}$$ [19]

Theorems stated in subsequent sections of this paper apply only to the restricted form of the Laplace operator obtained by omitting the term $e^{-xz} X(z)\, U(x)$. Hence a knowledge of the grade of $U(x)$, when such solutions exist, permits an immediate extension of these subsequent theorems

[17] See H. Poincaré, Bull. de la Soc. Math. de France, vol. 11 (1882–83), pp. 136–144.

[18] See O. Perron, loc. cit., pp. 31–32.

[19] We assume that there exists no factor common to all the $A_i(z)$. If such a factor, $B(z)$, does exist then the solution of (38) will contain as a subset the solutions of $B(z) \to U(x) = 0$, since $F(x, z) \to B(z) = [F \cdot B]\,(x, z) = F(x, z)\, B(z)$. If $v(x)$ is a solution of the equation reduced by removing the factor $B(z)$ then the general solution is obtained by the inversion of the differential equation of infinite order with constant coefficients: $B(z) \to U(x) = v(x)$.

to include the case of the general operator. Because, however, limitations of space and the extensive nature of the considerations involved preclude an adequate discussion in this place, we shall give only a resumé of the results for the homogeneous equation.

Proceeding by classical methods we assume the existence of a solution in the form of a Laplace transformation, $U(x) = \int_L e^{xt}\, Y(t)\, dt$. If we refer to equation (28) and set $i = 1$, we conclude that $U(x)$ will be a formal solution of (38) provided $Y(t)$ is a solution of the equation $H_0[Y](t) = 0$, more explicitly (14), and L is a path at the extremities of which we have,

$$H_i[Y](t)\, e^{xt}|_L = 0, \qquad i = 1, 2, \cdots, p. \tag{39}$$

The determination of the path L is the problem of the homogeneous equation. For the purposes of the ensuing discussion it will be sufficient merely to state that the solution of (38) falls into four convenient classes depending upon the following choices of L:

If n is the number of zeros of $A_p(z)$ (multiplicity counted but in no case exceeds p), then there exists:

(1) $n - p$ Pochhammer circuits about pairs of zeros of $A_p(z)$;

(2) s Cauchy loops corresponding to zeros of $A_p(z)$ in the neighborhood of which there exists a regular solution of equation (13);

(3) $p - s$ loops to infinity about zeros of $A_p(z)$ in the neighborhood of which the solution of (13) possesses a branch-point singularity;

(4) Circuits not included in the first three classes. These circuits are frequently infinite in number.

The classical equation, $(e^z - x) \to U(x) = 0$, is an example of this last class. This equation leads to the solution, $U(x) = \int_L e^{-e^z + xz}\, dz$, where the path may be any line from $+\infty$ to $-\infty$ within regions bounded by the lines $y = (4n - 1)\pi/2$ and $y = (4n + 1)\pi/2$, $n = 0, 1, 2, \cdots$. The solution yields the classical result $U(x) = \pi(x) \int_0^\infty e^{-t}\, t^{x-1}\, dt = \pi(x)\, \Gamma(x)$, where $\pi(x)$ is any function of unit periodicity. Because of the polar singularities of $\Gamma(x)$ the solution is of infinite grade.

Perron employing quite different methods has proved the following theorem which defines the solutions of finite grade:

If in equation (13) *all the functions $A_i(z)$ are analytic in the circle $|z| \leqq q$ and if within the domain considered there exist n zeros* (*multiplicity counted and in all instances assumed less than or equal to p*) *of $A_p(z)$, then there will exist $n - p + s$ linearly independent integrals of* (38) *the grades of which do not exceed q, where s is the number of linearly independent integrals of* (13) *analytic within the circle $|z| \leqq q$.*

Since classes (1) and (2) above coincide with the class of solutions of finite grade obtained by Perron, we are able to infer that solutions corresponding to the circuits defined by (3) and (4) are of infinite grade. Class (3), however, is closely related to classes (1) and (2) since the essential nature of these solutions is determined from the regular singularities of the operator. Class (4) on the contrary is determined by the essential singularities of the operator as one sees from the classical example stated above.

7. Concerning the grades of functions generated by special operators. We can now establish the following theorems relating to the grades of functions obtained by applying differential operators of Laplace type to functions of both finite and infinite grades:

THEOREM 4. *If $f(x)$ is a function of grade q and if $F(z)$ is an operator with constant coefficients which possesses a Laurent expansion about $z = 0$ within an annulus that contains $z = q$ within it or upon its interior boundary, then the function $F(z) \to f(x)$ is at most a function of grade q.*

Proof: Expanding $F(z)$ in its Laurent series in the annulus described we shall have,

$$F(z) = (a_0 + a_1 z + a_2 z^2 + \cdots) + (b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} + \cdots).$$

Operating upon the function $F(z) \to f(x)$ with z^n and taking the absolute value of the result we obtain,

$$\begin{aligned} |z^n \to \{F(z) \to f(x)\}| &= |F(z) \to \{z^n \to f(x)\}| \\ \leqq \{|a_0 f^{(n)}| + |a_1 f^{(n+1)}| + \cdots\} &+ |b_1 f^{(n-1)}| + |b_2 f^{(n-2)}| + \cdots + |b_n f| \\ &+ \left| \int_a^x [b_{n+1} + b_{n+2}(x-t) + b_{n+3}(x-t)^2/2! + \cdots] f(t)\, dt \right|. \end{aligned} \tag{40}$$

From (A) of section 5 we know that if x be restricted to some interval $(a\, b)$ we can find a sequence of positive numbers M_n, $n = 0, 1, 2, \cdots$, such that $\lim_{n=\infty} [M_n]^{1/n} = 1$, for which the following inequality holds:

$$|f^{(n)}(x)| \leqq M_n q^n.$$

Let us assume, moreover, that $M_n \leqq M_{n+1}$ so that the sequence $\{M_n\}$ is non-decreasing and thus has no null elements. Under this condition we know that $\lim_{n=\infty} [M_n]^{1/n} = 1$ implies the limit $\lim_{n=\infty} M_{n+p}/M_n = 1$ for every value of p.

Replacing the various terms of (40) by these dominating values and using the abbreviation $m_p(n) = M_{n+p}/M_n$, we then get,

$$|z^n \to \{F(z) \to f(x)\}| \leqq q^n M_n [I_1(n) + I_2(n)] + I_3(n),$$

where we abbreviate,

$$I_1(n) = \sum_{r=0}^{\infty} |a_r| \, m_r(n) q^r, \qquad I_2(n) = \sum_{r=1}^{n} |b_r| \, m_{-r}(n) q^{-r},$$

$$I_3(n) = \int_a^x \sum_{m=0}^{\infty} |b_{n+m+1}| (x-t)^m f(t) \, dt/m!.$$

From the condition, $\lim_{n=\infty} m_r(n) = 1$, which holds for every value of r, we are able to conclude that $m_r(n)$ is dominated by $Kq^{\delta r}$, where K is a constant independent of n and r, and δ is an arbitrarily small positive quantity. Since, moreover, q is an *interior point* of the annulus of convergence of the Laurent expansion of $F(z)$, the series $I_1 = K \sum_{r=1}^{\infty} |a_r| q^{(1+\delta)r}$ converges and we have the inequality, $I_1(n) \leqq I_1$, for all values of n.

Moreover we obtain from $M_n \leqq M_{n+1}$, the inequality $m_{-r} \leqq 1$, $0 \leqq r \leqq n$, and hence conclude that, $I_2(n) \leqq \sum_{r=1}^{\infty} |b_r| q^{-r} = I_2$, for all values of n.

Finally since $\lim_{r=\infty} |b_r| = 0$ and since $f(t)$ is bounded in (a, b) we can write, $I_3(n) \leqq FB \int_a^b e^{(x-t)} \, dt = I_3$, where $|f(t)| < F$ and $|b_r| < B$.

Taking the nth root of the inequality,

$$|z^n \to \{F(z) \to f(x)\}| \leqq q^n M_n (I_1 + I_2) + I_3,$$

and noting the inequality, $[A^n + B^n]^{1/n} < A + B$, for A and B positive, we obtain, $\lim_{n=\infty} |z^n \to \{F(z) \to f(x)\}|^{1/n} \leqq q$, from which the theorem follows as an immediate consequence.

Theorem 4 which applies to differential equations with constant coefficients may be extended to operators of the type introduced by the solution of the Laplace equation. This result may be stated as follows:

THEOREM 5. *If an operator $F(x, z)$ can be represented in the form,*

$$F(x, z) = e^{-xz} \int_0^z e^{xt} \, Y(t) \, dt,$$

then the function defined by $F(x, z) \to f(x)$ exists and is of grade q provided $f(x)$ is of grade q and $Y(z)$ is analytic throughout the interior of the circle $\varrho = R$, where R is greater than q.

Proof. Since $f(x)$ is of grade q it is dominated by a function of the type $P(x) e^{qx} + G(x)$, where $P(x)$ is an entire function of genus zero and $G(x)$ is a function of grade lower than q. We can, therefore, limit our attention to the operation $F(x, z) \to P(x) e^{qx}$.

We shall begin with the identity,[20]

(41) $F(x, z) \to (uv) = uF(x, z) \to v + u' F'(x, z) \to v + u'' F''(x, z) \to v/2! + \cdots.$

[20] See S. Pincherle, Funktionaloperationen und Gleichungen. Enc. Math. Wiss., II A 11, pp. 761–817, in particular, p. 769.

Specializing this formula by setting $u = P(x)$ and $v = e^{qx}$ we get

$$(42)\quad F(x, z) \to Pe^{qx} = \{PF(x, q) + P' F'_q(x, q) + P'' F''_q(x, q)/2! + \cdots\} e^{qx}.$$

Employing the abbreviation $D = d/dq$ we also note that $D^n \to F(x, q) = e^{-qx}(D - x)^n \to I(x, q)$, where we set,

$$(43)\qquad I(x, q) = \int_0^q e^{xt}\, Y(t)\, dt.$$

Substituting this in (42) we obtain,

$$(44)\qquad F(x, z) \to Pe^{qx} = \sum_{n=0}^{\infty} P^{(n)}(x)\,(D - x)^n \to I(x, q)/n!.$$

But we see from (43) that $D \to I(x, q) = e^{qx}\, Y(q)$, $D^2 \to I(x, q) = e^{qx}(D + x) \to Y(q)$, and in general,

$$(45)\qquad D^n \to I(x, q) = e^{qx}(D + x)^{n-1} \to Y(q).$$

Making use of this we are able to derive,

$$\begin{aligned}(D - x)^n &\to I(x, q)\\ &= e^{qx}\{(D+x)^{n-1} - nx(D+x)^{n-2} + n(n-1)x^2(D+x)^{n-3}/2! - \cdots \pm nx^{n-1}\}\\ &\qquad \to Y(q) + (-1)^n x^n I(x, q)\\ &= e^{qx}\{[D^n - (-1)^n x^n]/(D + x)\} \to Y(q) + (-1)^n x^n I(x, q).\end{aligned}$$

But since $Y(q)$ is analytic its n-th derivative, by (4) section 5, is dominated by $M_n a^n n!$ where $\lim_{n=\infty} (M_n)^{1/n} = 1$. Hence the function $\{[D^n - (-1)^n x^n]/(D + x)\} \to Y(q)$ is dominated by $M_n A^n n!$ where A is suitably chosen. But since $P(x)$ is an entire function of genus zero we can find a set of values m_n such that $|P^{(n)}(x)| < m_n/Q^n$, where Q is arbitrary and $(m_n)^{1/n} \to 1$. From this we see that the series expansion of $F(x, z) \to P(x)\, e^{qx} - P(0)\, I(x, q)$ is dominated by the majorant,

$$(46)\qquad \sum_{n=0}^{\infty} M_n\, m_n\, (A/Q)^n, \quad Q > A, \qquad \lim_{n=\infty} (M_n m_n)^{1/n} = 1.$$

We are thus able to conclude that (44) is uniformly convergent.

In order now to show that $F(x, z) \to Pe^{qx}$ is a function of grade q we observe the identity, $\sum_{n=0}^{\infty} P^{(n)}(x)\,(D - x)^n/n! = P(D)$, and write (44) in the form,

$$F(x, z) \to Pe^{qx} = P(D) \to I(x, q).$$

Making use of (45) we achieve the further simplification,

$$P(D) \to I(x, q) = P(0)\, I(x, q) + e^{qx}\, Q(D + x) \to Y(q),$$

where we write $Q(z) = [P(z) - P(0)]/z$.

Expanding $Q(D+x)$ as a series in D and operating upon $Y(q)$ we get,

$$Q(D+x)\to Y(q) = Q(x)\,Y(q) + Q'(x)\,Y'(q) + \cdots + Q^{(m)}(x)\,Y^{(m)}(q)/m! + \cdots.$$

Let us now operate upon this equation with z^n and discuss the result.

$$S_n(x) = z^n \to \{Q(D+x)\to Y(q)\} = Q^{(n)}(x)\,Y(q) + Q^{(n+1)}(x)\,Y'(q) + Q^{(n+2)}(x)\,Y''(q)/2! + \cdots + Q^{(n+m)}(x)\,Y^{(m)}(q)/m!\cdots.$$

Since $Y(z)$ is analytic throughout the interior of the circle $\varrho = R$, $Y^{(m)}(q)$ is dominated by $M_m\,a^m\,m!$, $q \leqq a < R$, $\lim_{m=\infty}(M_m)^{1/m} = 1$, $M_m > 0$. Moreover, since $Q(x)$ is an entire function of genus zero, $Q^{(n)}(x)$ is dominated by a sequence of positive values ψ_n which has the property that $\lim_{n=\infty} A^n\,\psi_n = 0$ for all finite values of A.[21] We can also assume without loss of generality that $\psi_n \geqq \psi_{n+1}$. Hence we attain the inequality,

$$|S_n(x)| \leqq \sum_{m=0}^{\infty} M_m\,\psi_{n+m}\,a^m.$$

Making use of the limiting property of the sequence $\{\psi_n\}$ we see that this series converges and hence furnishes a Weierstrass majorant for the function $S_n(x)$ for every value of n and for x in any finite interval.

Furthermore, from the assumption $\psi_n \geqq \psi_{n+1} > 0$, we obtain,

$$|S_n(x)| \leqq \psi_n \sum_{m=0}^{\infty} M_m\,(\psi_{n+m}/\psi_n)\,a^m.$$

But from the limiting property of the sequence $\{\psi_n\}$ there exists a positive quantity C, independent of n, such that $\psi_n < C/A^n$, $A > a$. Hence we can write,

$$|S_n(x)| \leqq A^n\,\psi_n \sum_{m=0}^{\infty} M_m\,(a/A)^m.$$

From the limitations upon M_m this series is seen to converge and the majorant thus obtained establishes the fact that $S_0(x) = Q(D+x)\to Y(q)$ is a function of genus zero in the variable x.

Combining these results we can write equation (44) in the form,

$$F(x, z)\to Pe^{qx} = P(0)\,I(x, q) + e^{qx}\,S(x), \tag{47}$$

where $S(x)$ is a function of genus zero.

We now need the lemma:

LEMMA. *The grade of $I(x, q)$ does not exceed q.*

The proof is immediately attained by use of the Schwarz inequality,

$$\int_0^q |u^2(t)|\,dt \int_0^q |v^2(t)|\,dt \geqq \left|\int_0^q u(t)\,v(t)\,dt\right|^2.$$

[21] See J. F. Ritt, loc. cit. [2], p. 34.

Forming the n-th derivative of $I(x,q)$, $I^{(n)}(x,q)=\int_0^q e^{xt}\,t^n\,Y(t)\,dt$, we specialize $u(t)=e^{xt}\,Y(t)$, $v(t)=t^n$, and thus obtain,

$$|\,I^{(n)}(x,q)\,|\leqq\left\{\int_0^q|\,e^{2xt}\,Y^2(t)\,|\,dt\right\}^{1/2}q^n\,(2n+1)^{-1/2}.$$

Since the integral exists by hypothesis we obtain at once the desired inequality, $\lim\limits_{n=\infty}\,[I^{(n)}(x,q)]^{1/n}=q$.

Employing this lemma and making use of Theorems (B) and (D) of section 5, we are able to conclude that the grade of the function defined by (47) is q.

COROLLARY 1. *If $F(x,z)$ is the operator,*

$$F(x,z)=e^{-xz}\int_a^z e^{xt}\,Y(t)\,dt,$$

then the function $F(x,z)\to f(x)$, where $f(x)$ is of grade q, is of grade not larger than the larger of the two numbers $|a|$ and q.

The proof is immediate if we write

$$F(x,z)=e^{-xz}\left\{\int_a^0 e^{xt}\,Y(t)\,dt+\int_0^z e^{xt}\,Y(t)\,dt\right\}$$

and note that

$$e^{-xz}\int_a^0 e^{xt}\,Y(t)\,dt\to f(x)=f(0)\int_a^0 e^{xt}\,Y(t)\,dt.$$

The result of the lemma proved above combined with Theorem 5 is a statement of the corollary.

COROLLARY 2. *If $F(x,z)$ is defined to be the operator of Theorem* 5 *and if $P(x)$ is a function of genus zero, then $F(x,z)\to P(x)$ is a function of genus zero.*

The situation with respect to the application of operators of Laplace type to functions of infinite grade is greatly complicated by the appearance of divergent series. However, if Borel summability be employed the validity of the operations can in many instances be restored. The counterparts of Theorems 4 and 5 in this more general situation are given by the two theorems which follow.

THEOREM 6. *Under the following assumptions*: (a) $f(x)=g(x)\,h(x)$, *where $g(x)$ is a function of finite grade g and $h(x)$ is of the form,* $h(x)=h_1/x+h_2/x^2+h_3/x^3\cdots$; (b) *the functions* $Q(t)=h_1+h_2\,t+h_3\,t^2/2!+\cdots$ *and* $F(z)=a_0+a_1z+a_2z^2+\cdots$ *are of finite grades Q and F respectively, then the operation $F(z)\to f(x)$ defines a series, in general divergent, which is summable by the method of Borel to the form,*

$$F(z)\to f(x)=\int_0^\infty e^{-xt}\,Q(t)\left\{\sum_{n=0}^{\infty}g^{(n)}(x)\,F^{(n)}(-t)/n!\right\}dt,$$

where the real part of x, $R(x)$, satisfies the inequality,

$$Q+F<R \leqq R(x) \leqq R'<\infty .$$

Proof. The proof slightly generalizes one previously employed by the author.[22] Formally expanding the function $V(x)=F(z)\rightarrow h(x)$ we get,

$$\begin{aligned} V(x) = {} & h_1(a_0/x - a_1/x^2 + 2!\,a_2/x^3 - 3!\,a_3/x^4 + \cdots) \\ & + h_2(a_0/x^2 - 2!\,a_1/x^3 + 3!\,a_2/x^4 - 4!\,a_3/x^5 + \cdots) \\ & + h_3(a_0/x^3 - 3!\,a_1/2!\,x^4 + 4!\,a_2/2!\,x^5 - 5!\,a_3/2!\,x^6 + \cdots) \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

In general this series will be divergent, but it is easily summable by the method of Borel.[23] This makes use of the identity $\int_0^\infty e^{-s} s^n\, ds = n!$ from which we obtain $V(x)$ in the form,

$$(48)\quad V(x) = (1/x)\int_0^\infty e^{-s}\, Q(s/x)\, F(-s/x)\, ds = \int_0^\infty e^{-xt}\, Q(t)\, F(-t)\, dt.$$

As is well known the existence of this function is confined to the exterior of a polygon formed by drawing through the singular points of $V(x)$ perpendiculars to the lines which join the singularities to the origin.

In an entirely similar manner we define the function,

$$V_n(x) = F^{(n)}(z)\rightarrow h(x) = \int_0^\infty e^{-xt}\, Q(t)\, F^{(n)}(-t)\, dt.$$

To discuss the region of existence of this integral we note from (A), section 5, and from hypothesis (b) that the function $F^{(n)}(-t)$ is of grade F. Hence $|F^{(n)}(-t)|$ is bounded by the following inequality:

$$|F^{(n)}(-t)| < S_n(t)\, e^{Ft}, \quad S_n(t) = \begin{cases} S_n, & 0 \leqq t \leqq t_0, \\ S_n\, e^{\delta t}, & t > t_0, \end{cases}$$

where S_n is a positive quantity independent of t such that $\lim_{n=\infty} (S_n)^{1/n} = F$ and δ is an arbitrarily small positive number independent of n.

There exists a similar majorante for $Q(t)$ where $T(t)$ is substituted for $S_n(t)$, T for S_n, and Q for F.

We are thus able to attain the inequalities,

$$\begin{aligned} |V_n(x)| & < S_n\, T\left\{\int_0^{t_0} e^{-tp}\, dt + \int_{t_0}^\infty e^{-t(p-2\delta)}\, dt\right\}, \\ & < S_n\, T\{(1-e^{-t_0 p})/p + e^{-t_0(p-2\delta)}/(p-2\delta)\} \leqq S_n\, T(1+e^{2\delta})/p, \end{aligned}$$

where we have $p = x - Q - F$, $R(x) > Q + F + 2\delta$.

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[22] American Journal of Mathematics, vol. 52 (1930), p. 104.

[23] See E. Borel, Leçons sur les séries divergentes, Paris, (1901), chapter 4.

Operating upon $f(x)$ with $F(z)$ and employing the Leibnitz formula (41) in which we write $u = V(x)$, $v = g(x)$, we derive

$$F(z) \to f(x) = \sum_{n=0}^{\infty} g^{(n)}(x)\, V_n(x)/n!.$$

Employing the inequality just written down and hypothesis (a), from which we have $|g^{(n)}(x)| \leqq R_n g^n$, where $\lim_{n=\infty} (R_n)^{1/n} = 1$, we may write,

$$|F(z) \to f(x)| \leqq \sum_{n=0}^{\infty} R_n\, g^n\, S_n\, T(1 + e^{2\delta})/n!\, p, \qquad R(x) > Q + F + 2\delta.$$

Hence the series for $F(z) \to f(x)$ converges uniformly for values of x in some region $Q + F < R \leqq R(x) \leqq R' < \infty$. Therefore, since each integral representation of $V_n(x)$ exists in the specified region, we may interchange integration and summation signs and thus attain the statement of the theorem.

A similar theorem applies to operators of the type introduced by the Laplace equation.

THEOREM 7. *Under the following assumptions*: (a) $f(x) = g(x)\, h(x)$ *is a function of finite grade g and $h(x)$ is of the form* $h(x) = h_1/x + h_2/x^2 + h_3/x^3 + \cdots$; (b) $Q(t) = h_1 + h_2 t + h_3 t^2/2! + \cdots$ *exists and defines a function* $P(s) = \int_s^{\infty} Q(t)\, dt$ *throughout the interval* $0 \leqq s \leqq \infty$; (c) $Y(t)$ *is a function such that the following inequalities hold*:

$$|P(t)\, Y^{(n)}(-t)| \leqq \begin{cases} C_n, & 0 \leqq t \leqq t_0, \\ C_n\, e^{mt}, & t_0 < t \leqq \infty, \end{cases}$$

where $\lim_{n=\infty} (C_n)^{1/n} = C$, $C_{n+1} > C_n$, $m > 0$, *then* $F(x, z) \to f(x)$, *where* $F(x, z)$ *is the operator of Theorem* 5, *defines a series, in general divergent, which is summable by the method of Borel to the form,*

$$F(x, z) \to f(x) = \int_0^{\infty} e^{-xt}\, Q(t) \left\{ \sum_{n=1}^{\infty} g^{(n)}(x)\, A_n(x, t)/n! - g(0)\, P(t)\, Y(-t) \right\} dt,$$

for x in the region, $m < R \leqq R(x) \leqq R' < \infty$, *where we employ the abbreviation,* $A_n(x, t) = [\{D^n - (-x)^n\}/(D + x) \to Y(z)]_{z=-t}$, $D = d/dz$.

Proof: By an argument which coincides with that of the previous theorem since it is unaffected by the parameter x in $F(x, z)$, we express the operation $F(x, z) \to f(x)$ in the form,

$$F(x, z) \to f(x) = \sum_{n=0}^{\infty} g^{(n)}(x)\, W_n(x)/n!$$

where we abbreviate

$$W_n(x) = \int_0^{\infty} e^{-xt}\, Q(t)\, F_z^{(n)}(x, -t) \cdot dt, \quad F_z^{(n)}(x, -t) = \partial^n F(x, z)/\partial z^n |_{z=-t}.$$

Since we have $F_z^{(n)}(x, z) = e^{-xz}(d/dz - x)^{(n)} \to I(x, z)$, where $I(x, z)$ is defined by (43), we get from (45) and its consequences,

$$W_n(x) = \int_0^\infty e^{-xt} Q(t) A_n(x, t) \, dt + (-x)^n \int_0^\infty Q(t) I(x, -t) \, dt,$$

where $A_n(x, t)$ is defined above.

Substituting this value in $F(x, z) \to f(x)$ and noting that $g(0) = \sum_{n=0}^{\infty} g^{(n)}(x)$ $(-x)^n/n!$, we get

$$\begin{aligned} F(x, z) \to f(x) &= \sum_{n=1}^{\infty} \int_0^\infty e^{-xt} Q(t) A_n(x, t) g^{(n)}(x) \, dt/n! \\ &\quad + g(0) \int_0^\infty Q(t) I(x, -t) \, dt. \end{aligned} \tag{49}$$

Considering the second integral we write,

$$\begin{aligned} \int_0^\infty Q(t) \, I(x, -t) \, dt &= -\int_0^\infty Q(t) \, dt \int_0^t e^{-xs} \, Y(-s) \, ds \\ &= -\int_0^\infty e^{-xs} \, P(s) \, Y(-s) \, ds. \end{aligned}$$

The second equality sign is seen to be justified by hypotheses (b) and (c) of the theorem which permit the interchange of the order of integration by Dirichlet's formula and insure the existence of the infinite integral.

Referring finally to the first integral of (49) we are able to attain from hypothesis (a), i. e. that $|g^{(n)}(x)| \leqq R_n g^n$, $\lim_{n=\infty} (R_n)^{1/n} = 1$, and from hypothesis (c), the inequality

$$\begin{aligned} &\left| \int_0^\infty e^{-xt} \, Q(t) \, A_n(x, t) \, g^{(n)}(x) \, dt \right| \\ &\quad \leqq R_n \, g^n \int_0^\infty e^{-xt} \sum_{r=0}^{n-1} | \, Q(t) \, Y^{(n-1-r)}(-t) | \, |x|^r \, |dt| \\ &\quad \leqq R_n \, g^n \, C_{n-1} \{(1 - |x|^n)/(1 - |x|)\} \, \{1 + e^{-t_0(x-m)}\}/|x - m| \\ &\quad \leqq R_n \, g^n \, C_{n-1} (1 + R')^n \, \{1 + e^{-t_0(x-m)}\}/|x - m|, \end{aligned}$$

where x is limited to the region $m < R \leqq R(x) \leqq R' < \infty$.

Hence within the stated region the integrals $I_n(x) = \int_0^\infty e^{-xt} \, Q(t) \, A_n(x, t) \, dt$ all exist and the series $\sum_{n=1}^{\infty} I_n(x) \, g^{(n)}(x)/n!$ converges uniformly. Therefore we can interchange summation and integral signs and thus attain the statement of the theorem.

8. **Solutions of unbounded grade not included in the preceding.** In the last section theorems were developed which apply when the Laplace

operator is of the form $F(x, z) = \int_a^z e^{xt}\, Y(t)\, dt$, where a is finite. When this is not the case it is still possible to discuss the solution provided $f(x)$ is limited to the class of functions of finite grade.

The first result we shall state as follows:

THEOREM 8. *If $Y(z)$ and $f(x)$ are functions of finite grades Q and q respectively, then equation* (1) *has at least one solution in the region $|x| > Q$, which is in general of infinite grade.*

Proof: To establish this theorem we employ formula (35), which for simplicity we limit to the case $p = 1$,

$$X_0(x, z) = X(z)\, \{Y(z)/x - Y'(z)/x^2 + Y''(z)/x^3 - \cdots\}. \tag{50}$$

From Theorem 4 we know that $X(z)\, Y(z) \to f(x)$ is a function of grade q and hence, from the assumption regarding $Y(z)$, we have $X(z)\, Y^{(n)}(z) \to f(x) \leq M_n Q^n$, where $\lim_{n=\infty} M_n^{1/n} = 1$. Hence we find a majorant for $X_0(x, z) \to f(x)$ in the series,

$$M_0/|x| + M_1 Q/|x|^2 + M_2 Q^2/|x|^3 + M_3 Q^3/|x|^4 + \cdots,$$

which from the fact that $\lim_{n=\infty} M_n^{1/n} = 1$ is seen to converge for $|x| > Q$.

We thus establish the existence of a function in a region which does not include the origin in its interior. Since, in general, the origin is a singular point for $u(x) = X_0(x, z) \to f(x)$, the solution is in general of infinite grade.

If we admit the validity of semi-convergent or asymptotic series of the form,

$$g(x)\, \{a_0 + a_1/x + a_2/x^2 + a_3/x^3 + \cdots\}, \tag{51}$$

to represent a function $u(x)$ asymptotically, where the expression in braces satisfies the Poincaré criterion[24], a further extension of the region of definition of $u(x)$ is possible. This we state in the following theorem:

THEOREM 9. *If $Y(z)$ is a function of unbounded grade but otherwise satisfies the conditions of Theorem* 3 *for the case $p = 1$ and if $f(x)$ is a function of grade q then $X_0(x, z) \to f(x)$, where $X_0(x, z)$ is defined by* (50), *yields a semi-convergent series which is the asymptotic representation of the integral,*

$$u(x) = X_0(x, z) \to f(x) = (1/x) \int_0^\infty e^{-t}\, Y(z - t/x) \to \{X(z) \to f(x)\}\, dt, \tag{52}$$

in the sense of (51), *where $g(x)$ is a function of grade not greater than q.*

[24] Acta Mathematica, vol. 8 (1886), pp. 295–344.

Proof. Representing by $S_n(x, z)$ the operator

$$S_n(x, z) = X(z) \sum_{m=1}^{n} (-1)^{m-1} Y^{(m-1)}(z)/x^m,$$

let us consider the remainder operator,

$$R_n(x, z) = X_0(x, z) - S_n(x, z) = (-1)^n e^{-xz} X(z) (1/x)^n \int_{-\infty}^{z} e^{xt} Y^{(n)}(t)\, dt.$$

Making use of equation (47) we obtain the inequality,

$$|x|^n\, |R_n(x, z) \to f(x)|$$
$$\leqq \left|(Xf) \int_{-\infty}^{0} e^{xt} Y^{(n)}(t)\, dt\right| + \left|P(0) \int_{0}^{q} e^{xt} Y^{(n)}(t)\, dt\right| + |e^{qx} S(x)|,$$

where (Xf) is the constant $e^{-xz} X(z) \to f(x)$.

By hypothesis $Y^{(n)}(t)$ is bounded by $M_n A^n n!$, $\lim_{n=\infty} M_n^{1/n} = 1$ and satisfies the assumptions of Theorem 3. Hence the inequality may be reduced to,

$$|x|^n\, |R_n(x, z) \to f(x)|$$
$$\leqq \{e^{qx} M_n A^n n! + |P(0)| e^{qx} M_n A^n n! |x| + e^{qx} |S(x)|\, |x|\}/|x|$$
$$< e^{qx}\, T(x) \{M_n A^n n!/|x|\},$$

where $T(x)$ is a positive function of zero grade which dominates both $|x|$ and $|S(x)|\,|x|$.

We then conclude that the function $g(x)$ of (51) is dominated by a function of grade q and hence is a function of grade which does not exceed q.

In order to attain the integral of which (50) is the asymptotic representation, we apply Borel's integral to (50) and thus obtain,

$$X_0(x, z) = X(z) (1/x) \int_{0}^{\infty} e^{-t}\, Y(z - t/x)\, dt.$$

The integral representation of $u(x)$, (52), follows as an immediate consequence.

As an example illustrating Theorem 9 consider the equation:

$$[(1 + x) + z + z^2 + z^3 + \cdots] \to u(x) = f(x). \tag{53}$$

Since we have $X(z) = 1 - z$, $Y(z) = 1/(1 - z)$, the solution is given by the operator,

$$\begin{aligned} u(x) &= e^{-xz} (1 - z) \int_{-\infty}^{z} \{e^{xt}/(1 - t)\}\, dt \to f(x) \\ &= \{1/x - 1!/(1-z) x^2 + 2!/(1-z)^2 x^3 - 3!/(1-z)^3 x^4 + \cdots\} \to f(x). \end{aligned} \tag{54}$$

If we set $f(x) = 1$ we replace z by zero in (54) and thus obtain the asymptotic development of the integral,

$$u(x) = \int_0^\infty \{e^{-s}/(x+s)\}\, ds.$$

If, however, we set $f(x) = e^{ax}$, $0 < a < 1$, we get

$$u(x) = e^{ax}\{1/x - 1!/(1-a)x^2 + 2!/(1-a)^2x^3 - 3!/(1-a)^3x^4 + \cdots\}. \tag{55}$$

This series is the asymptotic representation in the sense of (51), where $g(x) = e^{ax}$, of the formal solution of (53),

$$u(x) = e^{ax}(1-a)\int_0^\infty \{e^{-s}/[(1-a)x+s]\}\, ds.$$

9. **Convergence of the factorial operator.** We proceed finally to a consideration of the convergence of series (33) which for convenience we shall specialize for the case $p = 1$. Since the n-th term of the series is readily seen to be of the form,

$$\begin{aligned} w_n(z)\, e^{nz} &= (D-n+1)\cdots(D-1)\, D \to Y \\ &= (-1)^{n-1}(n-1)!\, D(1-D)(1-D/2)\cdots[1-D/(n-1)] \to Y(z), \end{aligned}$$

where $D = d/dz$ we are led to a consideration of the operator,

$$A_n(D) = D(1-D)(1-D/2)\cdots[1-D/(n-1)].$$

Product operators of the form,

$$D(1-D/a_1)(1-D/a_2)\cdots(1-D/a_n)\cdots,$$

where $\sum_{n=1}^{\infty} 1/|a_n|$ converges, have been extensively studied by J. F. Ritt,[25] but it is clear that the condition imposed by him is not satisfied in the case of the operator $\lim_{n=\infty} A_n(D)$.

We may then proceed as follows: Introducing convergence factors $e^{D/m}$ into the product A_n we have,

$$A_n(D) = [D(1-D)\, e^D(1-D/2)\, e^{D/2}\cdots\{1-D/(n-1)\}\, e^{D/(n-1)}]\; e^{-D\{1+1/2+\cdots 1/(n-1)\}} \tag{56}$$

For sufficiently large values of n the product in the brackets may be written, $-e^{CD}/\Gamma(-D) + e^{CD}\epsilon_n(D)$, where C is Euler's constant and $\epsilon_n(D)$

[25] Loc. cit. 2.

is a function that tends to zero as $n \to \infty$.[26] Similarly the coefficient of $-D$ in the exponent of e may be written $\log n + C_n$ where the difference $C_n - C$ becomes vanishingly small with increasing n.[27]

We can then write (56) in the form,

$$A_n(D) = [-1/\Gamma(-D) + \varepsilon_n(D)]\, e^{-D \log n}\, e^{D(C-C_n)}.$$

Employing Theorem 4 and recalling that $1/\Gamma(D)$ is analytic in the entire plane we see that $A_n(D) \to Y(z)$ is a function of grade less than or equal to Q provided $Y(z)$ is of grade Q. If in particular $Y(z) = e^{-Qz}$ where Q is a positive number, we shall have

$$A_n(D) \to Y(z) = \{-1/\Gamma(Q) + \varepsilon_n(-Q)\}\, e^{-Q(C-C_n)}\, e^{-Qz}\, n^Q,$$

from which we infer that,

$$A_n(D) \to Y(z) = O(n^Q).$$

More generally let us consider the case where $Y(z)$ is a function of grade Q. Operating upon $Y(z)$ with $1/\Gamma(-D)\, e^{D(C-C_n)}$ and recalling Theorem 4 we obtain a new function $R(z)$, which is also of grade Q. Operating upon $R(z)$ with $e^{-D\log n}$ we get $R(z - \log n)$. Since $R(z)$ is of grade Q it is dominated for large values of the argument by a function of the form $e^{Qz} P(z)$, where $P(z)$ is a suitably chosen positive function of genus 0. Hence we can write $|R(z - \log n)| \leqq n^Q e^{Qz} P(z + \log n)$ for n sufficiently large. But since $P(z)$ is a function of genus zero $P(z + \log n)$ is dominated by n^δ, where δ is an arbitrarily small positive constant,[28] and we have $|R(z - \log n)| < n^{Q+\delta} e^{Qz}$.

We are thus able to assert that

$$A_n(D) \to Y(z) = o(n^{Q+\delta}), \quad \delta > 0, \tag{57}$$

provided $Y(z)$ is a function of grade Q.

The situation is much more complicated if $Y(z)$ is a function of infinite grade. An important special case, however, is furnished by the series,

$$Y(z) = a_0 + a_1 e^z + a_2 e^{2z} + \cdots + a_n e^{nz} + \cdots, \tag{58}$$

where $a_n = O(1/n!)$, which from the example of section 4 is seen to include the classical operator $u(x+1) - xu(x)$, that is when $a_n = -1/n!$

Operating with $A_n(D)$ and noting that $A_n(D) \to e^{mz} = 0$ if $m \leqq n-1$, and $(-1)^{n-1}\Gamma(m+1)\, e^{mz}/\Gamma(n)\,\Gamma(m-n+1)$ if $m \geqq n$, we shall have

[26] Whittaker and Watson, Modern Analysis, 3rd ed. (1920), p. 236.

[27] Ibid.: p. 235.

[28] This follows from the fact that $\lim\limits_{n=\infty} P(z + \log n)/n^\delta = \lim\limits_{x=\infty} P(z+x)/e^{x\delta} = 0$.

$$A_n(D) \to Y(z) = (-1)^{n-1} n e^{nz} \{a_n + a_{n+1}(n+1) e^z/1! + a_{n+2}(n+1)(n+2) e^{2z}/2! + \cdots\}$$
$$= (-1)^{n-1} n a_n e^{nz} \{1 + a_{n+1}(n+1) e^z/1!\, a_n + a_{n+2}(n+1)(n+2) e^{2z}/2!\, a_n + \cdots\}.$$

Under the hypothesis that $a_n = O(1/n!)$ it is clear that the function within the braces is entire and bounded with n and hence that

$$A_n(D) \to Y(z) = O[e^{nz}/(n-1)!]. \tag{59}$$

We now turn to a consideration of series (33) when $p = 1$. Let us first examine the case where (57) applies.

From the operational identity $w_n(z) e^{nz} = (-1)^n (n-1)!\, A_n(D) \to Y(z)$ we write equation (33) in the form,

$$X_0(x,z) = X(z)\{Y(z)/x + (A_0 \to Y)/x(x+1) + (A_1 \to Y)1!/x(x+1)(x+2) + (A_2 \to Y)2!/x(x+1)(x+2)(x+3) + \cdots\}. \tag{60}$$

We now recall the fundamental convergence fact associated with a factorial series, $F(x) = \sum_{n=0}^{\infty} a_{n+1} n!/x(x+1)\cdots(x+n)$, namely, that this series converges with the exception of the points $0, -1, -2, \cdots$, for values of x the real part of which exceeds a value, λ, called the *abscissa of convergence*. If the series $A = \sum_{n=0}^{\infty} a_{n+1}$ diverges then $\lambda \geqq 0$ and is determined by the limit,

$$\limsup_{p=\infty} \log \left| \sum_{n=0}^{p} a_{n+1} \right| \Big/ \log p.$$

If the series A converges then $\lambda \leqq 0$ and is determined by the limit,

$$\limsup_{p=\infty} \log \left| \sum_{n=p+1}^{\infty} a_{n+1} \right| \Big/ \log p.[29]$$

The abscissa of convergence of (60) is easily obtained in the two cases already discussed: (1) equation (57); (2) equation (59).

In the first case the series $\sum_{n=0}^{\infty} n^{Q+\delta}$ diverges for $Q + \delta > -1$. Hence, employing the abbreviation $q = Q + \delta$, we compute,

$$\sum_{n=0}^{p} n^q = p^{q+1}/(q+1) + \tfrac{1}{2} p^q + q B_1 p^{q-1}/2! - \cdots.$$

[29] E. Landau, Über die Grundlagen der Theorie der Fakultätenreihen. Sitzungsber. Akad. München, (Math.-phys.), vol. 36, (1906), pp. 151–218. For a comprehensive survey of present knowledge with regard to these series see T. Fort, The General Theory of Factorial Series, Bulletin of the Amer. Math. Soc., vol. 36 (1930), pp. 244–258.

From this we obtain

$$\lambda_1 = \lim_{p=\infty} \sup [\log p^{q+1}/(q+1) \log p - 1] = q.$$

We are thus able to conclude that the abscissa of convergence for the case where $Y(z)$ is a function of grade Q is not *in general* smaller than $Q+\delta$, although the special case $Y(z) = e^{mz}$, where m is a positive integer, shows that it may be $-\infty$.

In the second case, equation (59), the series $\sum_{n=0}^{\infty} e^{nz}/(n-1)!$ obviously converges. Hence from the function,

$$\begin{aligned} f(p) &= \sum_{n=p+1}^{\infty} e^{nz}/(n-1)! \\ &= \{e^{(p+1)z}/p!\}\{1 + e^z/(p+1) + e^{2z}/(p+1)(p+2) + \cdots\}, \end{aligned}$$

we compute the abscissa of convergence to be,

$$\lambda_2 = -\lim_{p=\infty} \sup |\log f(p)|/\log p = -\lim_{p=\infty} |(p+1)z - \log p!|/\log p = -\infty.$$

We are thus able to state the theorem:

THEOREM 10. *If $Y(z)$ is a function of grade Q then an abscissa of convergence exists for series* (60) *which is in general not smaller than $\lambda = Q$; if $Y(z)$ is a function which has an expansion of the form* (58) *then the abscissa of convergence is $\lambda = -\infty$.*

ON RANGES OF INCONSISTENCY OF REGULAR TRANSFORMATIONS, AND ALLIED TOPICS.[1]

By Ralph Palmer Agnew.[2]

1. Several writers have given examples which show that not all *regular*[3] transformations used in the theory of summability are *mutually consistent*.[4] Hence it is known that there exist certain divergent sequences which are assigned unequal values by different regular transformations. The question as to whether any given divergent sequence may be assigned unequal values by different regular transformations, as well as further questions concerning the range of values which may be assigned to a given divergent sequence by different regular transformations, seems not to have been previously raised. These questions have led the writer to consider the following more general problem: *Given a sequence of a stated character, into what sequences may it be carried by regular sequence-to-sequence transformations?*

We find that any given bounded divergent sequence may be carried into any given bounded sequence by a transformation which is not only regular but satisfies also further important conditions. Theorems 4–9 are given to show what may be expected of specialized real regular transformations when they are applied to divergent sequences; the latter of these theorems supplement the *Kernsatz* given by Knopp.[5] It is also shown that a specialized regular transformation can be found which carries any given unbounded sequence into any given sequence; and that if two sequences converge to the same value, a regular transformation can be found which carries the first into the second.

An interesting corollary of this paper, which answers the first question raised in the introduction, is the following: Corresponding to each divergent sequence $\{s_n\}$ of complex constants and each complex constant σ, there is a regular transformation with a triangular matrix (satisfying important conditions in addition to those necessary to ensure regularity) which

[1] Received October 16, 1930. — Presented to the American Mathematical Society November 28, 1930.

[2] National Research Fellow.

[3] A transformation is said to be *regular* when it assigns to each convergent sequence the value to which it converges.

[4] Two transformations are *mutually consistent* if, whenever each transformation assigns a value to a sequence, the values are equal.

[5] Mathematische Zeitschrift, vol. 31 (Nov., 1929), p. 115.

evaluates $\{s_n\}$ to σ. If $\{s_n\}$ and σ are real the transformation may be taken real.

2. A transformation with a square matrix is a sequence-to-sequence transformation of the form

$$(S) \qquad \xi_n = a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots$$

which assigns to a sequence $\{x_n\}$ the value $\lim \xi_n$ when this limit exists. Such a transformation is said to have a triangular matrix, and is denoted by (T), when $a_{nk} = 0$ for $k > n$. According to the well-known Silverman-Toeplitz theorem, necessary and sufficient conditions that (S) and (T) be regular are

$$C_1: \quad \sum_{k=1}^{\infty} |a_{nk}| \text{ is bounded for all } n;$$

$$C_2: \quad \text{for each } k, \lim_{n\to\infty} a_{nk} = 0;$$

$$C_3: \quad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$$

3. Theorem 1. *Corresponding to each bounded divergent sequence $\{s_n\}$ of complex constants and each bounded sequence $\{\sigma_n\}$ of complex constants, there is a regular transformation with a square matrix which carries $\{s_n\}$ into $\{\sigma_n\}$.*

Any bounded divergent sequence $\{s_n\}$ must have at least two distinct finite limiting values, say z_1 and z_2. Choose a sequence $\mu_1 < \nu_1 < \mu_2 < \nu_2 < \mu_3 < \cdots$ of indices such that $s_{\mu_n} \to z_1$, $s_{\nu_n} \to z_2$, and $s_{\mu_n} \neq s_{\nu_n}$, $n = 1, 2, 3, \cdots$. Then the transformation

$$(1) \qquad \xi_n = \frac{s_{\nu_n} - \sigma_n}{s_{\nu_n} - s_{\mu_n}} x_{\mu_n} + \frac{\sigma_n - s_{\mu_n}}{s_{\nu_n} - s_{\mu_n}} x_{\nu_n}$$

carries $\{s_n\}$ into σ_n. We see by inspection that (1) satisfies C_2 and C_3; and the fact $\{\sigma_n\}$ is bounded, together with the fact that s_{μ_n} and s_{ν_n} approach distinct limits, enables us to show that (1) satisfies C_1. Hence (1) is regular and the theorem is proved. It is of interest to note that if $\{s_n\}$ and $\{\sigma_n\}$ are real, then the transformation may be taken real.

That transformations with triangular matrices do not have this property follows from a consideration of sequences $\{s_n\}$ and $\{\sigma_n\}$ of which $s_1 = 0$ and $\sigma_1 \neq 0$. That Theorem 1 exhibits the utmost to be expected of regular transformations when applied to bounded sequences follows from the fact that all regular transformations carry bounded sequences into bounded sequences.

Theorem 2. *Corresponding to each bounded divergent sequence $\{s_n\}$ of complex constants and each bounded closed set A of the complex plane, there*

is a regular transformation with a square (triangular) matrix which carries $\{s_n\}$ into a sequence having the set A for its set of limiting values.[6]

Since A is a closed set, there is a countable subset B: $b_1, b_2, b_3, \cdots$ of A such that the set B^0, consisting of the points and limit points of B, is identical with A. Let the sequence $\{\sigma_n\}$ consist of the successive elements

$$b_1;\, b_1, b_2;\, b_1, b_2, b_3;\, \cdots;\, b_1, b_2, \cdots, b_n;\, b_1, \cdots.$$

Then the set of limiting values of $\{\sigma_n\}$ is identical with B^0 and with A, and the truth of the theorem for transformations with square matrices follows from Theorem 1.

That the above theorem is true for transformations with triangular matrices results from a consideration of the transformation

$$\xi_\alpha = \begin{cases} 0 & 1 \leqq \alpha < \nu_1, \\ \dfrac{s_{\nu_n} - \sigma_n}{s_{\nu_n} - s_{\mu_n}} x_{\mu_n} + \dfrac{\sigma_n - s_{\mu_n}}{s_{\nu_n} - s_{\mu_n}} x_{\nu_n}, & \nu_n \leqq \alpha < \nu_{n+1} \end{cases}$$

which is related to (1), but has a triangular matrix and carries $\{s_n\}$ into a sequence composed of initial zeroes and the elements of $\{\sigma_n\}$ with repetitions. If $\{s_n\}$ and A are real, the transformations we have constructed are real.

If the closed set A of Theorem 2 is taken to be a single point σ, we obtain

THEOREM 3. *Corresponding to each bounded divergent sequence $\{s_n\}$ of complex constants and each complex constant σ, there is a regular transformation with a square (triangular) matrix which evaluates $\{s_n\}$ to σ. If $\{s_n\}$ and σ are real, the transformations may be taken real.*

4. The transformations of which Theorems 1, 2, and 3 establish the existence may or may not be real. That they need not always be real follows easily from the fact that real transformations can carry real sequences into only real sequences. We shall now give theorems which exhibit properties of real transformations.

THEOREM 4. *Corresponding to each bounded sequence $\{s_n\}$ of complex constants with at least three distinct limiting values represented by three non-collinear points, and each bounded sequence $\{\sigma_n\}$ of complex constants, there is a real regular transformation with a square matrix which carries $\{s_n\}$ into $\{\sigma_n\}$.*

Let z_1, z_2, and z_3 be three distinct non-collinear limiting points of $\{s_n\}$ and choose a sequence $\gamma_1 < \delta_1 < \epsilon_1 < \gamma_2 < \delta_2 < \epsilon_2 < \gamma_3 < \cdots$ of indices such that $s_{\gamma_n} \to z_1$, $s_{\delta_n} \to z_2$, $s_{\epsilon_n} \to z_3$, and for each n, s_{γ_n}, s_{δ_n}, and s_{ϵ_n} are non-collinear.

[6] The condition A is closed cannot be removed, for the set of limiting values of a sequence is always closed.

For each n, choose from the triangle $s_{\gamma_n}\, s_{\delta_n}\, s_{\varepsilon_n}$ a vertex, say s_{μ_n}, such that the line $s_{\mu_n}\, \sigma_n$ meets the opposite side $s_{\nu_n}\, s_{\pi_n}$ at a point y_n lying on the closed line-segment joining the vertices s_{ν_n} and s_{π_n}. Let α_n and β_n be such that $y_n = \alpha_n\, s_{\nu_n} + (1-\alpha_n)\, s_{\pi_n}$ and $\sigma_n = (1-\beta_n)\, s_{\mu_n} + \beta_n\, y_n$. Then the real transformation

$$\xi_n = (1-\beta_n)\, x_{\mu_n} + \alpha_n\, \beta_n\, x_{\nu_n} + (\beta_n - \alpha_n\, \beta_n)\, x_{\pi_n} \tag{2}$$

carries $\{s_n\}$ into σ_n. The transformation (2) evidently satisfies C_2 and C_3. It follows from the relative positions of y_n, s_{ν_n}, and s_{π_n}, and the relation defining α_n that $0 \leqq \alpha_n \leqq 1$ for all n; and using the fact that s_{γ_n}, s_{δ_n}, and s_{ε_n} approach three distinct limits together with the fact that $\{\sigma_n\}$ is bounded, we find from the relation defining β_n that β_n is bounded for all n; thus (2) satisfies C_1. Therefore the transformation is regular and the theorem is proved.

Using the preceding theorem and the method of proof of Theorem 2, we obtain

THEOREM 5. *Corresponding to each bounded divergent sequence $\{s_n\}$ of complex constants with at least three distinct limiting values represented by noncollinear points, and each bounded closed set A of the complex plane, there is a real regular transformation with a square (triangular) matrix which carries $\{s_n\}$ into a sequence having the set A for its set of limiting values.*

Letting A be a single point σ we obtain

THEOREM 6. *Corresponding to each bounded divergent sequence $\{s_n\}$ of complex constants with at least three distinct limiting values represented by noncollinear points, and each complex constant σ, there is a real regular transformation with a square (triangular) matrix which evaluates $\{s_n\}$ to σ.*

5. The conditions

$$C_4:\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 1,$$

$$C_5:\quad \text{for each } k,\ a_{nk} = 0 \text{ for almost all } n,$$

$$C_6:\quad \sum_{k=1}^{\infty} a_{nk} = 1 \text{ for all } n,$$

are of importance[7] in the theory of summability. It may be noted that the transformations which we have constructed to prove the preceding theorems satisfy C_5 and C_6 but not necessarily C_4. For real transformations the condition

$$C_7:\quad a_{nk} \geqq 0 \text{ for all } n \text{ and } k$$

[7] See the author's paper *The Behavior of Bounds and Oscillations of Sequences of Functions under Regular Transformations*, Transactions of the American Mathematical Society, vol. 32 (1930), pp. 669–708, and the papers there refered to in footnotes.

is of importance; C_7 may or may not be satisfied by such of the transformations we have constructed as are real. Hence it is natural to ask: *What may be expected of real regular transformations which satisfy* C_5, C_6, *and* C_7*?*[8]

6. Knopp[9] has defined the *core* R of a complex sequence $\{s_n\}$ as follows: Let R_n be the least convex closed region of the complex plane which contains the points $s_n, s_{n+1}, s_{n+2}, \cdots$; then the *core* R is the intersection of the sets $R_1, R_2, R_3, \cdots$. The question raised in § 5 is answered by

THEOREM 7. *Corresponding to each sequence* $\{s_n\}$ *of complex constants and each sequence* $\{z_n\}$ *of points of the core of* $\{s_n\}$, *there is a real regular transformation with a square matrix satisfying* C_5, C_6, *and* C_7 *which carries* $\{s_n\}$ *into a sequence* $\{\sigma_n\}$ *such that* $|z_n - \sigma_n| < 1/n$, $n = 1, 2, 3, \cdots$.[10]

To prove this theorem, let $R_{n,p}$ represent the least convex closed region containing the points $s_n, s_{n+1}, \cdots, s_{n+p}$. Let $R_{n,\infty}$ denote the union of the sets $R_{n,1}, R_{n,2}, \cdots$; then for each index p, $R_{n,p} \subset R_{n,p+1} \subset R_{n,\infty}$. It may be shown that $R_{n,\infty} \subset R_n$, and that R_n is included in the set $R^0_{n,\infty}$ consisting of the points and limit points of $R_{n,\infty}$. Hence

$$R^0_{n,\infty} = R_n. \tag{3}$$

Since z_n is a point of R, $z_n \subset R_n$ for each n. Owing to (3) we may choose, for each n, a point σ_n and an index p_n such that $|z_n - \sigma_n| < 1/n$ and $\sigma_n \subset R_{n,p_n}$. Since R_{n,p_n} is a convex closed polygon whose vertices are some or all of the points $s_n, s_{n+1}, \cdots, s_{n+p_n}$, we may choose three of its vertices, say $s_{n+\mu_n}, s_{n+\nu_n}, s_{n+\pi_n}$ so that the resulting triangle[11] contains σ_n as an inner or boundary point. Then we may choose constants α_n and β_n such that $0 \leqq \alpha_n \leqq 1$ and $0 \leqq \beta_n \leqq 1$ and

$$\sigma_n = (1 - \beta_n) s_{n+\mu_n} + \alpha_n \beta_n s_{n+\nu_n} + (\beta_n - \alpha_n \beta_n) s_{n+\pi_n}.$$

The transformation

$$\xi_n = (1 - \beta_n) x_{n+\mu_n} + \alpha_n \beta_n x_{n+\nu_n} + (\beta_n - \alpha_n \beta_n) x_{n+\pi_n}$$

has all of the properties demanded by the theorem and the theorem follows.

Using the preceding theorem and the method of proof of Theorem 2 we obtain

THEOREM 8. *Corresponding to each sequence* $\{s_n\}$ *of complex constants and each closed set A contained in the core of* $\{s_n\}$, *there is a real regular trans-*

[8] Owing to C_3, any regular transformation which satisfies C_7 also satisfies C_4.

[9] Mathematische Zeitschrift, vol. 31 (November 1929), pp. 97–127.

[10] There does not necessarily exist a regular transformation satisfying C_6 and C_7 which carries $\{s_n\}$ into $\{z_n\}$ itself, even when $\{s_n\}$ is assumed bounded.

[11] In case $s_n, s_{n+1}, \cdots s_{n+p_n}$ are collinear or coincident the polygon is a line or a point and two or all of the vertices of the triangle are coincident.

formation with a square (triangular) matrix satisfying C_5, C_6, *and* C_7 *which carries* $\{s_n\}$ *into a sequence having the set* A *for its set of limiting values.*[12]

Taking the set A to be a single point, we obtain

THEOREM 9. *Corresponding to each sequence* $\{s_n\}$ *of complex constants and each constant* σ *of the core of* $\{s_n\}$, *there is a real regular transformation with a square (triangular) matrix satisfying* C_5, C_6, *and* C_7 *which evaluates* $\{s_n\}$ *to* σ.

That the preceding theorems exhibit the utmost in dispersion of limiting values of sequences obtainable by using regular transformations satisfying C_7 follows from *Knopp's Kernsatz.*[13] *If a regular transformation satisfies* C_7, *then the core of* $\{\sigma_n\}$ *is contained in the core of* $\{s_n\}$.

7. In Theorems 7, 8, and 9 the $\{s_n\}$ sequences were not assumed bounded; hence we have learned what may be expected of regular transformations satisfying C_5, C_6, and C_7 when they are applied to unbounded sequences. We shall now give three theorems showing what may be expected of slightly less specialized regular transformations when they are applied to unbounded sequences.

THEOREM 10. *Corresponding to each unbounded sequence* $\{s_n\}$ *of complex constants and each sequence* $\{\sigma_n\}$ *of complex constants, there is a regular transformation with a square matrix satisfying* C_4, C_5, *and* C_6 *which carries* $\{s_n\}$ *into* σ_n.

Let $\{\theta_n\}$ be a sequence of positive numbers with the limit zero. Since $\{s_n\}$ is unbounded, we can choose, for each n, an index $p_n > n$ such that

$$\frac{|s_{p_n} - \sigma_n| + |\sigma_n - s_n|}{|s_{p_n} - s_n|} < 1 + \theta_n. \tag{4}$$

Then the transformation

$$\xi_n = \frac{s_{p_n} - \sigma_n}{s_{p_n} - s_n} x_n + \frac{\sigma_n - s_n}{s_{p_n} - s_n} x_{p_n} \tag{5}$$

carries $\{s_n\}$ into $\{\sigma_n\}$. It is easily seen that (5) satisfies C_5 and C_6, and it follows from (4) and the choice of $\{\theta_n\}$ that it satisfies C_4. Since C_4, C_5, and C_6 imply C_1, C_2, and C_3, (5) is regular and the theorem is proved. If $\{s_n\}$ and $\{\sigma_n\}$ are real, the transformation we have constructed is real.

Using the preceding theorem and the methods of the proof of Theorem 2, we obtain

THEOREM 11. *Corresponding to each unbounded sequence* $\{s_n\}$ *of complex constants and each closed set* A *of the complex plane, there is a regular trans-*

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[12] Since the core of any sequence is closed, the set A may be the core itself.

[13] Loc. Cit., p. 115.

formation with a square (triangular) matrix satisfying C_4, C_5, *and* C_6 *which carries* $\{s_n\}$ *into a sequence having the set* A *for its set of limiting values.*

If $\{s_n\}$ and A are real, the transformations may be taken real. Letting A be a single point σ, we have

THEOREM 12. *Corresponding to each unbounded sequence* $\{s_n\}$ *of complex constants and each complex constant* σ, *there is a regular transformation with a square (triangular) matrix satisfying* C_4, C_5, *and* C_6 *which evaluates* $\{s_n\}$ *to* σ. *If* $\{s_n\}$ *and* σ *are real, the transformations may be taken real.*

8. If a sequence $\{s_n\}$ converges, say to s, then each regular transformation must carry it into a sequence which converges to s. Here the question arises as to whether there is any restriction on the sequences $\{\sigma_n\}$ with the limit s into which $\{s_n\}$ may be carried by regular transformations. We will answer this question by proving the following theorems.

THEOREM 13. *Corresponding to each complex sequence* $\{s_n\}$ *which does not consist exclusively of repetitions of a single element for all sufficiently great* n *and which converges, say to* s, *and each sequence* $\{\sigma_n\}$ *which converges to* s, *there is a regular transformation with a square matrix satisfying* C_4, C_5, *and* C_6 *which carries* $\{s_n\}$ *into* $\{\sigma_n\}$. *If* $\{s_n\}$ *and* $\{\sigma_n\}$ *are real, the transformation may be taken real.*

Choose a sequence $1 > \theta_1 > \theta_2 > \theta_3 > \cdots$ of positive numbers with the limit zero. Under the hypotheses of the theorem, $\{s_n\}$ contains a subsequence of distinct elements with the limit s. Choose an element s_{p_1} different from s and then choose $p_2 > p_1$ such that $s_{p_2} \neq s$ and such that $|s_{p_2} - s|/|s_{p_2} - s_{p_1}| < \theta_1/4$; and in general when $p_1, p_2, \cdots, p_n$ are chosen, choose $p_{n+1} > p_n$ such that $s_{p_{n+1}} \neq s$ and such that

$$|s_{p_{n+1}} - s|/|s_{p_{n+1}} - s_{p_n}| < \theta_n/4. \tag{6}$$

Let $\alpha_1 = 1$ and choose, for each $n > 1$, an index $\alpha_n > \alpha_{n-1}$ such that

$$|\sigma_\alpha - s| < |s_{p_{n+1}} - s|, \qquad \alpha \geqq \alpha_n. \tag{7}$$

Then let

$$\xi_\alpha = \frac{s_{p_{n+1}} - \sigma_\alpha}{s_{p_{n+1}} - s_{p_n}} x_{p_n} + \frac{\sigma_\alpha - s_{p_n}}{s_{p_{n+1}} - s_{p_n}} x_{p_{n+1}} \qquad \alpha_n \leqq \alpha < \alpha_{n+1}. \tag{8}$$

The transformation (8) carries $\{s_n\}$ into $\{\sigma_n\}$ and satisfies C_5 and C_6. Using (6) and (7) we see that for $n > 1$ and $\alpha_n \leqq \alpha < \alpha_{n+1}$

$$\sum_{k=1}^{\infty} |a_{\alpha,k}| \leqq \frac{|s_{p_{n+1}} - s| + |s - \sigma_\alpha|}{|s_{p_{n+1}} - s_{p_n}|} + \frac{|\sigma_\alpha - s| + |s - s_{p_{n+1}}| + |s_{p_{n+1}} - s_{p_n}|}{|s_{p_{n+1}} - s_{p_n}|}$$

$$< 1 + \theta_n;$$

hence C_4 follows from the choice of $\{\theta_n\}$. Since C_4, C_5, and C_6 imply C_1, C_2, and C_3, (8) is regular. Finally, if $\{s_n\}$ and $\{\sigma_n\}$ are real, the transformation we have constructed is real and the theorem is proved.

If $\{s_n\}$ is a sequence such that $s_n = s \neq 0$ for $n \geqq N$, and $\{\sigma_n\}$ is any sequence with the limit s, then the transformation $\xi_n = (\sigma_n/s)x_N$, $n < N$; $\xi_n = (\sigma_n/s)x_n$, $n \geqq N$ carries $\{s_n\}$ into $\{\sigma_n\}$ and, in addition to being regular, satisfies C_4 and C_5.

Let $\{s_n\}$ be a sequence of which at least one element, say s_p, is different from zero, and such that $s_n = 0$ for $n \geqq N$; and let $\{\sigma_n\}$ be any sequence with the limit 0. Then the transformation $\xi_n = (\sigma_n/s_p)x_p + x_N$, $n < N$; $\xi_n = (\sigma_n/s_p)x_p + x_n$, $n \geqq N$, carries $\{s_n\}$ into $\{\sigma_n\}$ and, in addition to being regular, satisfies C_4.

Combining the preceding trivial results with Theorem 13, we obtain

THEOREM 14. *Corresponding to each convergent sequence $\{s_n\}$ having at least one element different from zero and each sequence $\{\sigma_n\}$ which converges to the same value, there is a regular transformation with a square matrix satisfying C_4 which carries $\{s_n\}$ into $\{\sigma_n\}$.*

It may be of academic interest to note that if we do not demand that the regular transformations used have square matrices, even the slight restriction placed on the $\{s_n\}$ sequences in Theorem 14 may be removed. In fact, a sequence composed exclusively of zeroes is carried into an arbitrary sequence $\{\sigma_n\}$ with the limit zero by the regular transformation $\xi_n = \sigma_n + x_n$, and we have

THEOREM 15. *If two sequences have a common limit, there is a regular transformation which carries the first into the second.*

CINCINNATI, OHIO.

ON THE NECESSARY CONDITION OF WEIERSTRASS IN THE MULTIPLE INTEGRAL PROBLEM OF THE CALCULUS OF VARIATIONS. II.[1]

BY E. J. McSHANE.[2]

The difference between the states of development of the theory of the single-integral and multiple-integral problems of the calculus of variation is well exhibited in the case of the necessary condition of Weierstrass. In contrast with the strong theorems known for the single integral problem, we have for the multiple integral problem in parametric form only the theorem of Kobb.[3] Kobb considered the problem of minimizing $\iint f(x, y, z, x_u, x_v, y_u, y_v, z_u, z_v)\, du\, dv$, and defined a function $\mathcal{E}(x, y, z, x_u, \cdots, z_v, \bar{x}_u, \cdots, \bar{z}_v)$ which he proved to be non-negative at points of a minimizing surface near which $\frac{\partial x}{\partial u}, \cdots, \frac{\partial z}{\partial v}$ are all continuous. This $\mathcal{E}$-function lacks however the homogeneity which is desirable in an analogue of Weierstrass' E-function, and has proven itself unfruitful of results. Radon[4] defined a different E-function which had the desired homogeneity properties, and utilized it in establishing a set of sufficiency conditions. However, to the best of my knowledge no necessary condition has been stated with regard to Radon's E-function. The object of the present paper is to establish such a condition.

1. **Preliminary remarks.** By a p-spread in n-space $(n > p)$ we shall mean a system of continuous functions $x^1(u^1, \cdots, u^p), \cdots, x^n(u^1, \cdots, u^p)$, where the point $(u^1, \cdots, u^p)$ ranges over a point set D. We shall often use the abbreviated symbol $x(u)$ to designate the above set of functions. We shall not have need in the following pages to consider properties which it might be desirable to require of D, nor to consider the definitions of distance of two spreads, identity of two spreads, limits of sequences of spreads, etc. Nor shall we attempt to give a definition of points of indifference which shall be a complete analogue of Tonelli's;[5] the following

[1] Received February 5, 1931.

[2] National Research Fellow.

[3] G. Kobb, *Sur les maxima et les minima des integrales doubles,* Acta mathematica, vol. 16 (1892), p. 65.

[4] J. Radon, *Über einige Fragen, betreffend die Theorie d. Max. u. Min. mehrfacher Integrale,* Monatshefte, vol. 22 (1911), p. 53.

[5] Tonelli, *Fondamenti di Calcolo delle Variazione,* vol. II, p. 84.

is adequate for our purposes: Given a point set A of x-space, a class $\mathfrak{K}$ of spreads S: $x = x(u)$ (u on D) lying in A, and a particular spread S_0: $x = x_0(u)$ (u on D_0); we say that the point $x_0(u_0)$ is a point of indifference of S_0 with respect to $\mathfrak{K}$ and A provided that there exist a positive number ε and a "cube (hypercube)" Q, contained in D_0 and having u_0 in its interior, such that every spread $\sum$: $x = \xi(u)$ (u on D_0) with the properties:

1) $\sum$ is in A;
2) $\xi(u) = x_0(u)$ for all u not in Q;
3) $(\xi(u) - x_0(u))$ satisfies a Lipschitz condition;
4) $|\xi^i(u) - x_0^i(u)| \leqq \varepsilon \quad (i = 1, \cdots, n)$

is also a spread of $\mathfrak{K}$.

The particular problem with which we are concerned is the following: Given a point set A in x-space, and a class of p-spreads S: $x = x(u)$ (u on D) lying in A, and given an integral

$$\mathfrak{F}(S) = \int \cdots \int F\left(x^1, \cdots, x^n, \frac{\partial x^1}{\partial u^1}, \cdots, \frac{\partial x^n}{\partial u^p}\right) du^1 \cdots du^p;$$

to find the spread S_0 of $\mathfrak{K}$ for which $\mathfrak{F}(S)$ assumes its minimum value. As is well known, in order that $\mathfrak{F}(S)$ be independent of the particular parameterization of S it is necessary and sufficient that the integrand F be a function of the Jacobians $J^\alpha(x)$ of the $x(u)$ with respect to the u's:

$$F\left(x^1, \cdots, x^n, \frac{\partial x^1}{\partial u^1}, \cdots, \frac{\partial x^n}{\partial u^p}\right) \equiv f(x^1, \cdots, x^n, J^1, \cdots, J^q)$$
$$(q = n!/(n-p)!\, p!)$$

and that it be positively homogeneous of degree 1 in the J's:

$$f(x^1, \cdots, x^n, kJ^1, \cdots, kJ^q) = kf(x^1, \cdots, x^n, J^1, \cdots, J^q) \quad (k \geqq 0).$$

Here and henceforward we assume that $f(x, J^\alpha)$ is continuous for all x in A and all J^α, and that all first and second partial derivatives $f_{J^\alpha}(x, J^\alpha)$ and $f_{J^\alpha J^\beta}(x, J^\alpha)$ exist and are continuous for all x in A and all J^α such that $\sum (J^\alpha)^2 > 0$. Since f_{J^α} is positively homogeneous of degree 0 in the J^α, it is undefined if $J^1 = \cdots = J^q = 0$.

We define $E(x, J^\alpha, \bar{J}^\alpha) \equiv f(x, \bar{J}^\alpha) - \sum_\alpha \bar{J}^\alpha f_{J^\alpha}(x, J^\alpha)$. As a consequence of the homogeneity relationships, this can also be written $f(x, \bar{J}^\alpha) - f(x, J^\alpha) - \sum_\alpha (\bar{J}^\alpha - J^\alpha) f_{J^\alpha}(x, J^\alpha)$. It is defined and continuous for all x in A, all $\bar{J}^\alpha$, and all J^α such that $\sum_\alpha (J^\alpha)^2 > 0$, and is positively homogeneous of degree 0 in the J^α and of degree 1 in the $\bar{J}^\alpha$.

The case with which we shall have most to do is that of n-spreads in $n+1$-space: $x^i = x^i(u^1, \cdots, u^n)$ $(i = 1, \cdots, n+1)$. For this case we find it convenient to establish a definite ordering of the J^α:

$$J^k(x) = (-1)^{k-1} \frac{D(x^1, \cdots, x^{k-1}, x^{k+1}, \cdots, x^{n+1})}{D(u^1, \cdots, u^n)} \qquad (k = 1, \cdots, n+1). \tag{1}$$

2. A convergence theorem.

THEOREM I. *If the functions $x_0(u)$ (that is, $x^1(u^1, \cdots, u^p), \cdots, x^n(u^1, \cdots, u^p)$) and the sequence $\{x_j(u)\}$ $(j = 1, 2, \cdots)$ are all defined and all satisfy the same Lipschitz condition on a parallelopiped Q: $(\alpha^i \leqq u^i \leqq \beta^i)$, and $\lim x_j(u) = x_0(u)$ uniformly on Q, then*

$$\lim_{j\to\infty} \left\{ \int_Q f(x_j, J^\alpha(x_j))\, du - \int_Q f(x_0, J^\alpha(x_0))\, du - \int_Q E(x_0, J^\alpha(x_0), J^\alpha(x_j))\, du \right\} = 0 \tag{2}$$

where the $J^\alpha(x_0)$ are replaced at the points where they are undefined or all vanish by any values such that the $J^\alpha(x_0(u))$ remain measurable on Q.

We note that the redefinition of $J^\alpha(x_0)$ required in the statement is always possible; in fact it is sufficient to replace them by the same set of constants at all points at which they are undefined or all vanish. We will establish the theorem by induction. It is known[6] to be true if there is only one independent variable u; we assume it demonstrated if the number of independent variables is $p-1$, and show that it is then true if $u = (u^1, \cdots, u^p)$. To do this we first establish a sequence of lemmas.

LEMMA 1. *If U be a parallelopiped or simplex of u-space, and the functions $\xi^1(u^1, \cdots, u^p), \cdots, \xi^p(u^1, \cdots, u^p)$ are quasi-linear on U, then*

$$-\frac{1}{n}\int \cdots \int \begin{vmatrix} \xi^1 & \xi^2 & \cdots & \xi^p \\ \dfrac{\partial \xi^1}{\partial v^1} & \dfrac{\partial \xi^2}{\partial v^1} & \cdots & \dfrac{\partial \xi^p}{\partial v^1} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial \xi^1}{\partial v^{p-1}} & \dfrac{\partial \xi^2}{\partial v^{p-1}} & \cdots & \dfrac{\partial \xi^p}{\partial v^{p-1}} \end{vmatrix} dv^1 \cdots dv^{p-1} = \int \cdots \int \frac{D(\xi^1, \cdots, \xi^p)}{D(u^1, \cdots, u^p)}\, du^1 \cdots du^p. \tag{3}$$

The functions ξ^i are said to be quasi-linear on U if they are continuous on U, and U can be divided into a finite number of simplexes on each

[6] This is established in my thesis, to be published in *Contributions to the Calculus of Variations*, 1930, The University of Chicago. For completeness it is repeated at the end of this section.

of which each ξ^i is linear in the u's. The v's in the left member of the equation are a rectangular coördinate system chosen in each face of U, the orientation being such that if $(P_1, P_2, \cdots, P_p)$ is a simplex in the face oriented positively with respect to the v's, and P_0 is interior to U, then the simplex $(P_0, P_1, \cdots, P_p)$ is positively oriented with respect to the axes of u-space.

This is established by direct computation when U is a simplex and the ξ^i linear. We have then only to notice that if the ξ^i are quasi-linear, each face interior to U contributes zero to the left member, since it brings in two integrals with oppositely oriented v-axes.

For compactness, let us henceforward designate the left member of equation (3) by $\int \cdots \int \Delta(\xi^1, \cdots, \xi^p)\, dv^1 \cdots dv^{p-1}$.

Lemma 2. *If Q: $(\alpha^i \leqq u^i \leqq \beta^i)$ be a parallelopiped of u-space, and the functions $x^1(u^1, \cdots, u^p), \cdots, x^p(u^1, \cdots, u^p)$ satisfy a Lipschitz condition on Q, then*

$$\begin{aligned} &-\frac{1}{n}\int \cdots \int \Delta(x^1, \cdots, x^p)\, dv^1 \cdots dv^{p-1} \\ &= \int \cdots \int \frac{D(x^1, \cdots, x^p)}{D(u^1, \cdots, u^p)}\, du^1 \cdots du^p. \end{aligned} \tag{4}$$

Let us effect a simplicial division of Q in the following manner: Divide each edge of Q into two equal segments. In each two-dimensional face of Q form the eight triangles determined by the midpoint of the face and the eight segments of its boundary. In each three dimensional face of Q form the forty-eight tetrahedra determined by the midpoint of the face and the triangles of its boundary. Proceed so until Q is completely divided into simplexes. We call this partition Π_0. Now divide Q into 2^p equal parallelopipeds by hyperplanes through the midpoint, and subdivide each parallelopiped as above to obtain partition Π_1. Again subdividing each parallelopiped into 2^p equal parts, we subdivide into simplexes to obtain Π_2; and so proceeding, we obtain $\Pi_3, \Pi_4, \cdots$. In order that each point of Q shall belong to just one simplex, we consider each parallelopiped half open $(a^i \leqq u^i < b^i)$ and assign the common edges, etc. of simplexes each to one simplex. (Modifications at the boundary of Q are obvious.) The method of construction makes it evident that for all simplexes U_k^j, the ratio of the diameter to the shortest side is bounded, say less than ϱ. Moreover, each U_k^j has one edge parallel to each coördinate axis.

Construct now a sequence of quasi-linear functions $\xi_j(u)$; each $\xi_j(u)$ is linear on each simplex of Π_j, and for each vertex $\bar{u}$ we take $\xi_j^i(\bar{u}) = x^i(\bar{u})$.

It is easily seen that each $\xi_j(u)$ satisfies the same Lipschitz condition as $x(u)$.

Since each $x^i(u)$ is Lipschitzian, it is almost everywhere totally differentiable in the sense of Stolz.[7] Let E be the set of all points of Q at which all $x^i(u)$ are totally differentiable; $m(Q-E)=0$. At each point $P=(u_0^1, \cdots, u_0^p)$ of E we have

$$
\begin{aligned}
& x^i(u_0^1+h^1, \cdots, u_0^p+h^p) \\
(5) \qquad & = x^i(u_0^1, \cdots, u_0^p)+\sum_k h^k \frac{\partial x^i}{\partial u^k}+\sqrt{\sum (h^i)^2}\cdot R^i(h^1, \cdots, h^p),
\end{aligned}
$$

where $|R^i(h^1, \cdots, h^p)| \leqq S(\sqrt{\sum (h^i)^2})$ and $S(t)$ tends to zero monotonically with t. $\left(\text{The } \frac{\partial x^i}{\partial u^k} \text{ are taken at } P.\right)$ For each partition Π_j the point P belongs to some simplex $U_j^{k(j)}$. Let $u_0+h_{j,1}$, $u_0+h_{j,2}$ designate the vertices of $U_j^{k(j)}$ which are the ends of the edge parallel to the u^α-axis; then $|h_{j,2}^\alpha - h_{j,1}^\alpha|$ is the length of that edge, and $h_{j,2}^i = h_{j,1}^i$ for all $i \neq \alpha$. Then, since each ξ_j^i is linear in $U_j^{k(j)}$, we have at P

$$
\frac{\partial \xi_j^i}{\partial u^\alpha} = \frac{\xi_j^i(u_0+h_{j,2})-\xi_j^i(u_0+h_{j,1})}{h_{j,2}^\alpha - h_{j,1}^\alpha} = \frac{x^i(u_0+h_{j,2})-x^i(u_0+h_{j,1})}{h_{j,2}^\alpha - h_{j,1}^\alpha};
$$

and recalling that the ratio of the diameter of $U_j^{k(j)}$ to $|h_{j,2}^\alpha - h_{j,1}^\alpha|$ is less than a constant ϱ, we have from this and equation 5:

$$
\left| \frac{\partial \xi_j^i}{\partial u^\alpha} - \frac{\partial x^i}{\partial u^\alpha} \right| \leqq 2\varrho S(\text{diam } U_j^{k(j)}),
$$

which tends to zero with $1/j$. Hence for almost all u we have $\lim \frac{\partial \xi_j^i}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}$; and since all the functions involved are uniformly bounded, it follows immediately that

$$
\begin{aligned}
& \lim_{j\to\infty} \int \cdots \int \frac{D(\xi_j^1, \cdots, \xi_j^p)}{D(u^1, \cdots, u^p)}\, du^1 \cdots du^p \\
(6) \qquad & = \int \cdots \int \frac{D(x^1, \cdots, x^p)}{D(u^1, \cdots, u^p)}\, d^1 v \cdots du^p.
\end{aligned}
$$

On the other hand, the integral $\int \cdots \int \Delta(x^1, \cdots, x^p)\, dv^1 \cdots dv^{p-1}$ involves only $p-1$ independent variables, satisfies the hypotheses of Theorem 1, and is linear in the Jacobians of the x's with respect to the v's, so that

[7] Rademacher, *Über partielle und totale Differenzierbarkeit,* Math. Annalen vol. 79, p. 340. The proof extends immediately to functions of n variables.

its E-function is identically zero; so that by Theorem 1 we have on each face of Q (and consequently on the whole boundary of Q)

$$(7)\ \lim \int \cdots \int \Delta(\xi_j^1, \cdots, \xi_j^p)\, dv^1 \cdots dv^{p-1} = \int \cdots \int \Delta(x^1, \cdots, x^p)\, dv^1 \cdots dv^{p-1}.$$

Comparing equations (3), (6), and (7), equation (4) is established.

LEMMA 3. *If the set of functions* $x_0(u)$ $(= x^1(u^1, \cdots, u^p) \cdots, x^p(u^1, \cdots, u^p))$ *and the sequence* $\{x_j(u)\}$ *all satisfy the same Lipschitz condition on the parallelopiped* Q: $(\alpha^i \leqq u^i \leqq \beta^i)$, *and* $\lim x_j(u) = x_0(u)$ *uniformly on* Q, *then*

$$\lim_{j \to \infty} \int \cdots \int \frac{D(x_j^1, \cdots, x_j^p)}{D(u^1, \cdots, u^p)}\, du^1 \cdots du^p$$
$$= \int \cdots \int \frac{D(x_0^1, \cdots, x_0^p)}{D(u^1, \cdots, u^p)}\, du^1 \cdots du^p.$$

We have merely to transform each integral by Lemma 2, and observe as in proving equation (7) that

$$\lim_{j \to \infty} \int \cdots \int \Delta(x_j^1, \cdots, x_j^p)\, dv^1 \cdots dv^{p-1} = \int \cdots \int \Delta(x_0^1, \cdots, x_0^p)\, dv^1 \cdots dv^{p-1}.$$

We now return to the proof of Theorem 1. By the definition of the E-function, the quantity in the brackets in equation 2 is equal to

$$\int_Q [f(x_j, J^\alpha(x_j)) - f(x_0, J^\alpha(x_j))]\, du$$
$$+ \sum_\alpha \int_Q [J^\alpha(x_j) - J^\alpha(x_0)]\, f_{J^\alpha}(x_0, J^\alpha(x_0))\, du.$$

The first integral here tends to zero, since $f(x, J^\alpha)$ is continuous and the arguments bounded and $\lim x_j(u) = x_0(u)$ uniformly. For the remaining terms, we have by Lemma 3 that for every parallelopiped Q' contained in Q

$$\lim_{j \to \infty} \int_{Q'} [J^\alpha(x_j) - J^\alpha(x_0)]\, du = 0;$$

hence[8] each term of the sum tends to zero, and our theorem is established.

For completeness, we give here the proof that Theorem 1 holds in the case of one independent variable; for the induction this was assumed true. For this case the quantity in brackets in equation (2) can be written

$$\int_a^b [f(x_j, x_{ju}) - f(x_0, x_{ju})]\, du + \sum_{k=1}^n \int_a^b \left(\frac{dx_j^k}{du} - \frac{dx_0^k}{du} \right) f_{x_u^k}(x_0, x_{0u})\, du,$$

[8] Hobson, *Theory of Functions of a Real Variable*, vol. II, § 279.

where x_u denotes the set $\left(\frac{dx^1}{du}, \cdots, \frac{dx^n}{du}\right)$. As above, the first integral tends to zero. For each interval $[\alpha, \beta]$ contained in $[a, b]$ we have

$$\int_\alpha^\beta \left(\frac{dx_j^k}{du} - \frac{dx_0^k}{du}\right) du = [x_j^k(\beta) - x_0^k(\beta)] - [x_j^k(\alpha) - x_0^k(\alpha)],$$

which tends to zero with $1/j$; applying the theorem already cited[9], we see that each term of the sum tends to zero.

3. The condition of Weierstrass. We now restrict ourselves to the case of n-spreads in $n+1$-space; $x(u) = (x^1(u^1, \cdots, u^n), \cdots, x^{n+1}(u^1, \cdots, u^n))$. Suppose that we are given a class $\mathfrak{K}$ of n-spreads all lying in a point set A of $n+1$-space, and that a particular spread $S_0 : x = x_0(u)$ (u on D) is a minimizing spread for $\int f(x, J^\alpha)\, du$ in the class $\mathfrak{K}$. We designate by L the set of points u such that

1) u is interior to D, and $x_0(u)$ is interior to A;
2) $x_0(u)$ is a point of indifference of S_0 with respect to $\mathfrak{K}$ and A;
3) in some neighborhood of u, the functions $x_0(u)$ satisfy some Lipschitz condition;
4) $J^\alpha(x_0)$ all are defined at u, and are not all zero.
5) The partial derivatives $\frac{\partial x_0^i}{\partial u^k}$ (and consequently the $J^\alpha(x_0)$) are all approximately continuous at u. (This condition rejects only a set of measure zero, since every measurable function is approximately continuous almost everywhere.)

We can then prove

THEOREM II. *For all points u in L, the inequality $E\left(x_0(u), J^\alpha(x_0(u)), \bar{J}^\alpha\right) \geqq 0$ holds for all sets of numbers $\bar{J}^\alpha$.*

In order to simplify notations, we shall give the proof only for the case in which condition 5) is replaced by the stronger condition

5′) All the derivatives $\frac{\partial x_0^i}{\partial u^k}$ are continuous at and near u.

This implies that 3) is necessarily satisfied. It will be seen without difficulty that the proof of the theorem as above stated requires merely the introduction of analytical details of a well-known type.

Suppose the theorem false; we can then find a point $P = (u_0^1, \cdots, u_0^1)$ in L and a set of numbers $\bar{J}^\alpha$ such that $E\left(x_0(u_0), J^\alpha(x_0(u_0)), \bar{J}^\alpha\right) = -2k < 0$. We can suppose without loss of generality that u_0 is the origin and that $\sum_\alpha \left[J^\alpha(x_0(0))\right]^2 = 1$; this can be brought about by a translation and

[9] Hobson, loc. cit.

a stretching $\bar{u}^i = Ku^i$, under which conditions 1), 2), 3), 4), 5') continue to be satisfied and E remains unchanged in value. Moreover, we can assume $\sum_\alpha (\bar{J}^\alpha)^2 = 1$, since E is homogeneous of degree 1 in the $\bar{J}^\alpha$.

We now change over to a more convenient system of parameters. The two distinct vectors $J^\alpha(x_0(0))$ and $\bar{J}^\alpha$ determine a flat $n-1$ dimensional space orthogonal to both; in this space we choose a system of $n-1$ mutually orthogonal unit vectors

$$\tilde{\xi}_1 = (\xi_1^1, \cdots, \xi_1^{n+1}), \qquad \tilde{\xi}_2, \cdots, \tilde{\xi}_{n-1}.$$

Let $\tilde{\xi}_{n+1}$ be the vector $J^\alpha(x_0(0))$, and $\tilde{\xi}_n$ a unit vector orthogonal to all the others. Then (by at most the reversal of one vector, say $\tilde{\xi}_1$) we will have the determinant $|\xi_j^i|$ orthogonal and of value 1. The hyperplane

$$x^i = \zeta(v) = x_0^i(0) + v^1 \xi_1^i + \cdots + v^n \xi_n^i$$

is orthogonal to $J^\alpha(x_0(0))$, and is therefore the tangent hyperplane

$$x^i = x_0^i(0) + \frac{\partial x_0^i}{\partial u^1} u^1 + \cdots + \frac{\partial x_0^i}{\partial u^n} u^n;$$

the u's are expressible in terms of the v's by means of a set of linear expressions of non-vanishing determinant; in fact, of determinant $+1$. For let us write $y(v) = x_0(u(v))$. The normal to the tangent hyperplane to $y(v)$ at the origin is the vector $\tilde{\xi}_{n+1}$, which is $J^\alpha(x_0(0))$; and

$$J^\alpha(y(0)) = J^\alpha(x_0(0)) \cdot \frac{D(u^1, \cdots, u^n)}{D(v^1, \cdots, v^n)},$$

so that the transformation determinant is $+1$. In terms of these new variables it is easily seen that all the $\dfrac{\partial y^i}{\partial v^k}$ are continuous at and near the origin.

The unit vector $\bar{J}^\alpha$ is coplanar with $\tilde{\xi}_n$ and $\tilde{\xi}_{n+1}$, and so can be represented as $\cos\vartheta \cdot \tilde{\xi}_{n+1} - \sin\vartheta \cdot \tilde{\xi}_n$. Then the surface

$$x^i = v^1 \xi_1^i + \cdots + v^{n-1} \xi_{n-1}^i + v^n [\cos\vartheta \cdot \xi_n^i + \sin\vartheta \cdot \xi_{n+1}^i]$$

is normal to $\bar{J}^\alpha$; in fact its Jacobians are the $\bar{J}^\alpha$.

We now define a set of auxiliary functions $\eta(v)$ in the cube Q: $-1 \leqq v^i \leqq +1$. The point set P for which $0 \leqq v^n \leqq 1$, $\sum_1^{n-1} |v^j| \leqq 1$ lies in Q, and for each positive $\varepsilon < 1/2$ the point set Π_ε for which $\sum_1^{n-1} |v^j| + \varepsilon^{-1} v^n \leqq 1$ $(v^n \geqq 0)$ lies in P. On Π_ε we set

$$\eta^i(v^1, \cdots, v^n) = v^n [(\cos\vartheta - 1)\, \xi_n^i + \sin\vartheta \cdot \xi_{n+1}^i];$$

on $P-\Pi_\varepsilon$,

$$\eta^i(v^1, \cdots, v^n)$$

$$= \varepsilon(1-v^n)\Big(1-\sum_1^{n-1}|v^j|\Big)\Big[1-\varepsilon\Big(1-\sum_1^{n-1}|v^j|\Big)\Big]^{-1}[(\cos\vartheta-1)\xi_n^i+\sin\vartheta\cdot\xi_{n+1}^1];$$

on $Q-P$, we set

$$\eta^i(v^1, \cdots, v^n) = 0.$$

This function is continuous and satisfies a Lipschitz condition on Q, and vanishes on the boundary. Moreover we readily verify

$$(8) \qquad \left|\frac{\partial \eta^i}{\partial v^k}\right| \leqq 16\varepsilon \ (v \text{ on } Q-\Pi_\varepsilon); \quad J^\alpha(\eta(v)+\zeta(v)) = \bar{J}^\alpha(v \text{ on } \Pi_\varepsilon).$$

The ratio $m(\Pi_\varepsilon)/m(Q)$ is easily calculated to be equal to $\frac{1}{2}\varepsilon n^{-n/2}$. We designate by D' the set into which D is carried by the transformation $v = v(u)$.

We now list two simple consequences of the definition of E and of the continuity of the $\dfrac{\partial y^i}{\partial v^k}$:

A) If $\eta^i(v) = v^n[(\cos\vartheta-1)\xi_n^i+\sin\vartheta\cdot\xi_{n+1}^1]$, there exists a positive number β such that for all v such that $|v^i| \leqq \beta$ $(i=1,\cdots,n)$, $y(v)$ is interior to D' and $E\big(y(v), J^\alpha(y(v)), J^\alpha(\eta(v)+y(v))\big) < -k$. (For by equation (8) $J^\alpha(\eta(0)+y(0)) = \bar{J}^\alpha\big)$.

B) There exist positive numbers β', $\delta < 1$ such that if $|v^i| \leqq \beta'$ and $\left|\dfrac{\partial \eta^k}{\partial v^i}\right| \leqq t < \delta$ for all i and k, then

$$\big|E\big(y(v), J^\alpha(y(v)), J^\alpha(\eta(v)+y(v))\big)\big| \leqq 2^{-8}K_1 t^2, \qquad K_1 \text{ a constant.}$$

(For $|J^\alpha(y(v)) - J^\alpha(\eta(v)+y(v))| < K_2 t$; take β' small enough so that $\sum_\alpha[J^\alpha(y(v))]^2 > \frac{1}{2}$, and $\delta = \frac{1}{4}K_2 n$, and express E as the remainder in a Taylor's expansion of $f(y, J^\alpha)$.)

Now about the origin let us construct a cube Q_1: $-q_1 \leqq v^i \leqq q_1$, having q_1 less than the smaller of β and β', and small enough to serve as the Q in the definition of points of indifference. For each integer j we divide Q_1 into $(2^n)^j$ smaller cubes, and define a function $y_j(v)$ in the following manner. Consider a small cube of the jth subdivision; its side will be $2q_1\cdot 2^{-j}$, and its center say $\bar{v}^1, \cdots, \bar{v}^n$. On this cube we define

$$y_j(v) = y(v) + q_1 2^{-j}\cdot\eta(q_1^{-1}2^j(v^1-\bar{v}^1), \cdots, q_1^{-1}2^j(v^n-\bar{v}^n)).$$

We do this for each small cube, and for all v not in Q we define $y_j(v) = y(v)$. The functions $y_j(v)$ are all continuous and tend to $y(v)$ uniformly; for large

enough values of j they define spreads of $\mathfrak{K}$, since $y(0)$ is a point of indifference of S_0 with respect to $\mathfrak{K}$ and A. Since $\dfrac{\partial y_j^h}{\partial u^k} = \dfrac{\partial y^h}{\partial u^k} + \dfrac{\partial \eta^h}{\partial u^k}$ is bounded, they all satisfy the same Lipschitz condition. In defining $y_j(v)$ we mapped $Q\colon -1 \leqq v \leqq 1$ linearly on each of the 2^{nj} small cubes; in doing so we mapped Π_ε on 2^{nj} point sets similar to it, of total measure $\frac{1}{2}\varepsilon n^{-n/2} m(Q_1)$. Let the sum of these point sets be called $\Pi_{\varepsilon j}$.

Now by Theorem 1 we can write

$$\begin{aligned}\overline{\lim_{j\to\infty}} &\left\{\int_{D'} f(y_j, J^\alpha(y_j))\,dv - \int_{D'} f(y, J^\alpha(y))\,dv\right\}\\ &= \overline{\lim_{j\to\infty}} \int_{Q_1} E(y, J^\alpha(y), J^\alpha(y_j))\,dv\\ &= \overline{\lim_{j\to\infty}} \left\{\int_{Q_1 - \Pi_{\varepsilon j}} + \int_{\Pi_{\varepsilon j}} E(y, J^\alpha(y), J^\alpha(y_j))\,dv\right\}.\end{aligned}$$

By property B) and inequality (8) the first integral is less than $K_1 \varepsilon^2 m(Q_1)$, provided that $\varepsilon \leqq \delta/16$. Since on $\Pi_{\varepsilon j}$ the Jacobians $J^\alpha(y_j)$ are identical with the $J^\alpha(\eta + y)$ where $\eta^i(v) = v^n[(\cos\vartheta - 1)\xi_n^i + \sin\vartheta\cdot\xi_{n+1}^i]$, we have by property A) that the second integral is at most

$$-k\cdot m(\Pi_{\varepsilon j}) = -k\cdot \tfrac{1}{2}\varepsilon n^{-n/2} m(Q_1);$$

the sum is therefore less than $m(Q_1)\,\varepsilon[K_1\varepsilon - \frac{1}{2}k n^{-n/2}]$, which is negative if ε be chosen as the smaller of $\delta/16$ and $\frac{1}{4}K_1^{-1} n^{-n/2} k$. This is in contradiction to the hypothesis that $x = x_0(u) = y(v)$ is a minimizing spread for $\int f(x, J^\alpha)\,du$ in the class $\mathfrak{K}$, and our theorem is established.

4. **The Legendre condition. Curves in n-space.** An immediate consequence of Theorem II is

THEOREM III. *For all points u in L, the quadratic form*

$$\sum_{\alpha,\beta} \bar{J}^\alpha\cdot f_{J^\alpha J^\beta}\big(x_0(u), J^\alpha(x_0(u))\big)\cdot \bar{J}^\beta \geqq 0$$

for all sets of numbers $\bar{J}^\alpha$.

For by Theorem II $E(x_0(u), J^\alpha, J^\alpha + r\bar{J}^\alpha) \geqq 0$ for all real numbers r. Expanding by Taylor's theorem,

$$\sum_{\alpha,\beta} r^2 \bar{J}^\alpha\cdot f_{J^\alpha J^\beta}(x_0, J^\alpha + \theta r\bar{J}^\alpha)\cdot\bar{J}^\beta \geqq 0 \qquad (0 < \theta < 1).$$

Letting now r tend to zero and observing that since $\sum_\alpha (J^\alpha)^2 > 0$ the $f_{J^\alpha J^\beta}$ are continuous, we have the desired result.

The methods developed above can be used almost without change to prove for curves in $n+1$-space the theorem corresponding to Theorem II.

If we adopt the usual hypothesis that $\mathfrak{K}$ is a class of rectifiable curves, then the minimizing curve can always be expressed by Lipschitzian functions: $x^i = x^i(s)$ $(0 \leqq s \leqq 1)$. Using this parameterization, we can prove

THEOREM IV. *If the curve* C_0: $x^i = x^i(s)$ *be a minimizing curve for* $\int_C f(x, x')\,ds$ *in a class* $\mathfrak{K}$ *of rectifiable curves lying in a point set* A *of* $n+1$*-space, then for almost all points of indifference of* C_0 *interior to* A *we have* $E(x_0, x_0', \bar{x}') \geqq 0$ *for all sets of numbers* $\bar{x}'$. *In particular, the inequality holds at all such points at which the derivatives* $x^{i'}$ *are approximately continuous and* $\sum (x^{i'})^2 > 0$.

We have merely to follow the lines of the proof of Theorem II, replacing $J^1, \cdots, J^{n+1}$ by $x^{1'}, \cdots, x^{(n+1)'}$; the change of variables from u to v becomes unnecessary, Π_ε becomes the interval $[0, \varepsilon]$ and P the interval $[0, 1]$, and the auxiliary functions η^i can be written

$$\eta^i(t) = t(\bar{x}^{i'} - x^{i'}) \qquad (0 \leqq t \leqq \varepsilon)$$
$$\eta^i(t) = \varepsilon(1-t)(1-\varepsilon)^{-1}(\bar{x}^{i'} - x^{i'}) \qquad (\varepsilon \leqq t \leqq 1).$$

Condition 3) in the definition of L is fulfilled everywhere, and condition 4) almost everywhere.

COLUMBUS, OHIO.
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SOME ARITHMETICAL PROPERTIES OF SEQUENCES SATISFYING A LINEAR RECURSION RELATION.[1]

By Morgan Ward.

1. Let

$$(U)_n\colon\quad U_0,\ U_1,\ U_2,\ \cdots$$

denote a sequence of integers satisfying the recursion relation

$$\Omega_{N+1+n} = P_1\,\Omega_{N+n} + P_2\,\Omega_{N+n-1} + \cdots + P_{N+1}\,\Omega_n \tag{1.1}$$

where $P_1, \cdots, P_{N+1}$ and the $N+1$ initial values $U_0, \cdots, U_N$ of $(U)_n$ are all fixed integers.

Let p denote a fixed prime, and assume furthermore that the characteristic function of (1.1)

$$F(x) = x^{N+1} - P_1 x^N - \cdots - P_{N+1}$$

is irreducible modulo p.

Denote the $N+1$ roots of the equation $F(x) = 0$ by $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_N$ and let S_m denote $\alpha_0^m + \alpha_1^m + \cdots + \alpha_N^m$. For convenience of printing, we shall occasionally write $\alpha(n)$ for α^n and $\{m\}$ for S_m.

In this paper, I give a number of congruences to the modulus p satisfied by particular solutions of (1.1) and by determinants relating to such solutions. The two main results are as follows:

If $(U)_n$ is any particular solution of (1.1), *then*[2]

$$U_{n+m} + U_{n+pm} + U_{n+p^2m} + \cdots + U_{n+p^Nm} \equiv U_n S_m \bmod p. \tag{I}$$

If $N+1$ is odd, and if $M(r_0, r_1, \cdots, r_N)$ denotes the determinant

$$|\alpha_i^{r_j}|, \qquad (i, j = 0, 1, \cdots, N)$$

where $r_0, r_1, \cdots, r_N$ are any fixed integers, then

$$M(r_0, r_1, \cdots, r_N) \equiv \sum_{(j)} (\pm)^j \{r_0 + p\,r_{j_1} + p^2 r_{j_2} + \cdots + p^N r_{j_N}\} \bmod p, \tag{II}$$

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[1] Received October 1, 1930, and February 3, 1931.

[2] If μ is the exponent to which α belongs, modulo p, the relations $U_a \equiv U_b$, $S_a \equiv S_b$ mod p if $a \equiv b \bmod \mu$ may be used to reduce the subscripts of U and S to values less than μ.

where the summation is extended over all the $N!$ permutations (j) of the integers $1, 2, \cdots, N$ and the sign $(\pm)^j$ is to be taken positive or negative according as the permutation is of even or odd parity.

For example, if $N+1 = 3$, we have

$$\begin{vmatrix} \alpha_0^{r_0}, & \alpha_0^{r_1}, & \alpha_0^{r_2} \\ \alpha_1^{r_0}, & \alpha_1^{r_1}, & \alpha_1^{r_2} \\ \alpha_2^{r_0}, & \alpha_2^{r_1}, & \alpha_2^{r_2} \end{vmatrix} \equiv S_{r_0+pr_1+p^2r_2} - S_{r_0+pr_2+p^2r_1} \mod p.$$

The proofs of these formulas are given in the next two sections of the paper. The final section contains some special cases of the first formula and some properties of the determinant $M(r_0, r_1, \cdots, r_N)$ regarded as a function of $r_0, r_1, \cdots, r_N$.

2. With a proper choice of notation, we may assume that

$$\alpha_i \equiv \alpha^{p^i} \mod p, \qquad (i = 0, 1, \cdots, N) \tag{2.1}$$

in the Galois Field[3] of order p^{N+1} associated with the root α of $F(x) = 0$.

Since $F(x)$ is irreducible, the general term of $(U)_n$ may be represented as

$$U_n = A_0\,\alpha_0^n + A_1\,\alpha_1^n + \cdots + A_N\,\alpha_N^n$$

where the constants A are independent of n. We shall take this formula as a definition of U_n when n is a negative integer. Thus

$$U_n \equiv \sum_{r=0}^{N} A_r\,\alpha^{p^r n} \mod p, \qquad S_m \equiv \sum_{s=0}^{N} \alpha^{p^s m} \mod p. \tag{2.2}$$

To prove formula I, we observe that

$$\sum_{s=0}^{N} \alpha^{p^{r+s}m} \equiv S_m \mod p$$

for any integer r. Hence

$$\sum_{s=0}^{N} U_{n+p^s m} \equiv \sum_{s=0}^{N}\sum_{r=0}^{N} A_r\,\alpha^{p^r(n+p^s m)} \equiv \sum_{r=0}^{N} A_r\,\alpha^{p^r n} \sum_{s=0}^{N} \alpha^{p^{r+s}m} \equiv U_n S_m \mod p.$$

3. Formula II may be proved as follows. With the notation explained in the introduction,

$$M(r_0, r_1, \cdots, r_N) = \sum_{(j)} (\pm)^j\,\alpha_0^{r_{j_0}}\,\alpha_1^{r_{j_1}} \cdots \alpha_N^{r_{j_N}}.$$

Hence by (2.1),

$$M(r_0, r_1, \cdots, r_N) \equiv \sum_{(j)} (\pm)^j\,\alpha(r_{j_0} + r_{j_1}p + \cdots + r_{j_N}p^N) \mod p. \tag{3.1}$$

[3] For the properties of Galois Fields which are assumed, see Dickson, *Linear Groups*, Teubner, (1901).

The $(N+1)!$ permutations (j) of the integers $0, 1, \cdots, N$ which occur in the subscripts of the r on the right hand side of (3.1) may be grouped into $N!$ classes

$$J_1, J_2, \cdots, J_{N!} \tag{3.2}$$

where each class contains exactly $N+1$ cyclic permutations. Suppose that

$$j_0, j_1, \cdots, j_{N-1}, j_N; \quad j_1, j_2, \cdots, j_N, j_0; \quad \cdots; \quad j_N, j_0, \cdots, j_{N-2}, j_{N-1}; \tag{3.3}$$

are the permutations of the class J. These permutations are either all even or all odd; for since $N+1$ is odd, each can be derived from its predescessor by an even number of transpositions.[4] Accordingly, the sign of the general term $\alpha(r_{j_0} + r_{j_1} p + \cdots + r_{j_N} p^N)$ in (3.1) is the same for all the permutations of a given class J.

Furthermore, since any one of the permutations (3.3) completely specifies the class J, we may choose our notation so that $j_0 = 0$. Make a similar change of notation for every other one of the classes (3.2). Then to each of the $N!$ permutations of the integers $1, 2, \cdots, N$ there corresponds a unique class J, and the parity of this permutation determines the parity of all the permutations of J.

The congruence (3.1) can now be written as

$$M(r_0, r_1, \cdots, r_N) \equiv \sum_{(j)} (\pm)^j \sum \alpha(r_0 + r_{j_1} p + \cdots + r_{j_N} p^N) \mod p \tag{3.4}$$

where the inner summation is taken over the $N+1$ permutations (3.3), while the outer summation (j) is taken over the $N!$ permutations of $1, 2, \cdots, N$, the sign being plus or minus according as (j) is even or odd.

But since $\alpha(rp^{N+1}) \equiv \alpha(r) \mod p$, the inner summation in (3.4) is congruent modulo p to

$$\begin{aligned} \alpha(r_0 + r_{j_1} p + \cdots + r_{j_N} p^N) + \alpha(p \cdot (r_0 + r_{j_1} p + \cdots + r_{j_N} p^N)) + \cdots \\ \cdots + \alpha(p^N \cdot (r_0 + r_{j_1} p + \cdots + r_{j_N} p^N)) \\ \equiv \{r_0 + r_{j_1} p + \cdots + r_{j_N} p^N\} \mod p, \end{aligned}$$

by formula (2.2). On substituting this last expression in (3.4), we obtain formula II.

4. The following special cases of formula I are of interest. First, if we write S for U in I, we obtain a multiplication formula for the function S:

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[4] It is at precisely this point that an attempted proof of a similar result for the case when the degree of $F(x)$ is even will break down.

$$S_n S_m \equiv S_{n+m} + S_{n+pm} + \cdots + S_{n+p^N m} \mod p. \tag{4.1}$$

Secondly, let $(Z^0)_n$; $(Z^{(1)})_n, \cdots, (Z^{(N)})_n$ denote the particular solutions of (1.1) with the initial values

$$1, 0, 0, \cdots, 0; \quad 0, 1, 0, \cdots, 0; \quad \cdots\cdots; \quad 0, 0, 0, \cdots, 1.$$

Then if the Kronecker symbol δ_{ij} is defined as usual by

$$\delta_{ij} = 0, \quad i \neq j; \qquad \delta_{ij} = 1, \quad i = j; \qquad (i, j = 0, 1, \cdots, N),$$

we have on taking $Z^{(i)}$ for U in I the curious formulas

$$Z^{(i)}_{j+m} + Z^{(i)}_{j+pm} + \cdots + Z^{(i)}_{j+p^N m} \equiv \delta_{ij} S_m \mod p,$$
$$(i, j = 0, 1, \cdots, N;\ m = 0, \pm 1, \cdots).$$

The function M has a multiplication theorem analogous to that given for S_m in formula (4.1). If for brevity we write $R_{(k)}$ for $r_{k_0} + r_{k_1} p + \cdots + r_{k_N} p^N$, then

$$\begin{aligned} & M(r_0, r_1, \cdots, r_N) \cdot M(u_0, u_1, \cdots, u_N) \\ & \quad \equiv \sum_{(k)} (\pm)^k M(R_{(k)} + u_0, u_1, \cdots, u_N) \mod p, \end{aligned} \tag{4.2}$$

where the summation (k) extends over all the $(N+1)!$ permutations of the integers $0, 1, \cdots, N$ and the signs are determined as in II by the parity of (k).

To prove (4.2), write $R_{(j)}$ for $r_0 + r_{j_1} p + \cdots + r_{j_N} p^N$. Then by formula II and formula (4.1)

$$\begin{aligned} & M(r_0, r_1, \cdots, r_N) \cdot M(u_0, u_1, \cdots, u_N) \\ & \quad \equiv \sum_{(j)} \sum_{(l)} (\pm)^j (\pm)^l \{R_{(j)}\} \{u_0 + u_{l_1} p + \cdots + u_{l_N} p^N\} \\ & \quad \equiv \sum_{(j)} (\pm)^j \sum_{t=0}^{N} \sum_{(l)} (\pm)^l \{R_{(j)} + p^t u_0 + p^t u_{l_1} p + \cdots + p^t u_{l_N} p^N\} \\ & \quad \equiv \sum_{(j)} (\pm)^j \sum_{t=0}^{N} M(R_{(j)} + p^t u_0, p^t u_1, \cdots, p^t u_N) \mod p. \end{aligned} \tag{4.3}$$

We now reverse the argument in section 3 by which we passed from the $(N+1)!$ permutations of $0, 1, 2, \cdots, N$ to the $N!$ permutations of $1, 2, \cdots, N$. Let $k_0 = j_t$, $k_1 = j_{t+1}, \cdots, k_N = j_{t-1}$ so that (k): $k_0, k_1, \cdots, k_N$ is a cyclic permutation of the subscripts $0, j_1, \cdots, j_N$ of the r in $R_{(j)}$. Then (k) is a permutation of $0, 1, \cdots, N$ which has the same parity as the permutation (j) of $1, 2, \cdots, N$.

If we now write $R_{(k)}$ for $r_{k_0}+r_{k_1}p+\cdots+r_{k_N}p^N$, then

$$\begin{aligned}(\pm)^j\, M(R_{(j)}+p^t u_0,\, p^t u_1,\, \cdots,\, p^t u_N) \\ \equiv (\pm)^k\, M(p^t R_{(k)}+p^t u_0,\, p^t u_1,\, \cdots,\, p^t u_N) \\ \equiv (\pm)^k\, M(R_{(k)}+u_0,\, u_1,\, \cdots,\, u_N) \quad \bmod p.\end{aligned}$$

On substituting this expression into (4.3), we obtain (4.2).

In conclusion, we may note that $M(0, 1, \cdots, N)$ is the square root of the discriminant of $F(x)$. Denoting this discriminant by Δ, we have from II after some obvious simplifications,

$$\sqrt{\Delta} \equiv \sum_{(j)} (\pm)^j \{j_1+j_2 p+\cdots+j_N p^{N-1}\} \quad \bmod p.$$

For $N+1=3$, this result assumes the simple form

$$\sqrt{\Delta} \equiv S_{1+2p} - S_{2+p} \quad \bmod p.$$

PASADENA,
September, 1930.

SOME APPLICATIONS OF POINT-SET METHODS.[1]

By Karl Menger.

In a mathematical seminary at Harvard University during the winter of 1930-31 some problems in different branches of mathematics were treated from the point of view of metrical geometry and the topology of point sets. The present paper contains results obtained in the following topics:

1. **Some remarks on the length of arcs and the area of surfaces.** In a recent paper,[2] I proved the following results: If A is an arc (a topological image of the line segment) in a euclidean space and if for each finite set F we denote by $\lambda(F)$ the length of the shortest polygonal line joining the points of F, then the length of A equals the upper bound of the numbers $\lambda(F)$ for all finite subsets F of A. Furthermore a set G which is the sum of a finite number of line segments whose endpoints lie in the finite set F may be called a *linear graph inscribed* in F. We call the length of the linear graph G the sum of the lengths of the segments whose sum is G. If F is a given finite set there is only a finite number of linear graphs inscribed in F and each of them has a length. We denote by $\varkappa(F)$ the length of the shortest connected graph inscribed in F. For each finite set F obviously $\varkappa(F) \leqq \lambda(F)$. The length of an arc A, it was proved, equals the upper bound of the numbers $\varkappa(F)$ for all finite subsets F of A.

This result has not only been proved for arcs in euclidean spaces but for arcs in general metrical spaces. A set of elements (called points) is called a *metrical space* if to each couple of elements p and q there corresponds a real number denoted by pq and called the distance between p and q subject to the conditions that $pq = qp$, that this number is >0 if $p \neq q$ and $= 0$ if $p = q$, and that for each three points, p, q, r

[1] Received February 26, 1931.

[2] Mathematische Annalen 103 (1930), pp. 466–501.

the *triangle inequality* i.e. $pq+qr \geqq pr$ holds. A finite subset F of a metrical space is called a graph if there is a rule which for each two points of F determines whether or not they are neighbor points. The length of such a graph is defined to be the sum of the distances of all pairs of neighbor points. A graph is called connected if it contains for each pair of points a chain of points whose end points are the given points and each two consecutive points of which are neighbor points. It has been proved that in this terminology the length of an arc A in a metrical space equals the upper bound of the numbers $\varkappa(F)$ for all finite subsets F of A where $\varkappa(F)$ denotes the length of the shortest connected graph inscribed in F.

This method of shortest polygonal lines and graphs of smallest length in connection with the problem of the length of arcs can be generalized to higher dimensions. We may consider polyhedral surfaces of smallest area inscribed in a finite set in order to get a definition of area of surfaces. In this way we get rid of these difficulties of the classical theory which are due to the discovery by Schwarz that it is possible to inscribe in a cylindrical surface polyhedra whose area surpasses the area of the cylinder and can be made, in fact, arbitrarily large. The reason why our proceedure gets rid of these difficulties is that even for the set of all vertices of a Schwarz polyhedron the polyhedron of *smallest* area inscribed in the finite set does *not* surpass the cylinder in area. A certain restriction however is necessary even if we consider polyhedra with *smallest* area. If we consider, e. g. in the plane a concave triangle then, if F denotes the set of the three vertices, the area of the smallest polyhedron which can be inscribed in F (i. e. the area of the rectilinear triangle with vertices in F) surpasses the area of the surface. This suggests that we restrict the method first of all to *convex* surfaces. For convex surfaces which are subsets of the plane it is indeed obvious that their area equals the upper bound of the smallest areas which can be inscribed in the finite subsets of the surface.

Mr. Whitney suggested to consider, more generally, surfaces which are *homeomorphic to the sphere* and to prove for them that their area equals the upper bound of the numbers $\lambda'(F)$ for all finite subsets F of S where $\lambda'(F)$ denotes the area of the smallest polyhedral surface which is homeomorphic to the sphere and can be inscribed in F. It seems probable that if $\lambda_g(F)$ denotes the area of the smallest polyhedral surface of genus g which can be inscribed in the finite set F, the area of a surface S of genus g equals the upper bound of the numbers $\lambda_g(F)$ for all finite subsets F of S.

For convex surfaces the problem can be divided. It has been proved by Mr. Littauer that if $\varkappa'(F)$ denotes the surface of the *convex* polyhedron

determined by the finite set F, the area of a convex surface S equals the upper bound of the numbers $\varkappa'(F)$ for all finite subsets F of S. This gives a very simple definition of area for convex surfaces.

The second question is wether the area of a convex surface S (which is always homeomorphic to a sphere) equals the upper bound of the numbers $\lambda'(F)$ where $\lambda'(F)$ denotes the area of the smallest polyhedron homeomorphic to the sphere which can be inscribed in F. This question is not answered by the first mentioned result, for the convex polyhedron determined by a finite set F is not necessarily the polyhedron with smallest area as Mr. Galbraith has shown by the following simple example. If we denote by F the set consisting of the five following points of three dimensional euclidean space; $p_1 = (0, 0, 0)$, $p_2 = (0, 0, 1)$, $p_3 = (0, 1, 0)$, $p_4 = (1, 0, 0)$, $p_5 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{10})$ then the concave surface consisting of the triangles (p_1, p_2, p_3), (p_1, p_2, p_4), (p_2, p_3, p_5), (p_2, p_4, p_5), (p_1, p_3, p_5), (p_1, p_4, p_5) has a smaller area than the convex polyhedron determined by F which consists of the triangles: (p_1, p_2, p_3), (p_1, p_2, p_4), (p_2, p_3, p_5), (p_2, p_4, p_5), (p_1, p_3, p_4), (p_3, p_4, p_5).

The set of all arcs, as is well known, is not compact (there exist converging sequences of arcs whose limits are not arcs) and in the system of all arcs the length of arcs is merely a *semi*-continuous but not a continuous function. Mr. Littauer remarked that the system of all *convex* arcs whose length surpasses a given $\varepsilon > 0$ contained in a plane square is compact and that, furthermore, the set containing all convex arcs and all points of a square is also compact, and *that in this compact system of all convex arcs the length of arcs is a continuous function.*

There is one problem unsolved concerning the length of arcs. We denote by $\varkappa(F)$ the length of the shortest graph inscribed in the finite set F. Let us denote by $\iota_R(F)$ the lower bound of all numbers $\varkappa(F')$ for all finite subsets F' of the space R which contain F as subset. If R is a euclidean space then $\iota_R(F)$ is the length of the shortest graph containing the set F. The graph may, however, also contain vertices outside the set F. Whereas there is only a finite number of graphs *inscribed* in a finite set F there are in an infinite metrical space (e. g. in euclidean space) infinitely many graphs *containing* the set F. This is why we have to define $\iota_R(F)$ as the *lower bound* of a set of numbers which, in general, may contain infinitely many numbers, whereas we could define $\varkappa(F)$ as the *smallest* length of a graph inscribed in F. The question which is still open is whether the length of an arc A in a metrical space R equals (or may surpass) the upper bound of the numbers $\iota_R(F)$. In other words we do not know whether or not in each metrical space R the length $l(A)$ of the arc A satisfies the condition

$$l(A) = \limsup_{F \subset A} \liminf_{F \subset F' \subset R} \varkappa(F'). \tag{1}$$

This relation certainly is true in the case that the metrical space which contains the arc A is identical with A. For in this case each graph containing a set F is inscribed in another finite subset F' of A. Hence, if the relation should not hold in general, then the term on the right side of (1) would depend not only on the arc A but also on the space R, and if this term were to be used as definition of length of A in R, an arc A could have different lengths in different spaces. This seems not very probable though it must be noticed that the analogous situation really occurs in two dimensions. We have denoted by $\lambda'(F)$ the area of the polyhedron of smallest surface which is homeomorphic to the sphere and can be inscribed in the set F. It is easy to see[3] that for each finite set F of a euclidean space there exist polyhedral surfaces homeomorphic to the sphere which contain F but have also some vertices outside of F and have an arbitrarily small area. The lower bound $\iota_R(F')$ of the numbers $\varkappa(F')$ for all finite subsets F' of the euclidean space containing a given set F is therefore 0 for each given set F. Hence, for a surface S, contained in a metrical space R the value of $\limsup_{F \subset S} \limsup_{F \subset F' \subset R} \lambda'(F')$ depends on the space R under consideration. The number is 0 if R is a euclidean space. It is >0 if R is identical with S.

2. **On convexity.** The term "convex" is, in the classic theory, used in three senses: a) An open set or the *closed cover of an open subset* of n-dimensional euclidean space is called convex if, for each pair of points, the set contains as a subset the line segment determined by the two points. b) An $(n-1)$-dimensional *hypersurface* of the n-dimensional euclidean space is called convex if it is the boundary of a convex open set. c) Also certain *real functions* are called convex.

The first of these three concepts has been generalized in a general theory of convexity[4] by the following definition: A subset S of a complete metrical space is called *convex* if S contains for each two points p and q of S at least one point r *between* p and q, i. e. a point which is different from p and q and satisfies the equality $pr + rq = pq$. It has been proved that a closed and convex subset of a complete metrical space contains for each two points p and q a subset containing p and q and congruent to a line segment whose length equals the distance pq. If we call *externally convex* a set S which for each two points p and q contains two points o and r such that p lies between o and q and that q lies between p and r then it has

[3] Blaschke, *Kreis und Kugel*, 1916, p. 81.

[4] Menger, Mathematische Annalen 100 (1928), pp. 75–163.

been proved that a closed subset of a complete metrical space which is convex and externally convex contains for each two points p and q a subset containing p and q and congruent with the straight line.

b) As to convex hypersurfaces of an n-dimensional euclidean space, it is for certain problems desirable to have a more intrinsic definition of these objects than the one that defines them as boundaries of convex regions. Let us consider, first, the case of convex curves in the plane. Three points of the plane which do not lie in a straight line determine three straight lines. The complement of these three lines is the sum of seven mutually exclusive open sets: one part whose boundary contains the three points (the interior of the triangle), three parts each of which contains in its boundary two of the three points, and three parts each of which contains in its boundary one of the three points. The three parts of the second type, if to each of them the open line segment between the two triangle points on its boundary is added, may be called the three parts which are *adherent* to the three given points. In this terminology we prove:

In order that a closed and bounded subset S of the plane which is not a subset of a straight line be a convex curve it is necessary and sufficient that for each three points of S which do not lie on a straight line each of the three adherent parts of the plane contains at least one point of S, whereas no other of the seven corresponding parts of the plane has points in common with S and that for each three points p, r, t of S which lie on a straight line (r between p and t) there exist two points q and s of S between p and r and between r and t respectively.

To prove the necessity of the condition let us suppose that a closed set S contains three points p, q, r such that one of the four non-adherent parts of the plane contains a point s of S. Then one of the four points lies in the interior of the triangle determined by the three others and, therefore, lies in the interior of each convex region whose boundary contains the three other points. Hence the points p, q, r, s cannot lie on the boundary of a convex region. If S contains three points such that one of the adherent regions does not contain any point of S then, evidently, S cannot be the boundary of a bounded region either. In a similar way it can be proved that if S contains three points of a straight line without containing the line-segments between them, S cannot be the boundary of a convex region.

In order to prove that the conditions are sufficient, let us suppose that S is a set satisfying them. As S is not a subset of a straight line, S contains three points p, q, r not lying on a straight line. Let s be some point of the interior of the triangle determined by p, q, r. Each half-line ending in s has *at most* one point in common with S: for if a half-line had

two points in common with S, then there would exist in the set composed of these two points and the three points p, q, r at least one which would not lie in a part of the plane adherent to three of the points. Each half-line ending in s has *at least* one point in common with S: for otherwise, S being closed, there would exist two half-lines L and L' ending in s and having in common with S two points l and l' respectively, such that no half-line ending in s which lies in the angle between L and L' would have a point in common with S. Then, however, l and l' together with one of the points p, q, r would be three points such that one of the adherent parts of the plane would not contain any point of S.

As each half-line ending in s has exactly one point in common with S it follows that S is a closed curve. In order to prove that S is a convex curve we have to show that the bounded region whose boundary is S is a convex region. Let a and b be two points of this region, and s a point of this region different both from a and b. The half-lines ending in s and containing a and b, contain two points a' and b' of S. Let c' be a point of S lying on a half-line which ends in s and whose opposite half lies in the angle asb. If t is some point of the line segment joining a and b then t lies in the interior of the triangle a', b', c' and, therefore, does not belong to S. Hence each point of the segment joining a and b belongs to the region. The region is therefore convex.

Among the closed curves of the plane the convex curves can also be characterized, as Mr. Galbraith remarked, by the property *that for each couple of pairs of points separating each other the joining segments intersect* or by the property *that for each two points of the curve either all points or no points of the segment joining them belong to the curve.*

The property which we proved above to be characteristic for convex curves of the plane can serve as the definition of these curves. This definition can immediately be generalized to higher dimensions. Four points of three dimensional space which do not lie in a plane determine fifteen parts of the space. If we call *adherent* to the four points those ten parts whose boundaries have in common with the tetrahedron either a surface or a segment, then we may call a closed set S of the three dimensional space which is not a subset of a plane a *convex surface* if for each four points of S each of the ten adherent parts of the space contains at least one point and each other part no point of S. The divisions of the n-dimensional space into $2^{n+1}-1$ parts by $n+1$ points which do not lie in an $(n-1)$-dimensional hyperplane are the starting point of a generalization of the linear relation of betweenness by J. Priebsch.[5] His research

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[5] Monatshefte f. Mathem. u. Physik 38 (1931), p. 29.

combined with the above mentioned principle of defining $(n-1)$-dimensional convex surfaces in the n-dimensional space suggests a systematic study of the existential properties of closed sets with respect to the parts of the space determined by the systems of $n+1$ of their points. It would be especially interesting to know whether or not it is also possible to define, in this way, m-dimensional closed convex sets of the n-dimensional space for integers m which are $< n-1$.

c) As to the concept of convex functions, the theory of convexity gives a mean of generalizing it for general metrical spaces. We call a real function defined in a metrical space a *convex function* if for each three points p, q, r of the space for which $pq+qr=pr$ the inequality $\frac{f(q)-f(p)}{f(r)-f(p)} \geqq \frac{pq}{pr}$ holds and therefore $f(q) \geqq \frac{pq \cdot f(r)+qr \cdot f(p)}{pr}$. If f and g are two convex functions and c and d two positive numbers then, obviously, $c \cdot f + d \cdot g$ is a convex function too. We could call a function *concave* in which $pq+qr=pr$ always implied $f(q) \leqq \frac{pq \cdot f(r)+qr \cdot f(p)}{pr}$. A function which is convex as well as concave could be called a *linear function.* Each constant function is linear. There are metrical spaces in which each linear function is constant, e. g. the convex circle and the convex tripod.[6] It is an important step on the way from general metrical spaces to the euclidean spaces to demand that on each convex and closed subset of the space *there exists a non-constant linear function.* These spaces seem to be closely related to the spaces with the properties called Zweitripeleigenschaft and the Dreitripeleigenschaft.[7]

3. Convexity and differential geometry in the large. The general concept of convexity[8] plays an important rôle in differential geometry in the large which has previously not been emphasised. The reason for the fact that many problems and theorems of this discipline are subordinate to the general theory of convexity is, that a bounded[9] surface S in which each pair of points is joined by an arc of finite length is a convex metrical space if for each two points p and q of S we define as distance pq the length of the shortest arc joining p and q in S (such an arc always exists).[10] For, with this definition, we have $pq=qr>0$ if $p \neq q$ and $pp=0$; besides, evidently, the triangle inequality $pq+qr \geqq pr$ is satisfied

[6] Menger, l. c.[4], p. 110.

[7] Menger, l. c.[4], p. 107.

[8] Cf. § 2 a) of this paper.

[9] A subset of a euclidean space is called *bounded,* if it is a subset of a sphere with finite radius.

[10] Menger, l. c.[2], p. 492.

for each three points p, q, r. Thus if we call S' the set of all points of S with this definition of distance, then S' is a metrical space. If p and q are two points of S' and r is a point of the shortest arc of the surface S joining p and q then r lies between p and q in S' and hence the metrical space S' is convex. For each two points the distance in S' is at least as great as their distance in S, so it follows that each sequence of points converging towards a point p in S' converges towards p also in S. If we suppose *that for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that each two points of the given surface* S *whose distance is* $< \delta$ *are joined by an arc whose length is* $< \varepsilon$ then, conversely, each sequence of points converging towards a point p in S converges towards p also in S'. Under this hypothesis on S, which will be assumed in what follows, the convex metrical space S' is *homeomorphic to* S and therefore compact.

Let us now consider Blaschke's problem of whether a surface S of a three dimensional euclidean space without singularities in which each point has a conjugate point, or, in other words, *a surface in which for each point* p *all geodesics ending in* p *concur in a point* p', is necessarily *a sphere.* The space S' being convex and compact a point q of the space S' lies between two points p and p' if and only if p and p' can be joined in S by a geodesic passing through q. This gives a simple criterion which tells when all geodesics ending in a point p concur in the point p': this is the case if and only if each third point lies between p and p'. Let us call two points p and p' of a metrical space *opposite* if each third point lies between them. A metrical space cannot contain more than one point opposite to a given point p. For if p' and p'' were both opposite to p then p' would lie between p and p'', and p'' would lie between p and p' which is impossible. If p, p' and q, q' are two pairs of opposite points then $pq + qp' = pq' + q'p' = pp'$ and $qp + pq' = qp' + p'q' = qq'$, hence $pp' = qq'$ and $pq = p'q'$. Thus each pair of opposite points of a metrical space has the same distance and, if for each point of the space there exists an opposite point, then the application of the space onto itself by which each point is transformed into its opposite point is a congruent transformation.

Let us call a *convex sphere* (hemisphere) a sphere (hemisphere) each two points of which have as distance the length of the shorter segment of the great circle joining them. Then Blaschke's question evidently is contained in the following general problem, I: *Is a convex metrical space which is homeomorphic to the sphere and which for each point contains an opposite point necessarily congruent to the convex sphere?* We need make no assumptions as to the absence of singularities in this general problem, as such singularities are merely properties of the imbedding of the surface in

the euclidean space, but not properties of the intrinsic convex metric of the surface. For instance a sphere in which a cap is pushed in corresponds, of course, to the same convex metric space S' as the sphere without singularity viz. to the convex sphere.

We see, first of all, that problem I is equivalent to the following problem II: *Is each space H satisfying the four following properties congruent to the convex hemisphere?* 1) *H is homeomorphic to the hemisphere,* 2) *H is convex,* 3) *H without its boundary* (i. e. the points corresponding to the boundary of the hemisphere) *is externally convex,*[11] 4) *each two points of H corresponding to diametrically opposite points of the hemisphere are opposite points and have the same distance d* (i. e. satisfy the condition $pq + qp' = pp' = d$ for each point q of S).

It is obvious that a negative answer to problem I implies a negative answer to problem II. Conversely, let us suppose that there exists a surface H satisfying the four conditions and not congruent to a hemisphere. By uniting H with a congruent surface $\bar{H}$ such that H and $\bar{H}$ have their boundary circle and no other points in common we obtain a surface S which is homeomorphic to the sphere. If p is a point of H and $\bar{q}$ a point of $\bar{H}$ then, if q is the point of H which corresponds to the same point of the hemisphere as the image of $\bar{q}$ in the center of $\bar{H}$, we define the distance $\overline{pq}$ to be $= d - pq$. In this way S is a metrical space in which each point p has an opposite point, for the point $\bar{q}$ of $\bar{H}$ and the point q of H are opposite. Besides S is convex, i. e. for each two points of S there exists a point between them. (If the two points both lie in H or in $\bar{H}$ then the existence of a point between them follows from the convexity of H and $\bar{H}$; if p lies in H and $\bar{q}$ in $\bar{H}$ then, due to the fact that H is externally convex, there exists a point r of H such that p lies between r and the point q which is opposite to $\bar{q}$, and as r lies between q and $\bar{q}$, the point r lies between p and $\bar{q}$.) But S is not congruent with a convex sphere as H is not congruent with a convex hemisphere. A negative answer to problem II thus implies a negative answer to problem I.

We show now that the answer to problem II is negative. Let us denote by C the convex hemisphere. Let us suppose that there is defined

a) A one-to-one correspondence between the system of all geodesic arcs of C (i. e. all great half circles) and a system $\mathfrak{S}$ of arcs of C according to the following two conditions: 1) Each geodesic line has the same endpoints as the corresponding arc; 2) For each two non-opposite points of C there exists exactly one arc of the system $\mathfrak{S}$ containing the two points.

[11] Cf. this paper p. 742.

b) For each geodesic of C there is defined a topological correspondence between the geodesic and the arc which corresponds to the geodesic according to the correspondence a).

Let us call H the set of all points of C with the following definition of distance: Each two opposite points p and q of C have the distance d; for each two non-opposite points p and q of C the distance in H equals the distance in C of the two points corresponding to p and q according to the transformation b) on that geodesic which according to the correspondence a) corresponds to the arc of the system $\mathfrak{S}$ passing through p and q. Evidently, according to this definition of distance in H, each two diametrically opposite points of H are opposite points and, besides, H is convex and without its boundary externally convex (for, if p and q are two non-opposite points of C, all points of the segment of the arc of the system $\mathfrak{S}$ joining p and q are between p and q). Thus if H satisfies the triangle inequality H is a metrical space, which gives a negative solution of problem II, provided that H is not congruent to a convex hemisphere.

It is actually possible to define correspondences a) and b) such that H is a metrical space not congruent to a convex hemisphere. Let us, in the manner of Hilbert's non-Desarguesian geometry, chose a small circle K in the interior of the hemisphere C and a point p in the interior of C outside of K. Let G be a geodesic of C, i. e. a great half circle between two opposite points g and $\overline{g}$ of C. If G has at most one point in common with K then G' may be defined to be identical with G. If G has two points k and $\overline{k}$ in common with K (k nearer to g than to $\overline{g}$) then G' may be the sum of the segments of G between g and k and between $\overline{k}$ and $\overline{g}$ and of the segment of the circle passing through the points $k, \overline{k}, p$ between k and $\overline{k}$. The correspondence between the geodesics G and the arcs G', obviously, satisfies the conditions of the correspondence a). If we define for each geodesic G a topological correspondence between G and G' by the postulate that to each point γ of G there corresponds that point of G' which is the projection of γ from the point p, then the space H satisfies the triangle inequality and is thus a metrical space with the properties postulated in problem II. The space H is, however, not congruent to the convex hemisphere for it contains four points which are not congruent to four points of a convex hemisphere. Let k and $\overline{k}$ be those points of K which lie on the great half circle G passing through p and the center of K, and let k^* and $\overline{k}^*$ be the two points of K which lie on the great half circle G^* through the center of K, which is orthogonal to G. Then if $\overline{\overline{k}}$ is the point of intersection of G and the circle passing through $k^*, \overline{k}^*, p$ the four points $k^*, k, \overline{k}, \overline{\overline{k}}$ are not congruent to four points of a convex hemisphere. For if $h^*, h, \overline{h}$ are three points of a convex hemisphere con-

gruent to $k^*, k, \bar{k}$ then $\bar{\bar{k}}$ has from k the distance which the mid-point of h and $\bar{h}$ has from h^*, whereas the distances between $\bar{\bar{k}}$ and k and between $\bar{\bar{k}}$ and $\bar{k}$ are unequal. Hence there does not exist a point of the convex hemisphere whose distances from $h^*, h, \bar{h}$ equal the distances between $\bar{\bar{k}}$ and $k^*, k, \bar{k}$ respectively.

The general concept of a convex surface in three dimensional space is the basis of problems concerning indeformability. A topological correspondence between two metrical spaces S and S' may be called a *deformation* if each arc of S corresponds with an arc of S' of the same length. The general question concerning these surfaces is: *Is a convex surface of three dimensional space congruent with each convex surface into which it can be deformed?* The positive answer has been proved for two cases which are extreme from the point of view of analyticity, viz. for polyhedra (thus for sets which are characterized by their singularities) and for ovaloids (which have no singularities at all). I do not doubt that a positive answer to the general question could be proved without any hypothesis about analyticity using purely metrical methods.

4. **On angles.** A set S such that to each pair of elements p and q (called "points") there corresponds a real number $pq = qp$ which is > 0 if $p \neq q$ and $= 0$ if $p = q$ is called a *semi-metrical space.* (A metrical space is a semi-metrical space satisfying the triangle inequality.) One could call a semi-metrical space an *angle-space* if for each point p and for each pair of points q, r (both different from p) there is defined a real number denoted by $\sphericalangle\, qpr$ and called the angle qpr (with the vertex p) satisfying the conditions

$$\sphericalangle\, qpr = \sphericalangle\, rpq \begin{cases} = 0 \text{ if } pq + qr = pr \quad \text{or} \quad pr + rq = pq, \\ > 0 \text{ otherwise,} \\ = \pi \text{ if } qp + pr = qr. \end{cases}$$

Euclidean space is an example of an angle-space. A one-to-one correspondence between two angle-spaces which preserves the angles could be called *conformal.*

There seem to be three possible ways of constructing a theory of angle-spaces.

1) For semi-metrical spaces the following theorems hold true:[12] In order that a semi-metrical space be congruent to a subset of the straight line it is necessary and sufficient that for each triple of points p, q, r of the space, one lies between the two others (i. e. one of the three equalities $pq + qr = pr$, $pr + rq = pq$, $pq + pr = qr$ holds true). In order that

[12] Menger, l. c. [4], p. 137.

a semi-metrical space satisfying this condition be congruent to the straight line itself it is necessary and sufficient that the space be complete, convex and externally convex. Furthermore there are known relations between the distances in a semi-metrical space, that are characteristic for those semi-metrical spaces which are congruent to subsets of the plane.[13] It would be desirable to know relations between the angles of an angle-space, characteristic for those angle-spaces which are conformal to a subset of the plane and to the plane itself.

2) In a metrical space (i. e. in a semi-metrical space in which for each three points the triangle inequality holds) each three points are congruent to three points of the plane and, therefore, the definition of distances induces a definition of angles (viz. the angles in the corresponding plane triangles). A problem (connected, of course, with the first mentioned) is to find conditions which are necessary and sufficient in order that in an angle-space, which at the same time is a metrical space, the angles are identical with the angles induced by the definition of distance.

3) Angles are a special case of n-dimensional angles, i. e. of numbers connected with each point p and each system of n points $q_1, q_2, \cdots, q_n$ all different from p. Distance is, in this sense, a 1-dimensional angle. One could develop a general axiomatic theory of these n-dimensional angles in which analogues of the triangle inequality would play an important rôle and which would be connected with the theory of n-dimensional metric which was sketched in another paper.[14]

5. Euclidean metric and quadratic forms. Two semi-metrical spaces are called *congruent* if there exists a one-to-one correspondence between them which preserves the distances.

It was one of the fundamental problems of metrical geometry to find conditions which are necessary and sufficient in order that a semi-metrical space be congruent to a subset of the R_n (i. e. the n-dimensional euclidean space). If $p_1, p_2, \cdots, p_k$ are k given points of a metrical space then the determinant of the bordered matrix of the squares of the k^2 distances

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & (p_1p_2)^2 & \cdots & (p_1p_k)^2 \\ 1 & (p_2p_1)^2 & 0 & \cdots & (p_2p_k)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (p_kp_1)^2 & (p_kp_2)^2 & \cdots & 0 \end{vmatrix},$$

which can be symbolized by

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[13] Menger, l. c. [4], p. 136.

[14] Menger, l. c. [4], pp. 142–163.

$$\begin{vmatrix} 0 & 1 \\ 1 & (p_i p_j)^2 \end{vmatrix} \qquad i, j = 1, 2, \cdots, k,$$

may be denoted by $D(p_1, p_2, \cdots, p_k)$. Furthermore we write $\operatorname{sgn} x = 1, 0, -1$ when x is respectively >0, $=0$, <0. Then the solution of the above mentioned problem is contained in the four following theorems.[15]

I. *A semi-metrical space containing more than $n+3$ points each $n+2$ of which are congruent to $n+2$ points of the R_n is congruent to a subset of the R_n.*

II. *In order that $n+2$ points $p_1, p_2, \cdots, p_{n+2}$ be congruent to $n+2$ points of the R_n it is necessary and sufficient that each $n+1$ of the $n+2$ points are congruent to $n+1$ points of the R_n and that $D(p_1, p_2, \cdots, p_{n+2}) = 0$.*

III. *In order that $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ be congruent to $n+1$ points of the R_n but not to $n+1$ points of the R_{n-1}, it is necessary and sufficient that for each integer k $(2 \leqq k \leqq n+1)$ and for each system of k points $p_{i_1}, p_{i_2}, \cdots, p_{i_k}$*

$$\operatorname{sgn} D(p_{i_1}, p_{i_2}, \cdots, p_{i_k}) = (-1)^k.$$

IV. *A space containing exactly $n+3$ points $p_1, p_2, \cdots, p_{n+3}$ is congruent to $n+3$ points of the R_n if and only if each $n+2$ of its points are congruent to $n+2$ points of the R_n and besides $D(p_1, p_2, \cdots, p_{n+3}) = 0$. If for a semi-metrical space containing exactly $n+3$ points $p_1, p_2, \cdots, p_{n+3}$ each $n+2$ points are congruent to $n+2$ points of the R_n then sgn $D(p_1, p_2, \cdots, p_{n+3}) = 0$ or $(-1)^n$. There exist for each integer n spaces containing exactly $n+3$ points each $n+2$ of which are congruent to $n+2$ points of the R_n and for which $D(p_1, p_2, \cdots, p_{n+3}) \neq 0$.* (They are called *pseudo-euclidean spaces.*)

Mr. Morse remarked that the conditions of Theorem III can be simplified if we interpret the bordered matrix

$$\begin{Vmatrix} 0 & 1 \\ 1 & (p_i p_j)^2 \end{Vmatrix} \qquad (i, j = 1, 2, \cdots, k)$$

as a matrix of a bordered quadratic form in $k+1$ variables, i. e. of the quadratic form $2x_0 \sum_{i=1}^{k} x_i + \sum_{i,j=1}^{k} (p_i p_j)^2 x_i x_j$, which may be denoted by $f[p_1, p_2, \cdots, p_k]$. The form $f[p_1, p_2, \cdots, p_k]$ is singular if and only if $D(p_1, p_2, \cdots, p_k) = 0$.

If f is a quadratic form with the bordered matrix $\begin{Vmatrix} 0 & 1 \\ 1 & r_{ij} \end{Vmatrix}$ then we call a *bordered minor form* of f each form whose matrix is a principal minor of the matrix of f containing the bordering, i. e. each form whose

[15] Cf. Menger, l. c.[4], pp. 114–141 and Amer. Journ. of Math. (1931).

matrix can be obtained from the matrix of f by omitting, for a certain set of integers $k_1, k_2, \cdots, k_l$ all of which are >1 and $\leqq k$, the k_i-th row and the k_i-th column of the matrix of f $(i = 1, 2, \cdots, l)$. All bordered minor forms of a form $f(p_1, p_2, \cdots, p_k)$ are non-singular if and only if for each integer r $(2 \leqq r \leqq k)$ and for each system of r points $p_{i_1}, p_{i_2}, \cdots, p_{i_r}$ the relation $D(p_{i_1}, p_{i_2}, \cdots, p_{i_r}) \neq 0$ subsists.

Let us call furthermore a quadratic form *almost negatively definite* if f transformed by a real transformation into a sum of squares, contains exactly one positive square and otherwise only negative squares. In this terminology we can prove:

In order that $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ be congruent with $n+1$ points of the R_n but not with $n+1$ points of the R_{n-1} it is necessary and sufficient that the corresponding quadratic form is almost negatively definite and has no singular bordered minor form.

In order to prove that the condition is sufficient we prove that the condition implies that of Theorem III. Indeed, if the condition concerning the form $f[p_1, p_2, \cdots, p_{n+1}]$ is satisfied and if $i_1, i_2, \cdots, i_{n+1}$ is some permutation of the integers $1, 2, \cdots, n+1$ we consider the following sequence of determinants of principal minors of the matrix of $f[p_1, p_2, \cdots, p_{n+1}]$

$$0, \quad D(p_{i_1}) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad D(p_{i_1}, p_{i_2}),\ D(p_{i_1}, p_{i_2}, p_{i_3}),\ \cdots,\ D(p_{i_1}, \cdots, p_{i_{n+1}}).$$

According to the hypothesis on $f[p_1, p_2, \cdots, p_{n+1}]$ none of these determinants vanishes. Hence, as f is supposed to be almost negatively definite, according to well known theorems on quadratic forms,[16] the signs of the determinants of this sequence must alternate and hence $\operatorname{sgn} D(p_{i_1}, p_{i_2}, \cdots, p_{i_k}) = (-1)^k$.

The condition is necessary. For if $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ are congruent to $n+1$ points of the R_n but not to $n+1$ points of the R_{n-1} then the conditions of Theorem III are satisfied. They contain, in particular, that in the sequence of determinants,

$$0, \quad D(p_1) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad D(p_1, p_2),\ D(p_1, p_2, p_3),\ \cdots,\ D(p_1, p_2, \cdots, p_{n+1})$$

the signs alternate. This implies according to the theorems on quadratic forms that the form $f[p_1, p_2, \cdots, p_{n+1}]$ is almost negatively definite.

The proof shows that in Theorem III the condition of Theorem III might be replaced by the following weaker condition: *For each integer k $(2 \leqq k \leqq n+1)$ and for each system of k points $p_{i_1}, p_{i_2}, \cdots, p_{i_k}$ the relation $D(p_{i_1}, p_{i_2}, \cdots, p_{i_k}) \neq 0$ subsists and* $\operatorname{sgn} D(p_1, p_2, \cdots, p_k) = (-1)^k$ $(k = 1, 2, \cdots, n+1)$.

In the same way the Theorems I, II and IV could be translated into theorems on bordered quadratic forms.

[16] Cf. e. g. Kowalewski, Einführung in die Determinantentheorie 1909, p. 234.

6. **On similarity of groups.** In a recently published paper[17] I introduced a concept of distance for the elements of a group. Let us denote by $a+b$ the element of the group G which is composed of a and b, by $-a$ the inverse of a, by 0 the unit element of the group, by $|a|=|-a|$ the couple a, $-a$ which we call the *modulus* of a and $-a$. Evidently, the modulus of an element a is a couple of two identical elements if and only if a is of order 2. The set of the moduli of all elements of the group G may be denoted by $|G|$. In particular, if G is the additive group of all real numbers then the moduli of the elements correspond to the usual concept of absolute value.

If G is a given Abelian group a set S is called *G-metrical* if there corresponds to each two elements p and q of S an element of $|G|$ denoted by pq and called the *distance* between p and q which, for each pair of elements of S, satisfies the conditions

$$(*) \qquad pq = qp \begin{cases} = |0|, & \text{if } p = q, \\ \neq |0|, & \text{if } p \neq q. \end{cases}$$

If the given group G is the additive group of all real numbers then S is G-metrical if to each two elements of S there corresponds the absolute value of a real number according to conditions $(*)$, in other words, if S is a semi-metrical space.

If G is a given Abelian group then, in particular, each subset of G is G-metrical if for each pair of elements p, q of G we define as their distance the modulus of their difference; i. e. if we set $pq = |p-q|$, where $p-q$ stands for $p+(-q)$. This definition satisfies the conditions $(*)$.

If G is a given group and S and S' are two G-metrical sets then a one-to-one correspondence between S and S' is called a *congruent correspondence* if it preserves the distances, that is to say, if the distance of each pair of elements of S' is the same element of G as is that of the corresponding pair of elements of S. This concept of congruence is a natural generalization of the concept of congruence for semi-metrical spaces. An example of a congruent correspondence between a group G and itself is a *translation* of G by which for a certain element t of G each element a of G corresponds to the element $a+t$. Two G-metrical sets S and S' are called *congruent* if there exists a congruent correspondence between them. In the above mentioned paper[17] conditions are indicated which are necessary and sufficient in order that a G-metrical set be congruent to a subset of the group G,—generalizations of the conditions that a semimetrical space be congruent to a subset of the straight line.[18]

[17] Mathematische Zeitschrift, 33 (1931), p. 396.

[18] Menger, l. c. [4], p. 136.

The concept of congruence only applies to two sets S and S' which are G-metrical with respect to the *same* group G. In this paper I wish to mention the analogue of the ordinary transformations of similarity which also applies to sets which are metrical with respect to different groups.

If G and G' are two Abelian groups and if S is G-metrical and S' G'-metrical then a one-to-one correspondence between S and S' is called a *transformation of similarity* if each pair of congruent subsets of S corresponds to a pair of congruent subsets of S'. Two sets S and S' are called *similar* if there exists a transformation of similarity between them.

If G and G' are two given groups then each isomorphism between G and G' that is, each one-to-one correspondence by which for each two elements a and b of G the corresponding elements a' and b' have a sum $a' + b'$ which corresponds to $a + b$ and by which, especially, the unit elements of G and G' correspond is, obviously, a transformation of similarity. A transformation of similarity between G and G' is, in turn, not necessarily an isomorphism. For instance, a translation of a group G into itself is a congruent transformation and, therefore, a transformation of similarity but not an isomorphism as the unit elements of the group does not correspond to itself. It is natural to ask what relations subsist between the concepts of similarity and isomorphism. The following answer to this question was communicated to me by Miss O. Taussky.

Let G and G' be two given groups. If there is given an isomorphism between G and G' by which there corresponds to the element a of G the element a' of G' and if there is given, besides, a congruent transformation of G' into itself by which there correspond to the element a' of G' the element a'' of G',—then the correspondence between G and G' by which there corresponds to the element a of G the element a'' of G' is, evidently, a transformation of similarity. We shall prove now that, conversely, *each transformation of similarity between two Abelian groups G and G' is an isomorphism between G and G' combined with a congruent self-transformation of G'* (more precisely: with a *translation* of G' into itself).

Suppose that there is given a transformation of similarity between two Abelian groups G and G' by which the element a of G corresponds to the element a' of G' and, especially, the unity 0 of G to the element t' of G'. The transformation of G' into itself by which $a'' = a' - t'$ corresponds to a' is a translation. Our proposition, therefore, is proved if we show that the correspondence between G and G' by which the element a of G corresponds to the element a'' of G' (and by which especially the unit elements 0 and $0'$ of G and G' correspond) is an isomorphism. Let a and b be two elements of G, a'', b'' the corresponding elements of G',

and $(a+b)''$ the element of G' which corresponds to $a+b$. Our proposition is:

$$(\dagger) \qquad (a+b)'' = a''+b''.$$

We consider the four elements $0, a, b, a+b$ of G and the corresponding elements $0', a'', b'', (a+b)''$ of G'. As $a+b$ has the same distance from a as b has from 0, it follows from the similarity of the transformation between G and G' that $(a+b)''$ has the same distance from a'' as b'' has from $0'$, viz. the distance b''. Thus $(a+b)''$ is identical either with $a''+b''$ or with $a''-b''$. In the first case $(\dagger)$ is true. Let us thus suppose that

$$(\dagger\dagger) \qquad (a+b)'' = a''-b''.$$

Since in the case $2b'' = 0$ the formula $(\dagger\dagger)$ is identical with $(\dagger)$ we may assume that $2b'' \neq 0$ and, as the argument is symmetrical in a and b, that $2a'' \neq 0$. The element $(b+a)''$ of G' which corresponds to $b+a$ has the same distance from b'' as a'' has from $0'$ and is thus identical either with $b''+a''$ or with $b''-a''$. The group G being Abelian $a+b$ and $b+a$ are identical. Therefore $(a+b)'' = (b+a)$ and hence $a''-b''$ either $= b''+a''$ or $= b''-a''$. As the first possibility would imply $2b'' = 0$ we may assume

$$(\dagger\dagger\dagger) \qquad 2a'' = 2b''.$$

Then we conclude from $(\dagger\dagger)$ that $2(a+b)'' = 0'$ and, therefore, $2(a+b) = 0$. We furthermore treat the elements $(a-b)'' = (-b+a)''$ which correspond to $a-b = -b+a$ in the same way as we treated $(a+b)'' = (b+a)''$ before and conclude that $2(a-b) = 0$. This relation together with $2(a+b) = 0$ implies $4a = 4b = 0$. As $2a \neq 0 \neq 2b$ both a and b have the order 4, whereas $a-b$ has the order 2. This implies the existence of an element d of order 2 such that $b = a+d$. If d'' is the element of G' which corresponds to d there is, as d'' has the order 2, only one element of G' which has the distance $|d''|$ from a''. As b has the same distance from a as d has from 0 we conclude that b'' has the same distance from a'' as d'' has from $0'$. Thus $b'' = a''+d''$ and hence $a''-b'' = d''$. From $(\dagger\dagger)$ follows $(a-b)'' = a''+b''$. As $a-b = d$ we have $d'' = a''+b''$, thus $a''-b'' = a''+b''$ and therefore $(\dagger\dagger\dagger)$ implies the proposition $(\dagger)$. This proposition is proved in any case.

We have therefore the result: *The transformations of similarity between two Abelian groups are identical with the isomorphisms between them each combined with a translation of one of the two groups into itself.* This contains as a corollary: that two Abelian groups are similar if and only if *they are isomorphic.*

The concept of similarity applies not only to groups but more generally to any pair of systems for each of which *congruence* is defined in some way.

7. **On implicit functions.** The most general theorem on implicit functions, as was proved by W. Gross,[19] has the following form: We consider n real functions $f_i(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)$ of $m+n$ variables (interpreted as points in a $(m+n)$-dimensional space, the m variables x_i varying in an m-dimensional space, the n variables y_i varying in an n-dimensional space), which in the neighborhood of the point $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m$ are continuous. We make, furthermore, the hypothesis H: The n functions $f_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m, y_1, y_2, \cdots, y_n)$ $(n = 1, 2, \cdots, n)$ have differentials in the point $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$, that is, they admit a representation

$$f_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m, y_1, y_2, \cdots, y_n) = \sum_{j=1}^{n} a_{ij}(y_j - \bar{y}_j) + g_i(y_1, y_2, \cdots, y_n)\, r(y_1, y_2, \cdots, y_n)$$

where

$$r(y_1, y_2, \cdots; y_n) = \sum_{j=1}^{n} (y_j - \bar{y}_j)^2,$$

where the n functions $g_i(y_1, y_2, \cdots, y_n)$ tend towards 0 with $r(y_1, y_2, \cdots, y_n)$ and where $|a_{ij}| \neq 0$.

Then the following proposition P is proved: For each point $x_1^*, x_2^*, \cdots, x_m^*$ of a certain neighborhood of the point $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m$ there exists at least one point $y_1^*, y_2^*, \cdots, y_n^*$ such that the n equalities subsist

$$f_i(x_1^*, x_2^*, \cdots, x_m^*, y_1^*, y_2^*, \cdots, y_n^*) = 0 \quad (i = 1, 2, \cdots, n).$$

Gross proved first that the hypothesis H implies the following fact F: There exists an n-dimensional cube C which contains the point $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ in its interior and is bounded by n pairs of hyperplanes $y_i = y_i'$ and $y_i = y_i''$ $(i = 1, 2, \cdots, n)$ such that (eventually permuting the indices of the n functions f_i) for each i the function $f_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m, y_1, y_2, \cdots, y_n)$ is > 0 for $y_i = y_i'$ and < 0 for $y_i = y_i''$. In consequence of the continuity of the n functions f_i the same inequalities hold true for the functions $f_i(x_1^*, x_2^*, \cdots, x_m^*, y_1, y_2, \cdots, y_n)$ for all points $x_1^*, x_2^*, \cdots, x_m^*$ in a sufficiently small neighborhood of the point $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m$. From this fact Gross deduces the proposition P by purely topological reasoning.

It may be remarked that the implication of P from F is an immediate consequence of a lemma by Lebesgue[20] which is of importance in the theory of dimension, viz.

LEMMA L. For each decomposition of an n-dimensional cube into a finite number of closed sets none of which has points in common with opposite

[19] Jahresbericht der deutschen Math. Ver., 26 (1918), p. 292.

[20] Fundam. Math. 2 (1921), pp. 256–285.

surfaces of the cube there exists at least one point of the cube contained in at least $n+1$ of the closed sets,—or the equivalent

LEMMA L'. If the n-dimensional cube C is bounded by n pairs of hyperplanes $y_i = y_i'$ and $y_i = y_i''$ $(i = 1, 2, \cdots, n)$ and decomposed into $n+1$ closed sets A_i $(i = 1, 2, \cdots, n, n+1)$ such that A_i contains the part of the boundary of C contained in $y_i = y_i'$ and has no point in common with the plane $y_i = y_i''$ $(i = 1, 2, \cdots, n)$ whereas A_{n+1} contains the point with the coördinates $y_1'', y_2'', \cdots, y_n''$, then there exists at least one point of C belonging to each of the $n+1$ sets A_i.

For, if we denote, for the n-dimensional cube C whose existence follows from the above mentioned fact F, by A_i the set of all points of C for which

$$f_i(x_1^*, x_2^*, \cdots, x_m^*, y_1, y_2, \cdots, y_n) \geqq 0 \qquad (i = 1, 2, \cdots, n)$$

and by A_{n+1} the set of all points of C for which the functions

$$f_i(x_1^*, x_2^*, \cdots, x_m^*, y_1, y_2, \cdots, y_n)$$

are $\leqq 0$ then these $n+1$ sets, evidently, satisfy the conditions of the Lemma L', and there exists thus, at least one point of C belonging to these $n+1$ sets, that is a point $y_1^*, y_2^*, \cdots, y_n^*$, for which the n equalities

$$f_i(x_1^*, x_2^*, \cdots, x_m^*, y_1^*, y_2^*, \cdots, y_n^*) = 0$$

subsist and which, therefore, satisfies the proposition P on implicit functions.

The Lemmas L and L' can be extended in this way: For each integer k between 1 and $n+1$, there exists a subset of C of a dimension $\geqq n-k+1$ belonging to at least k of the closed sets into which C is decomposed.[21] An application of this extension to the cube described in fact F shows that for each integer k the set of all points for which at least k of the n functions f_i are $= 0$ is of dimension $\geqq n-k$.

8. **On constructiveness.** It is not only asserted by intuitionists but also admitted by some opponents of intuitionism that there are just two kinds of mathematics: the classical formalistic mathematics which uses the law of excluded middle and claims merely to be free from contradiction, and the intuitionistic mathematics which is the system of mathematical constructions. The real position however, seems to be quite different. The formalistic postulate that mathematics should be non-contradictory is perfectly precise. What is meant by a postulate of constructiveness is still unclear and unexplained. It is a priori very likely that, if such a postulate can be precisely formulated at all, it can be formulated in different ways.

[21] Menger, l. c. [4], p. 269.

That this is really the case and that thus, besides the formalistic mathematics, there exist many different mathematics corresponding to different postulates of constructiveness may be shown here.

The chief argument of Brouwer against the classical theory of real functions as developed by the Paris school consists[22] in his critique of the theorem that any two real numbers are either equal or unequal. This theorem, following Brouwer, is not true. For, if r and r' are two real numbers then Brouwer says that r is *equal* to r' if a contradiction can be deduced from the assumption $r \neq r'$ and he says that r and r' are unequal if a whole number n can be constructed such that $|r - r'| > 1/n$. He then says that there exists a real number which, in this sense, is *neither equal to nor unequal to* 0. For let r be the real number with the decimal-expansion $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ where for each integer n the number a_n is defined to be $= 1$ if in the decimal expansion of π the nth term and the nine following terms are sevens, and where a_n is defined to be $= 0$ otherwise. We are not able to deduce a contradiction proving the absurdity of the assumption that the decimal expansion of π somewhere contains ten consecutive sevens and we are not able to construct a whole number n such that the nth term and the following nine terms in the decimal expansion of π are sevens. Hence we are neither able to prove that r is equal to 0 nor that r is unequal to 0.

Let us analyze this argument! The reason for Brouwer's saying that r is neither equal nor unequal to 0 is, as many writers have observed, that we are unable, at present, to prove either of these propositions in a finite number of operations. The reason for this latter fact, however, as so far has apparently been overlooked, is the fact that we are not able at present *to define the number r in a finite number of operations.* It is true that the decimal expansion of r can be calculated in a finite number of operations as far as we please, i. e. *to any given approximation.* But the corresponding fact is, merely, that it can be proved in a finite number of operations whether r and 0 *are equal or unequal at each given approximation,* and this can be done. The *whole* decimal expansion of r, though each of its beginning segments can be calculated in a finite number of operations, cannot be calculated in a finite number of operations. We have no constructive principle to survey the decimal expansion of r or π in its *totality* and this is the reason why we cannot decide in a finite number of operations whether the decimal fraction r is equal to 0 or not.

The basis of Brouwer's critique of the classical arithmetic and the theory

[22] Crelle's Journal, 154 (1925), pp. 1–7.

of real numbers is thus the fact *that his demands of constructiveness concerning proofs and definitions are of different degree.* He only admits proofs concerning sequences of integers which are performed by a finite number of operations, but he introduces entities which, in reality, are not defined by a finite number of operations; he admits sequences which, though each of their beginning segments is defined by a finite number of operations, cannot be defined in a finite number of operations in their totality.

The next question is which sequences of integers *can* be defined by a finite number of operations in their totality. Here again we meet our initial question about the precise definition of the word constructiveness with respect to sequences of integers and we see that the answer, as a priori could be expected, can be given *in different ways.* We may fix in different ways systems of operations applicable to sequences of integers and call them constructive operations. If then a sequence of integers is given we may call *at least as much constructive with respect to the fixed system of operations as the given sequence* each sequence which can be obtained from the given sequence by a finite number of operations. We may call *constructive with respect to the fixed system of operations* each sequence which can be obtained in this way from the sequence consisting of ones exclusively.

Examples of such operations which could be chosen as the points of departure of different algebras of sequences, are 1) The omission of the first term of the sequence and the introduction of a new one. 2) The formation for two given sequences of a sum-sequence. 3) The construction from k given sequences $\{a_n\}, \{b_n\}, \cdots, \{k_n\}$ of one new sequence in which the first term equals a_1, the $(k\cdot n+1)$st term equals a_n, the $(k\cdot n+2)$nd term equals b_n etc., and inversely the construction of k sequences from one given sequence. 4) The formation for each two given sequences $\{a_n\}$ and $\{b_n\}$ of the sequence whose n-th term is $\sum_{k=1}^{n} a_k b_{n+1-k}$. In this way we go further and obtain for each system of operations which are called constructive certain constructive sequences of integers. We have in this way a real definition of the words "constructively defined sequences", and at the same time, many different definitions. If e. g. we only call the operations 1) 2) and 3) constructive then we obtain as constructive those sequences which after a finite beginning segment are periodic. With each of these definitions there correspond different systems of *real numbers* which are constructively defined, corresponding to the different ways of defining real numbers by sequences of integers. We may e. g. consider those real numbers whose *decimal expansion* is a constructively defined

sequence or those whose *continued fraction* is constructively defined. We may, furthermore, try to define such a system of constructive operations for sequences that each real number which has a constructive decimal expansion has a constructive continued fraction and inversely etc.

Constructive arithmetics in this sense though they restrict the field of real numbers still more than the intuitionistic arithmetic does, differ from the formalistic arithmetic as to some of their laws less than the intuitionistic arithmetic does. E. g. for each real number whose decimal fraction is constructively defined with respect to the above mentioned constructive operations it can be proved in a finite number of operations that it is either equal to 0 or not. Besides, these ideas suggest a quite natural definition of *function* as a constructive operation by which there corresponds to each constructively defined real number a constructively defined real number. And, evidently, this concept of function, though in a certain sense much more constructive than the intuitionistic concept, admits discontinuous functions, as they exist in formalistic theory of functions, whereas according to intuitionism each function is continuous.

This paper should not be misunderstood. I do not try to replace the intuitionistic way of defining sequences of integers by the above mentioned ways which are more restricted and may be called *finitistic* definitions, in the same way as the intuitionists try to replace the formalistic mathematics by their constructions. For in my opinion the critical arguments of intuitionists against formalism are absolutely meaningless, as they are based on the proposition that by merely not contradictory reasoning "no mathematical values are created", a proposition which is obviously not of objective character but a mere description of a personal taste. It would be a repetition of the same meaningless critique to apply it on intuitionism and only to admit finitistically defined objects as "mathematical values". The purpose of this paper is merely to emphazise that if some one asks from mathematics more than that it is not contradictory he must first of all, say exactly what he asks, and to emphazise furthermore that additional postulates can be formulated in *different ways.* Intuitionism is not the constructive mathematics but simply one of many ways of constructing mathematics according to stronger postulates than the mere postulate of consistency. It is neither the only constructive mathematics, nor the most constructive, nor that which in its results resembles most the beautiful formalistic theory.

EINFACHER BEWEIS EINES DIMENSIONSTHEORETISCHEN ÜBERDECKUNGSSATZES.

Von L. Pontrjagin in Moskau.

(Aus einem Briefe an Herrn P. Alexandroff.)*

... Übrigens möchte ich Sie darauf aufmerksam machen, daß die ε-Überdeckungen eines n-dimensionalen kompakten metrischen Raumes F, die Sie in der Theorie der Projektionsspektra[1] betrachten, *von selbst* folgende Eigenschaft haben:

Der Durchschnitt von je r Mengen der Überdeckung ist höchstens $n - r + 1$-dimensional.

Es sei in der Tat

$$A_1, A_2, \cdots, A_m, \cdots$$

ein n-dimensionales Projektionsspektrum von F. Sie nennen (a. a. O.[2], S. 133) $\Phi_{i_1 i_2 \cdots i_m}$ die Menge aller Punkte von F, deren erste m Koordinaten (im Sinne des Projektionsspektrums) der Reihe nach die Eckpunkte

$$a_{i_1}, a_{i_1 i_2}, \cdots, a_{i_1 i_2 \cdots i_m}$$

enthalten; (die $a_{i_1 i_2 \cdots i_m}$ sind dabei die Eckpunkte von A_m). Die Mengen $\Phi_{i_1 i_2 \cdots i_m}$ bilden eine ε_m-Überdeckung U_m, wobei $\lim \varepsilon_m = 0$ ist. Wir können offenbar voraussetzen, daß schon $\varepsilon_1 < \varepsilon$ ist (wobei ε eine beliebige fest gewählte positive Zahl ist) und somit nur die Überdeckung U_1 betrachten. Ihre Elemente seien $F_1, F_2, \cdots, F_s$, wobei F_i die Menge aller Punkte ist, deren erste Koordinate den Eckpunkt a_i enthält.

Es seien nun r beliebige unter den Mengen F_i gewählt, etwa $F_1, F_2, \cdots, F_r$; wir wollen zeigen, daß

$$\dim (F_1 \cdot F_2 \cdot \cdots \cdot F_r) \leqq n - r + 1$$

ist.

Es sei x ein Punkt von $F_1 \cdot F_2 \cdot \cdots \cdot F_r$; die erste Koordinate von x bezeichnen wir mit X; X ist ein Simplex, welches jedenfalls die Punkte $a_1, a_2, \cdots, a_r$ unter seinen Eckpunkten besitzt; deswegen sind alle Ko-

* Received Mai 17, 1931.

[1] Vgl. P. Alexandroff, „Gestalt und Lage abgeschlossener Mengen“, diese Annals, (2) 30 (1928), S. 101–187, insbesondere Seiten 107, 130–133 u. 144. (Die Pagination der Separate beginnt mit 1 und geht bis 87.)

[2] Der Satz von der Existenz von beliebig feinen Überdeckungen, die die soeben erwähnte Eigenschaft haben, wurde auf einem sehr komplizierten Wege von Menger und Hurewicz bewiesen (vgl. Menger, Dimensionstheorie, Kap. V, S. 155 u. f.).

ordinaten des Punktes x mindestens $r-1$-dimensionale Simplexe, d. h. es ist, Ihren Bezeichnungen (S. 144) anschließend,

$$k(x) \geqq r-1.$$

Da x ein beliebiger Punkt von $F_1 \cdot F_2 \cdot \cdots \cdot F_r$ kommen also für Punkte dieser Menge als $k(x)$ nur die $n-r+2$ Zahlen

$$r-1, r, r+1, \cdots, n$$

in Frage; folglich sind alle Punkte der Menge $F_1 \cdot F_2 \cdot \cdots \cdot F_r$ in

$$P_{r-1} + P_r + P_{r+1} + \cdots + P_n \tag{2}$$

enthalten (Sie bezeichnen auf S. 144 mit P_i die Menge aller Punkte x von F, für die $k(x) = i$ ist); da Sie auf derselben Seite beweisen, daß jede der Mengen P_i nulldimensional ist, ist die Menge (2), also erst recht $F_1 \cdot F_2 \cdot \cdots \cdot F_r$ (als Summe von höchstens $n-r+2$ nulldimensionalen Mengen) höchstens $n-r+1$-dimensional, w. z. b. w.

THE MINIMIZING PROPERTIES OF GEODESIC ARCS WITH CONJUGATE END POINTS.[1]

By I. Schoenberg.[2]

Introduction. Let E_{01} be an extremal arc for the integral $I = \int f(x, y, y')\, dx$ whose end points 0 and 1 are conjugate. The question whether or not E_{01} actually minimizes our integral has been discussed by A. Kneser,[3] W. F. Osgood,[4] J. W. Lindeberg[5] and H. Hahn,[6] by studying the shape of the envelope of the extremals through the point 0. Quite another method has been used by L. Lichtenstein.[7]

It is the purpose of this paper to give a method of carrying through the criterion obtained by Osgood and Lindeberg. This will be done in § 2.

The application of this method to the case of a geodesic arc g_{01} on a surface with conjugate end points leads to a suitable criterion stated at the end of § 3, for the question whether or not g_{01} actually gives a relative shortest distance on the surface. The same criterion may be used for the construction of the envelope of the geodesic lines through a fixed point on a surface.

In § 4 the results for the general case of geodesic lines are applied to the geodesic lines of a surface of revolution.

1. **The results of Osgood and Lindeberg.**[8] Let $E_{\alpha\beta}$: $y = y(x)$, $\alpha \leqq x \leqq \beta$, be an extremal arc for the integral $I = \int f(x, y, y')\, dx$. We will make the following hypotheses:

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[1] Received March 2, 1931. — Presented to the American Mathematical Society, December 31, 1930.

[2] Fellow of the International Education Board.

[3] A. Kneser, *Zur Variationsrechnung*, Math. Annalen, vol. 50 (1898), pp. 27–50.

[4] W. F. Osgood, *On the existence of a minimum of the integral* $\int_{x_0}^{x_1} f(x, y, y')\, dx$ *when* x_0 *and* x_1 *are conjugate points, and the geodesics on an ellipsoid of revolution: A revision of a theorem of Kneser's*, Transactions of the American Math. Society, vol. 2 (1901), pp. 166–182.

[5] J. W. Lindeberg, *Zur Theorie der Maxima und Minima einfacher Integrale mit bestimmten Integrationsgrenzen*, Math. Annalen, vol. 59 (1904), pp. 321–331.

[6] H. Hahn, *Über Extremalenbogen, deren Endpunkt zum Anfangspunkt konjugiert ist*, Wiener Sitzungsberichte, Bd. 118 (1909), Abteilung IIa, pp. 99–116.

[7] L. Lichtenstein, *Zur Variationsrechnung*, Göttinger Nachrichten, 1919, pp. 161–192.

[8] We are reproducing in this section these results in the more special form given by Lindeberg who considers the analytic case only.

1. $f(x, y, y')$ is a regular and analytic function of x, y, y', in some neighborhood of the elements (x, y, y') of the extremal arc $E_{\alpha\beta}$.
2. Legendre's condition in the strict sense is satisfied along $E_{\alpha\beta}$: $f_{y'y'}(x, y(x), y'(x)) > 0$ for $\alpha \leqq x \leqq \beta$.

It is well known that a consequence of these hypotheses is that $E_{\alpha\beta}$ is a regular analytic arc.

Let E_{01}: $y = y(x)$, $(\alpha <)\ x_0 \leqq x \leqq x_1\ (< \beta)$, be a subarc of $E_{\alpha\beta}$, with the point 1 conjugate to 0.

Another consequence of our hypotheses is the existence of a family of extremals

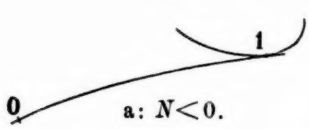
$$y = y(x, \lambda) \quad \text{for} \quad \alpha \leqq x \leqq \beta, \quad \lambda_0 - \delta < \lambda < \lambda_0 + \delta,$$

for $\delta > 0$ and sufficiently small, with

$$y(x_0, \lambda) \equiv y(x_0), \quad y(x, \lambda_0) \equiv y(x), \quad \lambda \equiv y_x(x_0, \lambda),$$

a: $N > 0$.

Fig. 1.



a: $N < 0$.

Fig. 2.

b.

Fig. 3.

c.

Fig. 4.

$y = y(x, \lambda)$ being regular and analytic in the region just stated.

We certainly have $\partial y / \partial \lambda (x_1, \lambda_0) = 0$, and we consider with Osgood and Lindeberg the very first derivative

$$N = \frac{\partial^n y}{\partial \lambda^n}(x_1, \lambda_0), \qquad (n \geqq 2),$$

which does not vanish at the point 1.

Lindeberg distinguishes four cases:

a) n is even,

b) n is odd and $N > 0$,

c) n is odd and $N < 0$,

d) n is infinite.

The results for these different cases are:

a) The envelope C of our family of extremals, which is always regular and analytic in a neighborhood of 1, has in 1 also a regular point and its shape is given by Fig. 1 or Fig. 2, according to the sign of N.

b) and c) The envelope has a cusp at the point 1 of two different types as shown in Fig. 3 and Fig. 4, respectively.

Using a terminology of Poincaré adopted by Hadamard,[9] we will refer to these different types of cusps as *foyer en pointe* (Fig. 3) and *foyer en talon* (Fig. 4).

[9] J. Hadamard, *Leçons sur le Calcul des Variations*, I, Paris 1910, pag. 111.

d) Every extremal of the family passes through the point 1 which Hadamard calls a *foyer absolu.*

The minimizing properties of E_{01} for these different cases are the following:

In the cases a) and c) we have no minimum for I. The envelope C has at 1 in both cases a branch projecting back towards 0 and therefore this statement can easily be proved by the usual geometric proof for the necessity of Jacobi's condition.[10]

In the case b) there is at least a weak relative minimum under our hypotheses, and also a strong relative minimum if, for instance, we strengthen Legendre's condition to

$$f_{y'y'}(x, y(x), p) > 0 \quad \text{for} \quad x_0 \leqq x \leqq x_1 \quad \text{and} \quad -\infty < p < +\infty.$$

The arc E_{01} may be indeed imbeded in a field of extremals and the method of Weierstrass can be applied.

Case d) gives surely at least a weak but improper relative minimum.

2. **A method to carry through the above criterion.** In order to determine the point 1 conjugate to 0 on $E_{\alpha\beta}$ we do not need the family of extremals through 0.

Let

$$F(y'', y', y, x) = 0 \text{ [11]} \tag{1}$$

be Euler's equation of our problem to minimize I. It is known that $\eta = y_\lambda(x, \lambda_0)$ satisfies the linear differential equation,

$$F_{y''}\eta'' + F_{y'}\eta' + F_y\eta = 0, \tag{2}$$

obtained by differentiating (1) with respect to λ, for $\lambda = \lambda_0$, and which is nothing else but Jacobi's equation for our extremal $E_{\alpha\beta}$.

As known since the time of Jacobi, we may use the linear equation (2), to determine $\eta = \eta(x)$ by the initial values $\eta(x_0) = 0$, $\eta'(x_0) = 1$ (or any other value > 0) and so find the conjugate point of 0 by the first root $x_1 (> x_0)$ of the equation $\eta(x) = 0$.

The same method can now be used to determine the value of $N = \partial^n y/\partial\lambda^n(x_1, \lambda_0)$ required by the criterion of § 1.

Let $\zeta = y_{\lambda\lambda}(x, \lambda_0)$. ζ is an integral of the equation,

$$F_{y''}\zeta'' + F_{y'}\zeta' + F_y\zeta + \sum_{i,k}^{0,1,2} F_{y^{(i)}y^{(k)}}\eta^{(i)}\eta^{(k)} = 0, \tag{3}$$

[10] See G. A. Bliss, *Calculus of Variations*, Chicago 1925, § 55.

[11] The methods of §§ 1 and 2 may be applied to discuss the envelope of the integral curves of any analytic differential equation of the second order $F(y'', y', y, x) = 0$, provided that $F_{y''} \neq 0$ along the curve under consideration.

obtained by differentiating now the equation (2) with respect to λ, for $\lambda = \lambda_0$, and defined by the initial conditions $\zeta(x_0) = 0$, $\zeta'(x_0) = 0$.

If $\zeta(x_1) \neq 0$, we are in the case a) of § 1 and our problem is solved.

If $\zeta(x_1) = 0$, then we have to determine $\omega = y_{\lambda\lambda\lambda}(x, \lambda_0)$ at the point $x = x_1$, and ω is the integral of the equation,

$$(4)\quad F_{y''}\,\omega'' + F_{y'}\,\omega' + F_y\,\omega + 3\sum_{i,k}^{0,1,2} F_{y^{(i)}y^{(k)}}\,\zeta^{(i)}\,\eta^{(k)} + \sum_{i,k,h}^{0,1,2} F_{y^{(i)}y^{(k)}y^{(h)}}\,\eta^{(i)}\,\eta^{(k)}\,\eta^{(h)} = 0,$$

obtained by differentiating (3) with respect to λ, for $\lambda = \lambda_0$, with the initial conditions $\omega(x_0) = 0$, $\omega'(x_0) = 0$.

We have now to discuss the sign of $\omega(x_1)$.

It will be seen farther, discussing special geodesics on surfaces of revolution that, if we have not to deal with the case of a *foyer absolu*, the consideration of the three equations (2), (3), and (4), will be sufficient to solve the problem in many special cases. But, of course, if $\omega(x_1)$ vanishes, then we have to differentiate (4) once more and so on. But that would be a very singular case.

Our differential equations (2), (3), and (4), for the recurrent determination of η, ζ and ω, are linear and of the second order and the homogeneous part of them is the first member of Jacobi's equation (2). Therefore if (2) is completely solved, then the equations (3) and (4) are solved too, by mere quadratures.

If Euler's equation (1) of our problem is quite solved, or if the family of extremals $y = y(x, \lambda)$ is known by any other way, then in general the criterion of § 1 can be applied directly, by calculating the required expression $N = \partial^n y / \partial \lambda^n (x_1, \lambda_0)$. But if we know only the extremal $E_{\alpha\beta}$ and nothing else, then the method of this section seems to me advisable. We shall see later (§ 3) that even in the first case for some special extremals it is advisable to apply this last method.

3. Application to the minimizing properties of geodesic arcs with conjugate endpoints. Let g be a geodesic line on a surface S, on which we take its length s as parameter. We consider the family of geodesic lines cutting the geodesic g orthogonaly and on every such geodesic line we take its length u as parameter ($u = 0$ on g, and $u > 0$ to the left side of g, with respect to a fixed side on S).

A well known theorem of Gauss says that the family of curves $u =$ const. is orthogonal to the family of geodesics $s =$ const.

Gauss's reduced form

$$d\sigma^2 = du^2 + C^2 ds^2, \quad \text{with} \quad C(0, s) = 1,$$

for the linear element on the surface follows from this at once. This system of coördinates (u, s) is certainly applicable for a sufficiently small neighborhood of g.

Let us seek the differential equation for geodesic lines. We have to minimize $\int (u'^2 + C^2)^{1/2}\, ds$ and Euler's equation

$$\frac{d}{ds}\left(\frac{u'}{(u'^2 + C^2)^{1/2}}\right) - \frac{CC_u}{(u'^2 + C^2)^{1/2}} = 0$$

becomes

$$Cu'' - 2C_u u'^2 - C_s u' - C^2 C_u = 0. \tag{5}$$

$u = 0$ has to be a geodesic line and therefore

$$C(0, s) = 1, \qquad C_u(0, s) = 0. \tag{6}$$

Let 0 and 1 be two conjugate points on g corresponding to $s = 0$ and $s = s_1$.

How about the minimizing properties of g_{01}?

Does g_{01} *give the shortest distance with respect to all the curves of its neighborhood joining* 0 *and* 1?

If C is analytic, regular and positive for $u = 0$, $0 \leqq s \leqq s_1$, then all our hypotheses of § 1 are satisfied, even the additional hypothesis sufficient to insure a strong minimum in the case of a *foyer en pointe*, and therefore we may apply the method of § 2.

We consider according to this method, u in (5) as a function of s and λ, $u \equiv 0$ for $\lambda = 0$, and we write η, ζ, and ω, for its first, second, and third partial derivative with respect to λ for $\lambda = 0$. If we differentiate (5) three times with respect to λ and put $\lambda = 0$ in those equations obtained in this way, and making use of the identities $C_s(0, s) = 0$, $C_{us}(0, s) = 0$, which follow from (6), we get the following three linear differential equations analogous to (2), (3), and (4):

$$\eta'' - C_{uu}\,\eta = 0, \tag{7}$$

$$\zeta'' - C_{uu}\,\zeta = C_{uuu}\,\eta^2, \tag{8}$$

$$\begin{aligned}\omega'' - C_{uu}\,\omega = {} & -3C_{uu}\,\eta^2\,\eta'' + 12C_{uu}\,\eta\,\eta'^2 + 3C_{suu}\,\eta^2\eta' + 6C_{uu}^2\,\eta^3 \\ & + C_{uuuu}\,\eta^3 + 3C_{uuu}\,\eta\,\zeta.\end{aligned} \tag{9}$$

Now we ought to express the coefficients of the equations (7), (8), and (9) in terms of Gauss' curvature of our surface.

A formula of Gauss gives

$$K = -\frac{1}{C} C_{uu}. \tag{10}$$ [12]

[12] See W. Blaschke, *Vorlesungen über Differentialgeometrie*, I, Berlin 1930, § 71 formula (26).

Differentiating with respect to u, we get

$$K_u = \frac{C_u}{C^2} C_{uu} - \frac{1}{C} C_{uuu}, \qquad K_{uu} = \frac{C_{uu}^2}{C^2} - \frac{1}{C} C_{uuuu} + \cdots$$

the points standing for terms vanishing for $u = 0$.

These last equations, together with (6), give for $u = 0$,

$$(11) \qquad C_{uu} = -K, \qquad C_{uuu} = -K_u, \qquad C_{uuuu} = K^2 - K_{uu}.$$

Differentiating (10) with respect to s we get

$$K_s = \frac{C_s}{C^2} C_{uu} - \frac{1}{C} C_{uus}$$

and since $C(0, s) = 1$, $C_s(0, s) = 0$, for $u = 0$

$$(12) \qquad C_{uus} = -K_s.$$

Introducing into (7), (8), and (9) the values given by (11), (12), and replacing in (9) η'' by its value $-K\eta$ given by (7), we get the system

$$(13) \quad \eta'' + K\eta = 0,$$

$$(14) \quad \zeta'' + K\zeta = -K_u \eta^2,$$

$$(15) \quad \omega'' + K\omega = (4K^2 - K_{uu})\eta^3 - 3K_s \eta' \eta^2 - 12 K \eta \eta'^2 - 3K_u \eta \zeta.$$

The first equation (13) is the Jacobi-Bonnet equation for geodesics.

Let $\eta = \eta(s)$ be a solution of (13) with $\eta(0) = \eta(s_1) = 0$ and $\eta(s) > 0$ for $0 < s < s_1$. This expresses that 0 and 1 are conjugate points.

According to the criterion of § 1 we need the value $\zeta(s_1)$, for $\zeta(s)$ defined by (14) and the initial conditions $\zeta(0) = 0$, $\zeta'(0) = 0$.

Multiplying respectively (14) and (13) with η and ζ, we get by substraction $\eta\zeta'' - \eta''\zeta = -K_u \eta^3$.

Integrating along g_{01} we get

$$\int_0^{s_1} (\eta\zeta'' - \eta''\zeta)\, ds = [\eta\zeta' - \eta'\zeta]_0^{s_1} = -\eta'(s_1)\zeta(s_1) = -\int_0^{s_1} K_u \eta^3\, ds$$

and writing $Z = -\eta'(s_1)\zeta(s_1)$ we get

$$(16) \qquad Z = -\int_0^{s_1} K_u \eta^3\, ds.$$

If $Z \neq 0$ our problem is solved.

If $Z = 0$, then we have to discuss the sign of $\omega(s_1)$, $\omega(s)$ being defined by (15) with the initial conditions $\omega(0) = 0$, $\omega'(0) = 0$.

By the same method which led to (16) we get now

$$(17) \qquad \begin{aligned} \Omega &= -\eta'(s_1)\,\omega(s_1) \\ &= \int_0^{s_1} \{(4K^2 - K_{uu})\eta^3 - 3K_s \eta' \eta^2 - 12K\eta\eta'^2 - 3K_u \eta\zeta\}\, \eta\, ds. \end{aligned}$$

This formula can be simplified very much.

Let (writing $K_s = K'$)

$$H = \int_0^{s_1} (4K^2\eta^4 - 3K'\eta'\eta^3 - 12K\eta^2\eta'^2)\, ds.$$

Equation (13) gives $-K = \eta''/\eta$, $-K' = (\eta\eta''' - \eta'\eta'')/\eta^2$ and introducing these values in H we get

$$H = \int_0^{s_1} (4\eta^2\eta''^2 + 3\eta^2\eta'\eta''' + 9\eta\eta'^2\eta'')\, ds.$$

But $(\eta^2\eta'\eta'')' = \eta^2\eta''^2 + \eta^2\eta'\eta''' + 2\eta\eta'^2\eta''$ and by integration

$$0 = \int_0^{s_1} (4\eta^2\eta''^2 + 4\eta^2\eta'\eta''' + 8\eta\eta'^2\eta'')\, ds,$$

which substracted from H gives

$$H = \int_0^{s_1} (\eta'\eta'' - \eta\eta''')\,\eta\eta'\, ds.$$

Introducing again $\eta'\eta'' - \eta\eta''' = K'\eta^2$, we get

$$H = \int_0^{s_1} K'\eta^3\eta'\, ds = \int_0^{s_1} K'\left(\frac{\eta^4}{4}\right)' ds = -\int_0^{s_1} K''\frac{\eta^4}{4}\, ds.$$

By this reduction, formula (17) becomes

$$\Omega = \int_0^{s_1} \left\{\left(K'\frac{\eta'}{\eta} - K_{uu}\right)\eta^4 - 3K_u\eta^2\zeta\right\} ds \tag{18}$$

or

$$\Omega = -\int_0^{s_1} \left\{\left(\frac{1}{4}K'' + K_{uu}\right)\eta^4 + 3K_u\eta^2\zeta\right\} ds. \tag{19}$$

Since $Z = 0$, ζ may be any integral of (14) vanishing for $s = 0$ and we need not require that $\zeta'(0) = 0$.

For the last term on the right side in (19), we have

$$\begin{aligned}\int_0^{s_1} -3K_u\eta^2\zeta\, ds &= 3\int_0^{s_1}(\zeta'' + K\zeta)\,\zeta\, ds = 3\int_0^{s_1}\frac{\eta\zeta'' - \eta''\zeta}{\eta}\,\zeta\, ds \\ &= 3\int_0^{s_1}\frac{\zeta}{\eta}(\eta\zeta' - \eta'\zeta)'\, ds = -3\int_0^{s_1}(\eta\zeta' - \eta'\zeta)\left(\frac{\zeta}{\eta}\right)' ds \\ &= -3\int_0^{s_1}\left(\zeta' - \eta'\frac{\zeta}{\eta}\right)^2 ds,\end{aligned}$$

and therefore this last term is always < 0, provided that we have not $K_u \equiv 0$ for $0 \leqq s \leqq s_1$, in which case this term vanishes.

We may now express these results in the following

THEOREM: *Let g_{01} be a geodesic arc of length s_1 with conjugate endpoints, and $\eta = \eta(s)$ be a solution of Jacobi-Bonnet's equation*

(13) $$\frac{d^2\eta}{ds^2} + K(s)\,\eta = 0$$

with $\eta(0) = \eta(s_1) = 0$, $\eta(s) > 0$ *for* $0 < s < s_1$.

Let

(16) $$Z = -\int_{g_{01}} \frac{\partial K}{\partial n}\,\eta^3\,ds.\ [13]$$

If $Z \neq 0$, *then* g_{01} *is not the shortest distance and the envelope of the geodesics through* 0 *has in* 1 *the shape of Fig.* 1 *or Fig.* 2, *according as* $Z > 0$ *or* $Z < 0$.

Suppose $Z = 0$. *Then we have to consider besides* (13) *the differential equation*

(14) $$\frac{d^2\zeta}{ds^2} + K(s)\,\zeta = -\frac{\partial K}{\partial n}\,\eta^2,$$

and to form with any solution of it vanishing for $s = 0$ *the expression*

(19) $$\Omega = -\int_{g_{01}} \left\{ \left(\frac{1}{4}\,\frac{\partial^2 K}{\partial s^2} + \frac{\partial^2 K}{\partial n^2} \right) \eta^4 + 3\,\frac{\partial K}{\partial n}\,\eta^2 \zeta \right\} ds.$$

If $\Omega \neq 0$, *then* g_{01} *is actually the shortest distance or not, and the envelope has in* 1 a foyer en pointe *or* a foyer en talon, *according as* Ω *is* $>$ *or* < 0.

The case $\Omega = 0$ *is undecided.*

The last term on the right side in (19) *is always* ≤ 0, *the equality sign occurring only if* $\partial K/\partial n \equiv 0$ *along* g_{01}. *Therefore its consideration becomes superfluous if the first term on the right side in* (19) *has been shown to be* < 0.

4. **Application to geodesic lines on surfaces of revolution.** Let $r = f(z)$, $r^2 = x^2 + y^2$, be the equations of a surface S of revolution around the z-axis. From $x = r\cos\varphi$, $y = r\sin\varphi$, $r = f(z)$ we get

(20) $$d\sigma^2 = dx^2 + dy^2 + dz^2 = [1 + f'^2(z)]\,dz^2 + f^2(z)\,d\varphi^2.$$

We obtain the geodesics minimizing the integral

$$\int \{[1 + f'^2(z)]\,z'^2 + f^2(z)\}^{1/2}\,d\varphi,$$

whose independence of φ leads to the usual formula for the geodesics

(21) $$\varphi = \varphi_0 + \int_{z_0}^{z} \frac{h\,dz}{f(z)} \left(\frac{1 + f'^2(z)}{f^2(z) - h^2} \right)^{1/2}, \qquad (h = \text{const.}).$$

[13] It was already known that $Z = 0$ is necessary if 1 is to be a foyer absolu of 0. See W. Blaschke, *Differentialgeometrie* I, Berlin 1930, § 102 form. (77). In our formula (16) as well as in (19), $\partial K/\partial n$, $\partial^2 K/\partial n^2$, stand for the first and second derivative of K along the geodesic arc orthogonal to g with respect to its length.

The envelope of the geodesics through the fixed point (φ_0, z_0) has been discussed by means of this formula (21) by von Braunmühl[14] and K. Fleischmann.[15] Fleischmann also discussed the cusps of the envelope in very general cases.

In general the criterion of our § 1 could be applied directly without our method of § 2, because equation (1) has been completely solved for this case, (21) representing its general integral. Nevertheless is the theorem of § 3 of a very easy application for two kinds of important special geodesics: The geodesic parallels and meridians of S, and leads to precise results in terms of the curvature K of the surface S.

a) *The geodesic parallels of* S. Let $z = z_0$ be a geodesic parallel of S. We have then $f'(z_0) = 0$. The geodesics orthogonal to g are just the meridians of our surface, and returning to our original variables of § 3, we have to take

$$u = \int_{z_0}^{z} [1 + f'^2(z)]^{1/2}\, dz, \qquad s = f(z_0) \cdot \varphi,$$

and (20) becomes

$$d\sigma^2 = du^2 + \left(\frac{f(z)}{f(z_0)}\right)^2 ds^2.$$

Therefore $C = f(z)/f(z_0)$ and formula (10) gives

$$K = -\frac{f''(z)}{f(z)\,[1 + f'^2(z)]^2}.\text{[16]} \tag{22}$$

From $f'(z_0) = 0$, we get at once along g

$$K_u = K_z, \qquad K_{uu} = K_{zz},$$

and therefore formula (22) gives along g (f stands for $f(z_0)$)

$$\begin{aligned} K &= -f''/f, \qquad K_u = -f'''/f, \\ K_{uu} &= -(ff^{\mathrm{IV}} - 4ff''^3 - f''^2)/f^2 = -\tilde{\omega}/f^2. \end{aligned} \tag{23}$$

The theorem of § 3 is of a very easy application because K, $K_u = \partial K/\partial n$, $K_{uu} = \partial^2 K/\partial n^2$ are constants along our parallel g. If we choose the origin 0 of the arc g_{01} as $\varphi_0 = s_0 = 0$, then Jacobi's equation (13) shows that there is a point 1 conjugate to 0, provided that $f''(z_0) < 0$.

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[14] A. v. Braunmühl, *Dissertation*, München 1878, and *Über Envelopen geodätischer Linien*, Math. Annalen, vol. 14 (1879), pp. 557–566.

[15] K. Fleischmann, *Die geodätischen Linien auf Rotationsflächen*, Dissertation, Breslau 1914, Chap. IV, § 28. See also A. R. Forsyth, *Differential Geometry*, Cambridge 1912, Chap. V, §§ 93–99 (compare footnote 21).

[16] This value can be also found applying directly Meusnier's theorem. See W. Blaschke, l. c., § 46 on pag. 96.

Suppose $f''(z_0) < 0$, that is, g is a maximum parallel, then we get $\eta = \sin(s\sqrt{K})$, $s_1 = \pi(-f/f'')^{1/2}$, $\varphi_1 = \pi(-ff'')^{-1/2}$ and formula (16) becomes

$$Z = -K_u \int_{g_{01}} \eta^3 \, ds.$$

If $K_u = 0$, it follows $Z = 0$, and then (19) gives

$$\Omega = -K_{uu} \int_{g_{01}} \eta^4 \, ds.$$

From the theorem of § 3 we get very easy the following results:

Let g be a maximum parallel of S and we suppose that not both derivatives, K_z and K_{zz}, of the curvature K of S vanishes along g. Let g_{01} be an arc of g limited by two conjugate points 0 *and* 1.[17]

If the curvature K has no maximum and no minimum on S along g, then the arc g_{01} does not give the shortest distance between 0 *and* 1, *and the envelope of the geodesics through* 0 *touches g in* 1 *on the side of g of larger curvature K.*

If the curvature K takes on an extreme value on g, then g_{01} gives actually the shortest distance or not, and the envelope of the geodesics through 0 *has in* 1 *a foyer en pointe or a foyer en talon, according as K has on S along g a maximum or a minimum value, or according as $\bar{\omega} = ff^{\mathrm{IV}} - 4ff'''^{3} - f''^{2}$ is $>$ or < 0.*

The case $K_z = K_{zz} = 0$ is undecided.

These results in terms of the curvature K are true for every geodesic g on every surface whatsoever, provided that K, K_u, and K_{uu}, are constant along g.[18]

Let us consider some special surfaces.

The *ellipsoid of revolution*

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$$

has the meridian

$$r = f(z) = a\left(1 - \frac{z^2}{c^2}\right)^{1/2} = a\left(1 - \frac{1}{2}\frac{z^2}{c^2} - \frac{1}{8}\frac{z^4}{c^4} + \cdots\right).$$

Therefore we get for $z = 0$:

$$f = a, \quad f' = 0, \quad f'' = -\frac{a}{c^2}, \quad f''' = 0, \quad f^{\mathrm{IV}} = -\frac{3a}{c^4}.$$

[17] We have conjugate points on g only for $K > 0$ along g, that is $f''(z_0) < 0$. Even for $f''(z_0) = 0$ there are no such points.

[18] Let us consider a surface generated by a helicoidal screw-motion of a plane curve around an axis of its own plane. A point of this curve of maximum distance from the axis generates a geodesic helice along which K, K_u, and K_{uu}, are constant and $K > 0$.

On the equator g defined by $z=0$, the point 1 conjugate to $\varphi=0$ is given by $\varphi_1=\pi(-ff'')^{-1/2}=\pi c/a$, and from (23) we get

$$\tilde{\omega}=ff^{\mathrm{IV}}-4ff''^3-f''^2=\frac{4a^2(a^2-c^2)}{c^6},$$

and therefore g_{01} is the shortest distance or not, and the envelope has at 1 a *foyer en pointe* or a *foyer en talon,* according as our ellipsoid is *oblate* or *prolate.* This is because the curvature K takes its maximum or minimum value on the equator g, according as the ellipsoid is oblate or prolate. This result of von Braunmühl's paper has been proved analytically by W. F. Osgood (l. c.).[19]

Let us consider the *anchor-ring* whose meridian is the circle $(r-d)^2+z^2=R^2$ $(d>0)$. We have

$$r=f(z)=d+R\left(1-\frac{z^2}{R^2}\right)^{1/2}=d+R\left(1-\frac{1}{2}\frac{z^2}{R^2}-\frac{1}{8}\frac{z^4}{R^4}+\cdots\right),$$

and for the equator $z=0$, we get

$$f=d+R,\quad f'=0,\quad f''=-\frac{1}{R},\quad f'''=0,\quad f^{\mathrm{IV}}=-\frac{3}{R^3},$$

$$\tilde{\omega}=ff^{\mathrm{IV}}-4ff''^3-f''^2=\frac{d}{R^3}>0,$$

and therefore the arc g_{01} joining the conjugate points, $\varphi_0=0$, and $\varphi_1=\pi(-ff'')^{-1/2}=\pi R^{1/2}(R+d)^{-1/2}$, actually gives the shortest distance and the envelope has at 1 a *foyer en pointe.*

b) *The meridians of a surface of revolution.* Let now g be a meridian of the surface of revolution S. We take on g its length s as parameter and let $r=f(z)=r(s)$.

Let

(13) $$\eta''+K\eta=0$$

be Jacobi's equation along our meridian g. Of course $r(s)\,d\varphi$ is a variation leading to a new geodesic and therefore $\eta=r(s)$ is a solution of the equation (13).

By means of a theorem of Sturm we see *the impossibility of the existence of two conjugate points on g, in the case that g does not meet the axis of revolution.*[20] Indeed, $\eta=r(s)$ has to vanish between two consecutive roots of some other solution of (13).

Let $r(0)=0$ and $r(s)<0$ for $s_0\leqq s<0$, $r(s)>0$ for $0<s\leqq s_1$, s_0 and s_1 corresponding to two consecutive conjugate points 0 and 1 on g.

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[19] Compare Fig. 1 and Fig. 3 of v. Braunmühl's paper, l. c., to which also Osgood refers to.

[20] Compare K. Fleischmann, l. c., § 27, pag. 58.

In order to avoid singular points on the surface, we have to suppose that the meridian g is symmetrical with respect to the axis of revolution. A solution of (13) linearly independent of $\eta = r(s)$ has the form

$$\eta_1 = r(s) \int \frac{ds}{r^2(s)}.$$

In some neighborhood of the vertex $s = 0$ we have the expensions

$$r(s) = s + a_3 s^3 + a_5 s^5 + \cdots, \qquad 1/r^2(s) = 1/s^2 + b_0 + b_2 s^2 + b_4 s^4 + \cdots$$

and therefore

$$\int \frac{ds}{r^2(s)} = -\frac{1}{s} + c_1 s + c_3 s^3 + \cdots + \text{const.} = \varphi(s) + \text{const.}$$

$\eta_1(s) = r(s)\,[\varphi(s) - \varphi(s_0)]$ vanishes for $s = s_0$, and since 1 is conjugate to 0, we get $\eta_1(s_1) = r(s_1)\,[\varphi(s_1) - \varphi(s_0)]$ and finally

$$\varphi(s_0) = \varphi(s_1).$$

In this way we may use this function $\varphi(s)$ in order to find the conjugate points on the meridian g.[21]

Along g_{01}, of course, we have $K_u \equiv 0$, as will be immediately shown. Therefore we get $Z = 0$ and we have to consider Ω, and since $K_u \equiv 0$, (18) gives

$$\Omega = \int_{s_0}^{s_1} \left(K' \frac{\eta'}{\eta} - K_{uu} \right) \eta^4 \, ds. \tag{24}$$

We have to calculate K_{uu} which is the second derivative of K with respect to the arc u of the geodesic line orthogonal to g.

Let P_0 be a point of g with $z = z_0$. We suppose that $f(z_0) > 0$, $f'(z_0) > 0$, since the same type of argument applies for $f'(z_0) < 0$.

Along the geodesic line orthogonal to g in P_0 we get from (20) and (21)

$$\frac{dz}{du} = \frac{1}{f(z)} \left(\frac{f^2(z) - f^2(z_0)}{1 + f'^2(z)} \right)^{1/2}$$

and therefore

$$K_u = K_z \frac{dz}{du} = K_z \frac{1}{f} \left(\frac{f^2 - f_0^2}{1 + f'^2} \right)^{1/2}.$$

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[21] A. R. Forsyth gives, l. c., a similar method. His result for the anchor-ring (pag. 143, Ex. 4) is not correct. He finds two diametrally opposite conjugate points on the circular meridian of the anchor-ring. Of course, there can not be any conjugate points at all, if the meridian does not cut the axis of revolution, according to our general remark stated above, and if there are conjugate points, those points are diametrally opposite only when the anchor-ring becomes a sphere.

It follows $K_u = 0$ at P_0. The second derivative is

$$K_{uu} = \frac{1}{f}\left(\frac{f^2-f_0^2}{1+f'^2}\right)^{1/2} \frac{d}{dz}\left\{K_z \frac{1}{f}\left(\frac{f^2-f_0^2}{1+f'^2}\right)^{1/2}\right\}.$$

Differentiating out, we get for $z = z_0$

$$K_{uu} = \frac{f' K_z}{f(1+f'^2)} \quad \text{at} \quad P_0.$$

Returning to the variable s we get

$$f' = f_s(ds/dz) = f_s(1+f'^2)^{1/2}, \qquad K_z = K_s(1+f'^2)^{1/2},$$

and therefore the last formula becomes

$$K_{uu} = \frac{K_s f_s}{f} = \frac{K' r'}{r}.$$

Now formula (24) gives

$$\Omega = \int_{s_0}^{s_1}\left(\frac{\eta'}{\eta} - \frac{r'}{r}\right) K' \eta^4 \, ds.$$

But η and r are both solutions of (13) and therefore $r\eta' - \eta r' = \text{const.}$ and we may suppose that $r\eta' - \eta r' = -1$. Therefore $\eta'/\eta - r'/r = -1/r\eta$ and finally

$$\text{(25)} \qquad \Omega = -\int_{g_{01}} \frac{K'}{r} \eta^3 \, ds.\ [22]$$

We have got the following result:

We consider a meridian g of a surface of revolution and we suppose that g cuts the axis of revolution and is symmetrical with respect to this axis. Let g_{01} be an arc of g whose endpoints 0 and 1 are conjugate. Let $\eta = \eta(s)$ be a solution of Jacobi-Bonnet's equation (13) *for g, vanishing at* 0 *and* 1, *positive between these points.*

The arc g_{01} actually gives the shortest distance between 0 *and* 1 *or not, and the envelope of the geodesics through* 0 *has in* 1 *a foyer en pointe or a foyer en talon, according as Ω, given by formula* (25),[23] *is $>$ or <0. The case $\Omega = 0$ is undecided.*

Let us consider, for instance, a *paraboloid of revolution*

$$z = -\frac{1}{2p}(x^2 + y^2), \qquad (p > 0).$$

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[22] This integral is only apparently singular because of K' vanishing with r at the vertex $s = 0$.

[23] In formula (25) $K' = dK/ds$ along g, and $r = r(s)$ is the distance to the axis with its proper sign.

The meridian is $r = f(z) = (-2pz)^{1/2}$. Formula (22) gives $K = (p-2z)^{-2}$ and therefore K has its maximum value at the vertex of the surface. Hence $K' > 0$ for $s < 0$ and $K' < 0$ for $s > 0$ and this gives $K'/r < 0$ for $-\infty < s < +\infty$.

Therefore formula (25) always gives $\Omega > 0$: *If g_{01} is an arc of the meridian with conjugate endpoints, then* 1 *is a foyer en pointe of* 0.[24]

We may add the following remark. Let g_{01} be a geodesic arc on the paraboloid with conjugate endpoints and which is not a meridian. $d\varphi/ds$ never changes its sign on g, because our geodesic g can never touch any meridian which is a geodesic line, too. Therefore, if for instance $d\varphi/ds > 0$ along g, we always have $K_u > 0$ along g_{01}. Therefore formula (16) gives $Z < 0$, so that the envelope of the geodesics through 0 is situated in 1 above our geodesic line.

Of course the same arguments are true for any surface of revolution of the same shape and whose curvature K always increases towards the vertex where it has a maximum.

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[24] See H. v. Mangoldt, Journal für Math., vol. 91 (1881), pp. 46–47, and K. Fleischmann, l. c., pag. 70 and Fig. 8.

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(ADDED IN PROOF, AUGUST 1931.) After the present paper was already in type, I noticed Prof. B. F. Kimball's paper *Geodesics on a Toroid,* American Journal of Math., vol. LII (1930), pp. 29–52, which, though different in purpose, has many points in common with the paper here presented. Prof. Kimball generalizes Lindeberg's envelope theory to the successive envelopes of a family of geodesics through a fixed point on a surface. He also uses the differential equation (3) of the second variation for the discussion of these envelopes. The method set forth in our §§ 2 and 3 may be applied without much change to Prof. Kimball's generalization of Lindeberg's theorem, just as in the case of the first envelope, which alone seemed to me of importance in the question of the minimizing properties of geodesic arcs. Essentially new in the present paper seems to be the consideration of the equations for the higher variations and especially of the equation (4) of the third variation which, as was shown in § 4, indicates in many cases the nature of the cusp of the envelope of the family of geodesics through a fixed point.

ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY LINEAR COMBINATIONS OF CONTINUOUS FUNCTIONS.[1]

BY W. SEIDEL.[2]

1. In the present paper we propose to solve the following problem: *Let there be given a sequence of arbitrary linearly independent*[3] *functions* $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots$, *defined and continuous in the interval* $a \leqq x \leqq b$. *What is a necessary and sufficient condition that there exist linear combinations* $\sum_{i=1}^{n} a_i^{(n)} \varphi_i(x)$ *of the* $\varphi_i(x)$ *which converge uniformly toward any preassigned function* $f(x)$, *defined and continuous in the interval* $a \leqq x \leqq b$?

This problem was first proposed by E. Schmidt[4] in his thesis. Schmidt obtains two conditions, one necessary and the other sufficient. The necessary, but not sufficient, condition is as follows: there does not exist any continuous function $\psi(x)$, not identically zero, which is orthogonal to all the functions $\varphi_i(x)$. In other words, if the sequence of functions $\varphi_i(x)$ has the above property that there exist linear combinations of them converging uniformly to any preassigned function $f(x)$, then there does not exist any continuous function $\psi(x)$, not identically zero, satisfying the equations

$$\int_a^b \varphi_i(x)\,\psi(x)\,dx = 0, \qquad (i = 1, 2, \cdots).$$

The sufficient, but not necessary, condition of Schmidt assumes that the second derivatives of all the functions $\varphi_i(x)$ exist and are continuous, and that there does not exist any continuous function, not identically zero, orthogonal to all these second derivatives. Then he shows that each continuous function $f(x)$ may be uniformly approximated by linear combinations of the functions 1, x, and $\varphi_i(x)$.

A necessary and sufficient condition was first obtained by F. Riesz[5] by the use of Stieltjes integrals. This condition is that every function of bounded variation $\alpha(x)$ satisfying the system of equations

[1] Received March 19, 1931. — The author wishes to thank Professor J. L. Walsh for the latter's many helpful suggestions and criticisms.

[2] National Research Fellow.

[3] A set of linearly independent functions $\{\varphi_n(x)\}$ is one for which the identity $\sum_{i=1}^{m} c_i \varphi_i(x) \equiv 0$ implies $c_1 = c_2 = \cdots = c_m = 0$.

[4] *Entwickelung willkürlicher Funktionen nach Systemen vorgeschriebener.* (Inaugural-Dissertation.) Göttingen (1905); Mathematische Annalen, vol. 63 (1907), pp. 433–476.

[5] F. Riesz, Annales de l'École Normale Supérieure (3), vol. 28 (1911), pp. 33–62.

$$\int_a^b \varphi_i(x)\, d\alpha(x) = 0, \qquad (i = 1, 2, \cdots),$$

shall be constant in the interval $a \leqq x \leqq b$ save perhaps for a denumerable set of values of x different from a and b.

Our treatment of the problem is purely geometric and obtainable by altogether elementary considerations in Minkowski's geometry of convex bodies.[6] At the end of the paper we shall see how the condition of Riesz and our own are related to one another.

2. Before proceeding with the solution of our problem, we shall briefly recall those elementary properties of convex bodies in n-dimensional euclidean space which we shall need in what follows.

A point X of n-dimensional space R_n is given if one knows its n coördinates $(x_1, \cdots, x_n)$. A closed point set K in R_n will be called *convex* if the line segment joining two arbitrary points A and B of K consists only of points belonging to K. A point not belonging to K is called an exterior point. A point of K which is limit point of exterior points will be called a *boundary point* of K. The set of all boundary points of K will be called the *boundary* of K. An $(n-1)$-dimensional plane will be called a *plane of support* of K if it contains at least one point of K and if every point of K lies either in the plane or on the one side of it.

In what follows we shall make use of two well-known properties of bounded convex bodies.[7]

LEMMA 1. *Through every boundary point of K may be passed at least one plane of support of K.*

Consider, furthermore, a closed and bounded point set S in R_n. It may be shown that there always exists a *smallest convex body* $K(S)$ containing S; that is, a convex body containing S and such that every other convex body containing S also contains $K(S)$.

LEMMA 2. *Every point of the smallest convex body $K(S)$ containing a closed and bounded point set S may be considered as the center of gravity of at most $(n+1)$ positive masses of total mass 1 and lying in S. Conversely, the center of gravity of at most $(n+1)$ masses of total mass 1 and lying in S belongs to the smallest convex body $K(S)$ containing the set S.*

3. We can now return to our problem. We shall first consider a somewhat different form of it. Let there be given a sequence of arbitrary linearly independent functions $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots$ defined and con-

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[6] An analogous treatment of Tschebyscheff approximation is given by A. Haar, Mathematische Annalen, vol. 78 (1918), pp. 294–311.

[7] An admirable account of these properties will be found in C. Carathéodory, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 1–25.

tinuous in the interval $a \leqq x \leqq b$ and let $f(x)$ be a continuous function defined in the same interval. What is a necessary and sufficient condition that there exist linear combinations $\sum_{i=1}^{n} a_i^{(n)} \varphi_i(x)$ which converge uniformly in the interval $a \leqq x \leqq b$ toward $f(x)$?

Consider in $(n+1)$-dimensional euclidean space R_{n+1} with the coördinates $(x_0, x_1, \cdots, x_n)$ the following two curves:

$$\Gamma_{n+1}^{+}\colon \begin{cases} x_0 = f(x), \\ x_1 = \varphi_1(x), \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n = \varphi_n(x), \end{cases} \quad \text{and} \quad \Gamma_{n+1}^{-}\colon \begin{cases} x_0 = -f(x), \\ x_1 = -\varphi_1(x), \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n = -\varphi_n(x), \end{cases}$$

as well as the smallest convex body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ containing these two curves Γ_{n+1}^{+} and Γ_{n+1}^{-}. It is clear from the definition that the origin O belongs to the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ and that this body is symmetric in the origin O. By this we mean that if a point $(a_0, a_1, \cdots, a_n)$ lies in the interior or on the boundary of the body, then the point $(-a_0, -a_1, \cdots$ $\cdots, -a_n)$ likewise lies in the interior or on the boundary of the body, respectively.

4. Let us first consider the necessary condition. We know by hypothesis that there exist linear combinations $\sum_{i=1}^{n} a_i^{(n)} \varphi_i(x)$ of the $\varphi_i(x)$ which converge uniformly toward $f(x)$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i^{(n)} \varphi_i(x) = f(x). \tag{4.1}$$

There are two cases possible according as $f(x)$ is itself a linear combination of the $\varphi_i(x)$:

$$f(x) = \sum_{i=1}^{n} b_i \varphi_i(x), \tag{4.2}$$

or is not.

Suppose a relation of the type (4.2) exists for some n. For the same n consider the $(n+1)$-dimensional space R_{n+1} and the $(n+1)$-dimensional convex body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ defined in section 3. Consider the n-dimensional plane through the origin

$$x_0 - b_1 x_1 - b_2 x_2 - \cdots - b_n x_n = 0. \tag{4.3}$$

In consequence of (4.2) and the definition of Γ_{n+1}^{+} and Γ_{n+1}^{-} these two curves lie in the plane (4.3). Since the plane itself is a convex body, the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ being the *smallest* convex body which contains the two curves Γ_{n+1}^{+} and Γ_{n+1}^{-}, has to lie wholly in the plane (4.3). Hence,

the plane (4.3) is a plane of support of $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ and since the origin is contained as well in the plane (4.3) as in the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$, the origin must be a boundary point of $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$.

But the converse is also true. If the origin lies on the boundary of $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$, then the function $f(x)$ may be represented as a linear combination of the functions $\varphi_1(x), \cdots, \varphi_n(x)$. Since the origin lies on the boundary, we may, according to Lemma 1 of section 2, pass through it a plane of support of the convex body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$. Let the equation of the plane be

$$c_0 x_0 - c_1 x_1 - \cdots - c_n x_n = 0. \tag{4.4}$$

Since the origin lies on the boundary of $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$, the whole body lies in the plane (4.4) and therefore also the curves Γ_{n+1}^{+} and Γ_{n+1}^{-}. This means that the equation

$$c_0 f(x) = \sum_{i=1}^{n} c_i \varphi_i(x)$$

holds. If now $c_0 = 0$, this would mean that the functions $\varphi_1(x), \cdots, \varphi_n(x)$ are linearly dependent, a case that we excluded from the start. Hence, $c_0 \neq 0$ and

$$f(x) = \sum_{i=1}^{n} \frac{c_i}{c_0} \varphi_i(x).$$

We have, then, the result that *a necessary and sufficient condition that $f(x)$ be a linear combination of the linearly independent functions $\varphi_1(x)$, $\varphi_2(x), \cdots, \varphi_n(x)$ is that the origin lie on the boundary of the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$.*

5. The second case is the one of equation (4.1), where the convergence is uniform in the interval $a \leqq x \leqq b$. This means that to every positive ε, no matter how small, there exists a positive integer $N(\varepsilon)$, independent of x, such that for $n \geqq N(\varepsilon)$

$$-\varepsilon < \sum_{i=1}^{n} a_i^{(n)} \varphi_i(x) - f(x) < \varepsilon. \tag{5.1}$$

Consider the region defined by the inequality

$$-\varepsilon < \sum_{i=1}^{n} a_i^{(n)} x_i - x_0 < \varepsilon. \tag{5.2}$$

Since the region (5.2) is itself convex, the smallest convex body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ containing Γ_{n+1}^{+} and Γ_{n+1}^{-} must lie in the interior of the region (5.2). The straight line which joins the origin with the point $(1, 0, \cdots, 0)$ will intersect the boundary of the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ in a point A_{n+1} such that the distance $\overline{OA_{n+1}} < \varepsilon$, where $n \geqq N(\varepsilon)$.

If, now, n is allowed to run through all values $1, 2, \cdots$ and if for each of these values the corresponding body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ is constructed, it is seen that

$$\lim_{n\to\infty} \overline{OA_{n+1}} = 0.$$

This is a necessary condition which will now be proved sufficient.

Consider a function $f(x)$ and a sequence of functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$ such that for each arbitrarily small positive ε there exists an integer $N(\varepsilon)$ having the property that when $n \geqq N(\varepsilon)$, the distance $\overline{OA_{n+1}} < \varepsilon$. We wish to show that there exist linear combinations of the functions $\{\varphi_n(x)\}$ which converge uniformly toward $f(x)$ in the interval $a \leqq x \leqq b$. If $\overline{OA_{n+1}} < \varepsilon$ and if B_{n+1} denotes the symmetric image point of A_{n+1} in the origin, $\overline{A_{n+1}B_{n+1}} < 2\varepsilon$ and B_{n+1} likewise lies on the boundary of the body. Through A_{n+1} and B_{n+1} let planes of support of the body be passed. From the symmetry of the body it follows that these planes may be chosen parallel. Let their equations be, respectively,

$$\sum_{i=0}^{n} a_i^{(n)} x_i = a_0^{(n)} \eta$$

and

$$\sum_{i=0}^{n} a_i^{(n)} x_i = -a_0^{(n)} \eta,$$

where η is a positive number smaller than ε. The body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$, therefore, lies in the region

$$\left| \sum_{i=0}^{n} a_i^{(n)} x_i \right| < |a_0^{(n)}| \varepsilon.$$

If $a_0^{(n)} = 0$, the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ lies in the plane $\sum_{i=1}^{n} a_i^{(n)} x_i = 0$, from which we deduce, as in section 4, that $f(x)$ is a linear combination of the functions $\varphi_1(x), \cdots, \varphi_n(x)$. If $a_0^{(n)} \neq 0$, the body lies in the region

$$\left| \sum_{i=0}^{n} \frac{a_i^{(n)}}{a_0^{(n)}} x_i \right| < \varepsilon.$$

Hence, the curve Γ_{n+1}^{+} lies in the same region, and we have

$$\left| f(x) + \sum_{i=1}^{n} \frac{a_i^{(n)}}{a_0^{(n)}} \varphi_i(x) \right| < \varepsilon,$$

which proves our assertion.

We, thus, have the result:

A necessary and sufficient condition that there exist linear combinations of the functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$ *which converge uniformly toward* $f(x)$ *is that* $\lim\limits_{n\to\infty} \overline{OA_{n+1}} = 0$.[8]

6. From this result it is easy to derive the theorem of Riesz, mentioned in section 1. According to Lemma 2 of section 2, every point $(a_0, a_1, \cdots, a_n)$ of the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ can be represented in the following manner:

$$(6.1)\qquad \begin{cases} a_0 = \sum\limits_{i=0}^{n+1} \lambda_i\, f(x_i) - \sum\limits_{j=0}^{n+1} \mu_j\, f(x_j), \\ a_k = \sum\limits_{i=0}^{n+1} \lambda_i\, \varphi_k(x_i) - \sum\limits_{j=0}^{n+1} \mu_j\, \varphi_k(x_j), \end{cases}$$

where $\lambda_0 \geqq 0,\ \lambda_1 \geqq 0, \cdots, \lambda_{n+1} \geqq 0,\ \mu_0 \geqq 0,\ \mu_1 \geqq 0, \cdots, \mu_{n+1} \geqq 0$ and $\sum\limits_{i=0}^{n+1} (\lambda_i + \mu_i) = 1$. We observe that the sums (6.1) can be represented as Stieltjes integrals

$$(6.2)\qquad a_0 = \int_a^b f(x)\, d\alpha(x), \quad a_k = \int_a^b \varphi_k(x)\, d\alpha(x), \qquad (k = 1, 2, \cdots, n),$$

where $\alpha(x)$ is a step function of total variation 1: $\int_a^b |d\alpha(x)| = 1$ with the jumps λ_i in the points $x = x_i$ and the jumps $-\mu_j$ in the points $x = x_j$.

It is clear that if more than $(n+1)$ masses lie on Γ_{n+1}^{+} and more than $(n+1)$ masses lie on Γ_{n+1}^{-} of total mass 1, then the center of gravity of all the masses lies in the convex body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$. This is a direct consequence of the property of convexity. A transition to the limit shows immediately that the center of gravity of a continuous distribution of masses of total mass 1 on the curves Γ_{n+1}^{+} and Γ_{n+1}^{-} likewise lies in the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$. Similarly, if $\alpha(x)$ is an arbitrary function of bounded variation of total variation 1: $\int_a^b |d\alpha(x)| = 1$, then the point $(a_0, a_1, \cdots, a_n)$, defined by the equations (6.2), lies in the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$.

Furthermore, if, $\alpha(x)$ being an arbitrary function of bounded variation, the point $(a_0, a_1, \cdots, a_n)$, where the a_k are given by (6.2), lies on the boundary of $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$, then the total variation $\int_a^b |d\alpha(x)| \geqq 1$. In order to prove this, denote the point $(a_0, a_1, \cdots, a_n)$ by A. If it should happen that $\int_a^b |d\alpha(x)| = c < 1$, then the point $\left(\frac{a_0}{c}, \frac{a_1}{c}, \cdots, \frac{a_n}{c}\right)$ will lie outside of $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ on the line joining O with A. According to (6.2), the point can be represented by the equations

[8] This theorem remains valid even when the functions $f(x)$ and $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots$ are defined on an arbitrary compact set S instead of on the interval $a \leqq x \leqq b$.

$$\frac{a_0}{c} = \int_a^b f(x)\, d\left(\frac{\alpha(x)}{c}\right), \qquad \frac{a_k}{c} = \int_a^b \varphi_k(x)\, d\left(\frac{\alpha(x)}{c}\right), \qquad (k = 1, 2, \cdots, n),$$

where $\int_a^b \left| d\left(\frac{\alpha(x)}{c}\right)\right| = 1$. According to the remark of the last paragraph, however, a point $\left(\frac{a_0}{c}, \frac{a_1}{c}, \cdots, \frac{a_n}{c}\right)$ with such a representation would have to lie in the body $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$, which is a contradiction.

This also shows that if $\alpha(x)$ is a function of bounded variation and if $(a_0, a_1, \cdots, a_n)$ is an arbitrary point A of the space R_{n+1}, then the total variation of $\alpha(x)$ cannot be less than the constant $\frac{\overline{OA}}{\overline{OB}}$, where B is the point in which the line joining O and A cuts the boundary of the body $K_n(f, \varphi_1, \cdots, \varphi_n)$:

$$\int_a^b |d\alpha(x)| \geqq \frac{\overline{OA}}{\overline{OB}}.$$

Let us apply our result to the point $(1, 0, \cdots, 0)$. We obtain the theorem: *If there exists a solution $\alpha(x)$ of the equations*

$$(6.3) \qquad 1 = \int_a^b f(x)\, d\alpha(x), \qquad 0 = \int_a^b \varphi_k(x)\, d\alpha(x), \qquad (k = 1, 2, \cdots, n),$$

then $\int_a^b |d\alpha(x)| \geqq \frac{1}{\overline{OB}}$, *where B is defined as above.*

7. Let us now assume that $f(x)$ can be represented as the limit of a uniformly convergent sequence of linear combinations of the functions $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots$. As we have seen at the end of sections 4 and 5, this implies that either $\overline{OA_{n+1}} = 0$ for some n or $\lim_{n\to\infty} \overline{OA_{n+1}} = 0$ with $\overline{OA_{n+1}} \neq 0$ for all n.

The first case corresponds to the fact, as we have already seen, that $K_{n+1}(f, \varphi_1, \cdots, \varphi_n)$ lies in an n-dimensional plane P which does not pass through the point $(1, 0, \cdots, 0)$. In that case, however, the point $(1, 0, \cdots, 0)$ can never be represented in the form (6.3), for that would imply that it can be considered as the center of gravity of a set of masses lying in the plane P.

In the second case, $\lim_{n\to\infty} \overline{OA_{n+1}} = 0$, we can apply the result of section 6 and assert that if $\alpha(x)$ is a solution of equations (6.3), then $\int_a^b |d\alpha(x)| \geqq \frac{1}{\overline{OA_{n+1}}}$. If there exists a solution $\alpha(x)$ of the equations

$$(7.1) \qquad 1 = \int_a^b f(x)\, d\alpha(x), \qquad 0 = \int_a^b \varphi_k(x)\, d\alpha(x), \qquad (k = 1, 2, \cdots),$$

let its total variation be V. Taking n so large that $\overline{OA_{n+1}} < \frac{1}{V}$, we immediately obtain a contradiction. There exists, therefore, no function $\alpha(x)$ of bounded variation satisfying the equations (7.1).

Conversely, if the equations (7.1) have no solution $\alpha(x)$ of bounded variation, $\lim_{n\to\infty} \overline{OA_{n+1}} = 0$. For, if that were not the case, there would exist a positive number ε and infinitely many $\overline{OA_{n_i}} > \varepsilon$. Hence, by preceding considerations there would exist for each i a function $\alpha_i(x)$ of total variation $< \frac{1}{\varepsilon}$ satisfying the system of equations

$$(7.2) \qquad 1 = \int_a^b f(x)\, d\alpha_i(x), \qquad 0 = \int_a^b \varphi_k(x)\, d\alpha_i(x), \quad (k = 1, 2, \cdots, n_i).$$

We can assume that $\alpha_i(a) = 0$ for all values of i, for the addition of a constant to $\alpha_i(x)$ does not affect the values of the integrals in (7.2). Furthermore, we can select a subsequence of $\alpha_i(x)$ such that the numbers $\alpha_i(b)$ converge toward a definite limit. According to a well-known theorem of Helly,[9] there exists a further subsequence of the functions $\alpha_i(x)$ converging everywhere save perhaps in a denumerable set of points toward a function $\alpha(x)$ of total variation $\leqq \frac{1}{\varepsilon}$. According to a further theorem of Helly,[10] the equations

$$(7.3) \qquad \begin{aligned} \lim_{i\to\infty} \int_a^b f(x)\, d\alpha_i(x) &= \int_a^b f(x)\, d\alpha(x), \\ \lim_{i\to\infty} \int_a^b \varphi_k(x)\, d\alpha_i(x) &= \int_a^b \varphi_k(x)\, d\alpha(x) \end{aligned}$$

hold under these conditions. Hence, the limit (7.3), carried out in (7.2), gives a solution of $\alpha(x)$ of (7.1), which is a contradiction.

We, thus, obtain the following theorem of F. Riesz:

A necessary and sufficient condition that there exist linear combinations of the functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$ *converging uniformly toward* $f(x)$ *is that there shall exist no solution* $\alpha(x)$ *of the equations* (7.1).

From this follows at once the theorem of Riesz, enunciated in section 1, which we set out to prove.

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[9] E. Helly, Sitzber. der Wiener Akad., vol. 121 (1912), pp. 265–297.

[10] E. Helly, ibid.

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ON THE FUNCTIONAL OF MR. DOUGLAS.*

By Tibor Radó.

Mr. Douglas published recently a new solution of the problem of Plateau,[1] based on an interesting functional introduced by him. The object of the present paper is to develop some remarks which might simplify the handling of this functional and throw some more light upon its nature.

§ 1. The Simultaneous Problem.

1. In this paragraph we shall state the problems we shall be concerned with.

The problem of the least area requires to determine a surface, bounded by a given Jordan curve and having an area as small as possible. It is shown in the calculus of variations that such a surface, if it is sufficiently regular, is a minimal surface (that is to say, its mean curvature H vanishes identically). Lebesgue has shown, on the other hand, that the problem has solutions which are not minimal surfaces.[2]

The problem of Plateau requires to determine a surface bounded by a given Jordan curve. Schwarz has given examples of portions of minimal surfaces which do not have a smallest area.[3]

2. Lebesgue[4] proposed the problem to show that the problem of the least area has always a solution which is a minimal surface (or, what is the same thing, to show that the problem of Plateau has always a solution with a minimum area). It might be convenient to have a name for this problem; we shall call it *the simultaneous problem.* This problem may be stated as follows.

The simultaneous problem. Given, in the xyz-space, a Jordan curve Γ, determine three functions $x(u, v)$, $y(u, v)$, $z(u, v)$ with the following properties.

I. $x(u, v)$, $y(u, v)$, $z(u, v)$ are harmonic for $u^2+v^2<1$.

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* Received April 20, 1931.

[1] J. Douglas, Solution of the problem of Plateau, Transactions of the American Mathematical Society, vol. 33 (1931), pp. 263–321.

[2] H. Lebesgue, Intégrale, longueur, aire, Annali di Matematica, Serie IIIa, vol. 7 (1902), pp. 231–359.

[3] H. A. Schwarz, Untersuchung der zweiten Variation des Flächeninhalts von Minimalflächenstücken, Gesammelte mathematische Abhandlungen, vol. 1. See in particular pp. 161–163.

[4] Loc. cit. [2].

II. $x(u, v), y(u, v), z(u, v)$ satisfy the relations

$$E = G, \ F = 0 \quad \text{for} \quad u^2+v^2<1,$$

where

$$E = x_u^2+y_u^2+z_u^2, \qquad F = x_u x_v + y_u y_v + z_u z_v, \qquad G = x_v^2+y_v^2+z_v^2.$$

III. $x(u, v), y(u, v), z(u, v)$ remain continuous for $u^2+v^2 = 1$, and the equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v)$$

carry the unit circle $u^2+v^2 = 1$ in a one-to-one way into the given Jordan curve Γ.

IV. The area of the surface

$$S\colon \ x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \qquad u^2+v^2 \leqq 1$$

is minimum.

The meaning of condition IV depends of course upon the class of surfaces in which S is required to minimize the area, and upon the definition of the area itself. We shall consider *the class of all continuous surfaces bounded by the given Jordan curve* Γ. Such a surface is defined by equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \qquad u^2+v^2 \leqq 1,$$

where $x(u, v), y(u, v), z(u, v)$ are continuous for $u^2+v^2 \leqq 1$ and where the above equations carry $u^2+v^2 = 1$ in a one-to-one way into the given curve Γ. As to the area of such a surface, we adopt the definition of Lebesgue, and we shall denote the area, in the sense of Lebesgue, of the continuous surface S by $\mathfrak{A}(S)$.[5]

3. As it follows from classical investigations of Schwarz, condition IV of the simultaneous problem is not a consequence of the conditions I, II, III, save of course in the trivial case when the area of all continuous surfaces bounded by Γ is $+\infty$.

We have now the theorem that *the simultaneous problem is solvable for any Jordan curve which bounds some continuous surface with a finite area.* There are at present two demonstrations of this theorem, one by myself and one by Mr. Douglas.

I have shown, in a first paper,[6] that the problem of Plateau is solvable for any rectifiable Jordan curve, and in a subsequent paper[7] I have shown

[5] See, for a systematic presentation of the theory of the area in the sense of Lebesgue, the author's paper Über das Flächenmaß rektifizierbarer Flächen, Mathematische Annalen, vol. 100 (1928), pp. 445–479.

[6] T. Radó, On the problem of Plateau, Annals of Mathematics, vol. 31 (1930), pp. 457–469.

[7] T. Radó, The problem of the least area and the problem of Plateau, Mathematische Zeitschrift, vol. 32 (1930), pp. 763–796.

that by means of slight modifications the methods used give a solution of the simultaneous problem. One of my main arguments was a classical theorem on the conformal mapping of polyhedrons.[8] Commenting on my first paper mentioned above, Mr. Douglas[9] observed that this polyhedron-theorem allows to complete his own solution of the problem of Plateau in the same direction; indeed, he shows[10] that this theorem allows to verify that the minimal surface, obtained by means of the new functional introduced by him, has a least possible area, that is to say that it solves the simultaneous problem also.

4. It is interesting, and for the sequel necessary, to compare briefly the two methods. The central and most elaborate part of the method of Mr. Douglas is the discussion of the first variation of his functional. A total absence of any first variation of any integral is characteristic for my method. Indeed, in terms of the calculus of variations, my method consists of solving the variation problem $\iint (EG-F^2)^{1/2} =$ minimum and its Euler-Lagrange equation $H = 0$ *simultaneously*; the discussion of the first variation drops out this way completely.[11] A common feature of both methods is the use of a strong theorem on conformal mapping of polyhedrons; the two methods differ however in this respect also. Indeed, I use this theorem at the very first step, in the construction of an approximate solution, while Mr. Douglas only wants it to verify the least area property of the minimal surface obtained by means of his functional.

5. In the following study of the functional of Mr. Douglas, we shall make use, in § 2, of the theorem that the simultaneous problem is solvable for any Jordan curve which bounds some continuous surface with a finite area; as my proof of this theorem is independent of any discussion of any first variation, there is no danger of a vicious circle. §§ 2 and 3 give an independent new method to discuss the first variation of the functional of Mr. Douglas.

§ 2. The Functional of Mr. Douglas.

1. Let there be given a Jordan curve Γ in the xyz-space. Consider all the one-to-one and continuous correspondences between Γ and the unit

[8] Loc. cit. [6], p. 459, and loc. cit. [7], p. 773.

[9] J. Douglas, Existence of a surface of absolutely least area bounded by a given contour, Bulletin of the American Mathematical Society, vol. 36 (1930), p. 796.

[10] Loc. cit. [1], part V.

[11] The method consists of showing first that for a Jordan curve which bounds some continuous surface with a finite area the simultaneous problem can be immediately solved in a certain approximate sense, and second that the exact solution may then be obtained by a passage to the limit.

circle $u^2+v^2=1$. Such a correspondence may be defined by a set of equations

(1) $x=\xi(\theta), \quad y=\eta(\theta), \quad z=\zeta(\theta), \quad 0<\theta\leqq 2\pi,$

where θ is the polar angle determining the position of a point varying on $u^2+v^2=1$. Mr. Douglas considers the integral

$$(2)\quad A(g)=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{[\xi(\theta)-\xi(\varphi)]^2+[\eta(\theta)-\eta(\varphi)]^2+[\zeta(\theta)-\zeta(\varphi)]^2}{4\sin^2\dfrac{\theta-\varphi}{2}}\,d\theta\,d\varphi,$$

where g denotes the one-to-one and continuous correspondence between $u^2+v^2=1$ and Γ, defined by (1). This integral $A(g)$ is the functional of Mr. Douglas; its argument g is a variable one-to-one and continuous correspondence between $u^2+v^2=1$ and Γ, or, what is the same thing, a variable parametric representation of Γ.[12]

Mr. Douglas assumes that the given Jordan curve Γ admits of some parametric representation g with a finite $A(g)$, and arrives to results which may be summed up as follows.

I. There exists a parametric representation g of the given Jordan curve Γ minimizing the functional $A(g)$.

II. Denote by

$$g\colon\ x=\xi(\theta), \quad y=\eta(\theta), \quad z=\zeta(\theta), \quad 0<\theta\leqq 2\pi$$

any parametric representation of Γ which minimizes $A(g)$. Consider the harmonic functions $x(u,v)$, $y(u,v)$, $z(u,v)$, defined for $u^2+v^2\leqq 1$ by the condition that they reduce to $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ for $u^2+v^2=1$. Then these harmonic functions satisfy the relations

$$E=G, \quad F=0 \quad \text{for} \quad u^2+v^2<1,$$

where

$$E=x_u^2+y_u^2+z_u^2, \quad F=x_ux_v+y_uy_v+z_uz_v, \quad G=x_v^2+y_v^2+z_v^2.$$

That is to say, the surface

$$S\colon\ x=x(u,v), \quad y=y(u,v), \quad z=z(u,v), \quad u^2+v^2\leqq 1$$

is a minimal surface bounded by the given Jordan curve Γ.

III. The above surface S has a least possible area with respect to all continuous surfaces bounded by Γ; that is to say, S solves also the simultaneous problem.

[12] Mr. Douglas considers $A(g)$ for improper parametric representations also. This is necessary for the details of the demonstration, the results however are only concerned with proper representations (one-to-one and continuous correspondences).

IV. The assumption that Γ admits of some parametric representation g with a finite $A(g)$, is equivalent to the assumption that Γ bounds some continuous surface with a finite area.

2. These results contain, in particular, the theorem that the simultaneous problem is solvable for any Jordan curve which bounds some continuous surface with a finite area. Besides, they contain interesting facts concerning the functional $A(g)$ itself, as e. g. the striking fact that *any* solution of the variation problem $A(g) = \text{minimum}$ gives a minimal surface, while the classical area-integral problem $\iint (EG - F^2)^{1/2} = \text{minimum}$ has always solutions which are not minimal surfaces.

The central part of the method of Mr. Douglas is his discussion of the first variation of his functional $A(g)$. We are first going to show that this discussion can be completely avoided. We shall namely see that the statements I to IV in the preceding No. 1 are immediate consequences of the existence of the solution of the simultaneous problem; and we observed (§ 1, No. 4) that the simultaneous problem can be solved without any discussion of any first variation.

3. It is convenient for our purposes to use another one of the many expressions given by Mr. Douglas for his functional $A(g)$. Consider a parametric representation

$$g\colon\; x = \xi(\theta), \qquad y = \eta(\theta), \qquad z = \zeta(\theta), \qquad 0 < \theta \leqq 2\pi$$

of the given Jordan curve Γ and denote by $x(u, v)$, $y(u, v)$, $z(u, v)$ the harmonic functions, defined for $u^2 + v^2 \leqq 1$ by the condition that they reduce on $u^2 + v^2 = 1$ to $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ respectively. Then

$$\begin{aligned} A(g) &= \iint \frac{1}{2}(E + G)\, du\, dv \\ &= \frac{1}{2}\left[\iint (x_u^2 + x_v^2) + \iint (y_u^2 + y_v^2) + \iint (z_u^2 + z_v^2)\right], \end{aligned} \tag{3}$$

where (and also in the sequel) the integrations are to be extended over $u^2 + v^2 < 1$, and

$$E = x_u^2 + y_u^2 + z_u^2, \qquad G = x_v^2 + y_v^2 + z_v^2.$$

That is to say, $A(g)$ is equal to half of the sum of the Dirichlet integrals of the harmonic functions with the boundary values $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$.

It is needless to consider the computations leading from (2) to (3), because in the present paper we only use the form (3), which therefore constitutes, for our purposes, the definition of $A(g)$.

4. We are now going to show that the results of Mr. Douglas summed up in the statements I to IV in No. 1, are immediate consequences of the

theorem that the simultaneous problem (see § 1, No. 2) is solvable for any Jordan curve Γ which bounds some continuous surface with a finite area.

Consider, indeed, such a curve Γ and denote by $x_0(u, v)$, $y_0(u, v)$, $z_0(u, v)$ a solution of the simultaneous problem. Denote by $\xi_0(\theta)$, $\eta_0(\theta)$, $\zeta_0(\theta)$ the boundary values of $x_0(u, v)$, $y_0(u, v)$, $z_0(u, v)$. On account of § 1, No. 2, III the equations $x = \xi_0(\theta)$, $y = \eta_0(\theta)$, $z = \zeta_0(\theta)$ constitute a parametric representation of Γ which we shall denote by g_0:

$$g_0:\quad x = \xi_0(\theta), \quad y = \eta_0(\theta), \quad z = \zeta_0(\theta), \quad 0 < \theta \leqq 2\pi.$$

We denote further by S_0 the surface

$$S_0:\quad x = x_0(u, v), \quad y = y_0(u, v), \quad z = z_0(u, v), \quad u^2 + v^2 \leqq 1.$$

Using (3) and § 1, No. 2, we have

$$(4)\qquad A(g_0) = \iint \frac{1}{2}(E_0 + G_0) = \iint (E_0 G_0 - F_0^2)^{1/2} = \mathfrak{A}(S_0) = \mathfrak{a}(\Gamma),$$

where $\mathfrak{a}(\Gamma)$ denotes the minimum of the areas of all continuous surfaces bounded by Γ. As $\mathfrak{a}(\Gamma)$ is finite by assumption, (4) shows that $A(g_0)$ is finite also, that is to say that Γ admits of a parametric representation with a finite $A(g)$.

Conversely, suppose that Γ admits of a parametric representation

$$g:\quad x = \xi(\theta), \quad y = \eta(\theta), \quad z = \zeta(\theta), \quad 0 < \theta \leqq 2\pi,$$

with a finite $A(g)$. Consider the surface

$$S_g:\quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leqq 1,$$

where $x(u, v)$, $y(u, v)$, $z(u, v)$ are the harmonic functions with the boundary values $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$. We have then

$$\mathfrak{A}(S_g) = \iint (EG - F^2)^{1/2} \leqq \iint E^{1/2} G^{1/2} \leqq \iint \frac{1}{2}(E + G) = A(g).$$

Thus $\mathfrak{A}(S_g)$ is finite, that is to say Γ bounds a continuous surface with a finite area.

The preceding remarks prove statement IV in No. 1. Denote now by

$$g:\quad x = \xi(\theta), \quad y = \eta(\theta), \quad z = \zeta(\theta), \quad 0 < \theta \leqq 2\pi$$

any parametric representation of Γ, by $x(u, v)$, $y(u, v)$, $z(u, v)$ the harmonic functions, defined for $u^2 + v^2 \leqq 1$ by the condition that they reduce to $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ on $u^2 + v^2 = 1$, and denote by S_g the surface

$$S_g:\quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leqq 1.$$

Denote again by

$$S_0:\ x = x_0(u, v), \quad y = y_0(u, v), \quad z = z_0(u, v), \quad u^2 + v^2 \leqq 1$$

a solution of the simultaneous problem, by $\xi_0(\theta)$, $\eta_0(\theta)$, $\zeta_0(\theta)$ the boundary values of $x_0(u, v)$, $y_0(u, v)$, $z_0(u, v)$, and by g_0 the parametric representation of Γ defined by

$$g_0:\ x = \xi_0(\theta), \quad y = \eta_0(\theta), \quad z = \zeta_0(\theta), \quad 0 < \theta \leqq 2\pi.$$

We have then

$$\begin{aligned} A(g_0) &= \iint \frac{1}{2}(E_0 + G_0) = \iint (E_0 G_0 - F_0^2)^{1/2} = \mathfrak{A}(S_0) \leqq \mathfrak{A}(S_g) \\ &= \iint (EG - F^2)^{1/2} \leqq \iint E^{1/2} G^{1/2} \leqq \iint \frac{1}{2}(E + G) = A(g). \end{aligned} \tag{5}$$

From this it follows that $A(g_0) \leqq A(g)$; that is to say, g_0 minimizes the functional $A(g)$, which proves statement I in No. 1.

We come now to the most striking result of Mr. Douglas, namely to his theorem that *any* solution of the variation problem $A(g) =$ minimum gives a solution of the problem of Plateau and also of the simultaneous problem. To prove this, consider any solution g of the variation problem $A(g) =$ minimum. From the minimizing property of g it follows that $A(g) \leqq A(g_0)$, while (5) shows that $A(g_0) \leqq A(g)$. So we have $A(g_0) = A(g)$, and from this we infer that for a minimizing g we must have the sign of equality everywhere in (5). This gives immediately the relations

$$\mathfrak{A}(S_g) = \mathfrak{A}(S_0), \tag{6}$$

$$0 = \iint \left[\frac{1}{2}(E + G) - E^{1/2} G^{1/2}\right] = \frac{1}{2} \iint (E^{1/2} - G^{1/2})^2, \tag{7}$$

$$0 = \iint [E^{1/2} G^{1/2} - (EG - F^2)^{1/2}]. \tag{8}$$

From (7) and (8) it follows that $E = G$, $F = 0$. That is to say, the surface S_g is a minimal surface, and (6) shows that S_g has also a minimum area. This proves statements II and III in No. 1.

5. So we see that the results of Mr. Douglas concerning his functional $A(g)$ are immediate consequences of the existence theorem for the simultaneous problem. In particular, we see that these results can be proved without any discussion of the first variation. As however the first variation of $A(g)$ is certainly interesting in itself, we are going to carry out the discussion of the first variation of $A(g)$ in a way which might throw some more light on the situation. Instead of using the expression (2) as Mr. Douglas did, we shall use the form (3). The result appears then as a very particular

case of a simple and general theorem on the conformal mapping of surfaces. As this theorem might be interesting in itself, we shall state and prove it in § 3 in a much more general form than the application to $A(g)$ would require.

§ 3. A Lemma on Conformal Mapping.

1. Let there be given three functions $x(u, v)$, $y(u, v)$, $z(u, v)$, defined and continuous for $u^2+v^2 \leqq 1$. We suppose that these functions have continuous partial derivatives of the first and second order for $u^2+v^2<1$. We say then that the equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \qquad u^2+v^2 \leqq 1$$

define a continuous surface S of class C_2 (x, y, z being considered as Cartesian coördinates).

Note that we do not suppose that the above equations define a one-to-one correspondence between $u^2+v^2 \leqq 1$ and the points of the surface S; this surface may for instance reduce to a single point.

2. Given a continuous surface of class C_2

$$(9) \qquad S\colon \ x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \qquad u^2+v^2 \leqq 1,$$

we define its *equivalent representations* as follows. Put

$$(10) \qquad u = u(\alpha, \beta), \qquad v = v(\alpha, \beta),$$

where $u(\alpha, \beta)$, $v(\alpha, \beta)$ satisfy the following conditions. The functions $u(\alpha, \beta)$, $v(\alpha, \beta)$ and their first and second partial derivatives are continuous for $\alpha^2+\beta^2<1$ and $u(\alpha, \beta)$, $v(\alpha, \beta)$ remain continuous even on $\alpha^2+\beta^2 = 1$; the determinant

$$\begin{vmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{vmatrix}$$

is different from zero for $\alpha^2+\beta^2<1$. Finally we suppose that the equations (10) define a one-to-one correspondence between $u^2+v^2 \leqq 1$ and $\alpha^2+\beta^2 \leqq 1$.

If we substitute, in (9), for u and v the functions $u(\alpha, \beta)$, $v(\alpha, \beta)$, we obtain new equations

$$(11) \quad x = \bar{x}(\alpha, \beta), \qquad y = \bar{y}(\alpha, \beta), \qquad z = \bar{z}(\alpha, \beta), \qquad \alpha^2+\beta^2 \leqq 1,$$

where $\bar{x}(\alpha, \beta) = x(u(\alpha, \beta), v(\alpha, \beta))$ and so on. We say that (11) is an *equivalent representation* of the surface (9).

It follows from these definitions that the relation between equivalent representations is a reciprocal one. Note that the region in which the coördinate functions are considered is all the time the unit circle.

3. Let (9) be any of the equivalent representations of S. We put as usual

$$E = x_u^2 + y_u^2 + z_u^2, \quad F = x_u x_v + y_u y_v + z_u z_v, \quad G = x_v^2 + y_v^2 + z_v^2.$$

These first fundamental quantities E, F, G have then continuous first derivatives for $u^2 + v^2 < 1$; nothing is supposed as to their behavior for $u^2 + v^2 \to 1$. Consider now the integrals

$$\iint (EG - F^2)^{1/2} \quad \text{and} \quad \iint \frac{1}{2}(E+G). \tag{12}$$

The first integral is the area $\mathfrak{A}(S)$ of S; we shall only use the fact that this integral has the same value for all the equivalent representations of S. The value of the second integral (12) depends upon the choïce of the equivalent representation. This integral is a functional, the argument of which is a variable equivalent representation of S.

4. An equivalent representation satisfying the relations

$$E = G, \quad F = 0 \quad \text{for} \quad u^2 + v^2 < 1$$

will be called a *conformal representation* of S. For a conformal representation we have obviously

$$\mathfrak{A}(S) = \iint (EG - F^2)^{1/2} = \iint E = \iint G = \iint \frac{1}{2}(E+G).$$

5. Our object in this paragraph is to prove the following

THEOREM. *Suppose that the given continuous surface S of class C_2 admits of an equivalent representation for which*

$$\iint \frac{1}{2}(E+G) < +\infty.$$

Then an equivalent representation

$$S\colon\ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leqq 1$$

is a conformal representation if and only if it minimizes the functional

$$\iint \frac{1}{2}(E+G). \tag{13}$$

The second half of the theorem is trivial. Indeed, we have for any equivalent representation the obvious relations

$$\mathfrak{A}(S) = \iint (EG - F^2)^{1/2} \leqq \iint E^{1/2} G^{1/2} \leqq \iint \frac{1}{2}(E+G). \tag{14}$$

We have therefore for any equivalent representation

$$\mathfrak{A}(S) \leqq \iint \frac{1}{2}(E+G),$$

while for a conformal representation we have (see No. 4)

$$\mathfrak{A}(S) = \iint \frac{1}{2}(E+G).$$

So we see that any conformal representation minimizes the functional (13).

The first half of the theorem would also be trivial if we would know that S admits of a conformal representation

$$\begin{aligned} &S\colon\quad x = x_0(u,v), \quad y = y_0(u,v), \quad z = z_0(u,v), \quad u^2+v^2 \leqq 1, \\ (15)\qquad &\qquad\qquad E_0 = G_0, \quad F_0 = 0. \end{aligned}$$

The preceding reasoning would then first give the relation

$$\min \iint \frac{1}{2}(E+G) = \iint \frac{1}{2}(E_0+G_0) = \mathfrak{A}(S).$$

Consequently, for a representation minimizing the functional (13) we must have the sign of equality everywhere in (14), which gives immediately $E = G$, $F = 0$.

As a matter of fact, we only used a consequence of the existence of a conformal representation, namely the relation

$$\min \iint \frac{1}{2}(E+G) = \mathfrak{A}(S).$$

Our theorem would therefore be trivial if we would know that this relation holds true.

It might be observed that a surface of class C_2, as defined in No. 1, does *not* admit generally of a conformal representation, the main reason being that we did not suppose that $EG - F^2 \neq 0$. It would certainly be interesting however, to investigate the possibility of basing an existence proof, for some conveniently restricted class of surfaces, on the functional (13).

6. We are now going to prove the first (not trivial) half of the theorem stated in No. 5.[13] Let

$$(16)\quad S\colon\quad x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad u^2+v^2 \leqq 1,$$

[13] The following developments are based on the so-called *method of the variation of the independent variables.* For a similar application of this method see the author's paper Bemerkung über die Differentialgleichungen zweidimensionaler Variationsprobleme, Acta Szeged, vol. 2 (1925), pp. 147–156, where the method is used to study the analytic character of minimal surfaces.

be an equivalent representation minimizing the functional (13). This minimum being finite by assumption, it follows, on account of the inequalities $F^2 \leqq EG$ and $|F| \leqq E^{1/2} G^{1/2} \leqq \frac{1}{2}(E+G)$, that the three integrals

$$\iint E, \quad \iint |F|, \quad \iint G \tag{17}$$

exist.

We introduce new parameters α, β according to the restrictions of No. 2. We may write

$$u = u(\alpha, \beta) \qquad v = v(\alpha, \beta),$$

or

$$\alpha = \alpha(u, v), \qquad \beta = \beta(u, v),$$

and by differentiating we obtain the relations

$$u_\alpha = \frac{\beta_v}{\Delta}, \qquad v_\alpha = -\frac{\beta_u}{\Delta}, \qquad u_\beta = -\frac{\alpha_v}{\Delta}, \qquad v_\beta = \frac{\alpha_u}{\Delta}, \tag{18}$$

where $\Delta = \alpha_u \beta_v - \alpha_v \beta_u$. We restrict ourselves to new parameters with

$$\Delta = \alpha_u \beta_v - \alpha_v \beta_u > 0. \tag{19}$$

Substituting into (16), we obtain an equivalent representation

$$\begin{cases} S: \; x = \bar{x}(\alpha, \beta), \quad y = \bar{y}(\alpha, \beta), \quad z = \bar{z}(\alpha, \beta), \quad \alpha^2 + \beta^2 \leqq 1, \\ \bar{E} = \bar{x}_\alpha^2 + \bar{y}_\alpha^2 + \bar{z}_\alpha^2, \quad \bar{F} = \bar{x}_\alpha \bar{x}_\beta + \bar{y}_\alpha \bar{y}_\beta + \bar{z}_\alpha \bar{z}_\beta, \quad \bar{G} = \bar{x}_\beta^2 + \bar{y}_\beta^2 + \bar{z}_\beta^2. \end{cases} \tag{20}$$

From the relations $x_\alpha = x_u u_\alpha + x_v v_\alpha$ and so on, we get

$$\bar{E} = E u_\alpha^2 + 2 F u_\alpha v_\alpha + G v_\alpha^2, \qquad \bar{G} = E u_\beta^2 + 2 F u_\beta v_\beta + G v_\beta^2.$$

The functional (13), for the representation (20), is therefore given by

$$\iint \frac{1}{2}(\bar{E} + \bar{G}) = \iint \frac{1}{2}[E(u_\alpha^2 + u_\beta^2) + 2F(u_\alpha v_\alpha + u_\beta v_\beta) + G(v_\alpha^2 + v_\beta^2)]\, d\alpha\, d\beta.$$

In the right-hand integral, we change back to the variables u, v. Using (18) and (19), we obtain

$$\iint \frac{1}{2}(\bar{E} + \bar{G}) = \iint \frac{1}{2\Delta}[E(\alpha_v^2 + \beta_v^2) - 2F(\alpha_u \alpha_v + \beta_u \beta_v) + G(\alpha_u^2 + \beta_u^2)]\, du\, dv,$$

and so the extremal property of the representation (16) is expressed by the inequality

$$\begin{aligned} &\iint \frac{1}{2}(E + G)\, du\, dv \\ &\leqq \iint \frac{1}{2\Delta}[E(\alpha_v^2 + \beta_v^2) - 2F(\alpha_u \alpha_v + \beta_u \beta_v) + G(\alpha_u^2 + \beta_u^2)]\, du\, dv, \end{aligned} \tag{21}$$

holding true for any couple of functions $\alpha(u, v)$, $\beta(u, v)$ for which the inverse of the transformation $\alpha = \alpha(u, v)$, $\beta = \beta(u, v)$ satisfies the conditions required in No. 2.

7. Let now $\varphi(u, v)$ be any function with continuous first and second derivatives for $u^2 + v^2 \leqq 1$, and denote by ε a small parameter. Put

$$\begin{aligned} \alpha(u, v) &= u \cos \varepsilon \varphi(u, v) - v \sin \varepsilon \varphi(u, v), \\ \beta(u, v) &= u \sin \varepsilon \varphi(u, v) + v \cos \varepsilon \varphi(u, v). \end{aligned} \tag{22}$$

We have then

$$\Delta = 1 + \varepsilon(u \varphi_v - v \varphi_u), \tag{23}$$

and as φ_u, φ_v are bounded by assumption, Δ will be different from zero for small values of ε. From (22) it follows that $\alpha^2 + \beta^2 = u^2 + v^2$, so the transformation $\alpha = \alpha(u, v)$, $\beta = \beta(u, v)$ carries concentric circles into concentric circles, and in a one-to-one way on account of $\Delta \neq 0$. So the equations $\alpha = \alpha(u, v)$, $\beta = \beta(u, v)$ define a one-to-one and continuous transformation of the unit circle into itself. The first and second derivatives of $\alpha(u, v)$, $\beta(u, v)$ are obviously continuous in $u^2 + v^2 \leqq 1$.

Substituting (22) into (21), the right-hand side of (21) becomes a function $Y(\varepsilon)$ of ε which has a minimum for $\varepsilon = 0$. So $Y'(\varepsilon)$ vanishes for $\varepsilon = 0$. In order to compute $Y'(\varepsilon)$, we differentiate under the integral sign,[14] remembering that ε occurs only in Δ, α_u, α_v, β_u, β_v, but not in E, F, G. The differential coefficients

$$\frac{d\Delta}{d\varepsilon}, \quad \frac{d\alpha_u}{d\varepsilon}, \quad \ldots, \quad \frac{d\beta_v}{d\varepsilon}$$

are obtained from (22) and (23). The easy details of this computation may be left to the reader; the result is

$$Y'(0) = \frac{1}{2} \iint \{[v(E-G) - 2uF]\varphi_u + [u(E-G) + 2vF]\varphi_v\} = 0, \tag{24}$$

this relation holding true for all functions $\varphi(u, v)$ with continuous first and second derivatives in $u^2 + v^2 \leqq 1$.

8. Let now $\psi(u, v)$ be any function with continuous first and second derivatives for $u^2 + v^2 \leqq 1$ and vanishing for $u^2 + v^2 = 1$. Put

$$\alpha(u, v) = u + \varepsilon \psi(u, v), \qquad \beta(u, v) = v. \tag{25}$$

On account of the boundedness of the derivatives of $\psi(u, v)$, $\alpha_u = 1 + \varepsilon \psi_u$ is positive for small values of ε. From this and from the vanishing of

[14] This may be justified immediately on account of the existence of the integrals (17) and of the boundedness of φ_u, φ_v.

$\psi(u, v)$ on $u^2+v^2=1$ it follows that any secant $v=$ constant of the unit circle is carried in a one-to-one and continuous way into itself. So the whole unit circle is carried into itself in a one-to-one and continuous way. This point being established, the same reasoning as in No. 7 leads to the equation

$$\iint [(E-G)\psi_u + 2F\psi_v] = 0, \tag{26}$$

holding for any function $\psi(u, v)$ with continuous first and second derivatives in $u^2+v^2 \leqq 1$ and vanishing for $u^2+v^2=1$.

9. The relation (26) has the usual form of the vanishing of the first variation in case of double integrals and so it gives the equation

$$\frac{\partial}{\partial u}(E-G) + \frac{\partial}{\partial v}(2F) = 0 \quad \text{for} \quad u^2+v^2<1; \tag{27}$$

as the familiar devices to pass from (26) to (27) only use functions $\psi(u, v)$ which vanish on *and* near to $u^2+v^2=1$, it does not make any difference that we do not know anything about E, F, G for $u^2+v^2 \rightarrow 1$.

Let us now consider the equation (24). This equation also holds for functions $\varphi(u, v)$ which do *not* vanish on $u^2+v^2=1$. We use it, however, for the moment only for functions $\varphi(u, v)$ which do vanish on $u^2+v^2=1$ and we obtain by the same reasons as in the case of (26), the equation

$$\frac{\partial}{\partial u}[v(E-G)-2uF] + \frac{\partial}{\partial v}[u(E-G)+2vF] = 0 \text{ for } u^2+v^2<1. \tag{28}$$

10. From (28) it follows the existence of a single-valued function $\Omega(u, v)$ for which

$$\Omega_v = v[E-G]-2uF, \qquad \Omega_u = -u(E-G)-2vF \text{ for } u^2+v^2<1. \tag{29}$$

From (29) it follows by differentiation that

$$\begin{aligned} \Omega_{uu}+\Omega_{vv} &= v\left[\frac{\partial}{\partial v}(E-G) - \frac{\partial}{\partial u}(2F)\right] \\ &\quad - u\left[\frac{\partial}{\partial u}(E-G) + \frac{\partial}{\partial v}(2F)\right]. \end{aligned} \tag{30}$$

Simplifying (28) we get

$$v\left[\frac{\partial}{\partial u}(E-G) + \frac{\partial}{\partial v}(2F)\right] + u\left[\frac{\partial}{\partial v}(E-G) - \frac{\partial}{\partial u}(2F)\right] = 0. \tag{31}$$

From (27), (31), and (30) it follows that

$$\Omega_{uu}+\Omega_{vv} = 0 \quad \text{for} \quad u^2+v^2<1,$$

that is to say $\Omega(u, v)$ is a harmonic function. Hence $\Omega(u, v)$ is the real part of a power series of $w = u + iv$, convergent in the unit circle, which gives for $\Omega(u, v)$ an expansion of the form

$$\Omega(u, v) = a_0 + \sum_{n=1}^{\infty} \varrho^n (a_n \cos n\theta + b_n \sin n\theta), \quad u = \varrho\cos\theta, \quad v = \varrho\sin\theta, \tag{32}$$

uniformly convergent in any smaller concentric circle (the coefficients a, b being real constants).

11. We use now the fact that (24) holds also for functions $\varphi(u, v)$ not vanishing on $u^2 + v^2 = 1$. We can choose therefore

$$\varphi(u, v) = \Re(u + iv)^n = \varrho^n \cos n\theta, \quad u = \varrho\cos\theta, \quad v = \varrho\sin\theta, \quad n \geqq 1. \tag{33}$$

(24) may be written, on account of (29),

$$\iint (\Omega_v \varphi_u - \Omega_u \varphi_v)\, du\, dv = 0. \tag{34}$$

Let R be a positive number less than 1; we obtain by partial integration

$$\begin{aligned} \iint_{u^2+v^2<R^2} (\Omega_v \varphi_u - \Omega_u \varphi_v)\, du\, dv &= -\int_{u^2+v^2=R^2} \Omega\, d\varphi \\ &= -\int_{u^2+v^2=R^2} \Omega \frac{\partial \varphi}{\partial \theta} d\theta. \end{aligned} \tag{35}$$

On account of (34) we have

$$\iint_{u^2+v^2<R^2} (\Omega_v \varphi_u - \Omega_u \varphi_v)\, du\, dv \to 0 \text{ for } R \to 1,$$

and so it follows from (35), for $R \to 1$, that

$$\int_{u^2+v^2=R^2} \Omega \frac{\partial \varphi}{\partial \theta} d\theta \to 0 \text{ for } R \to 1.$$

Substituting (32) and (33), this relation gives $a_n = 0$. Using $\varphi(u, v) = \Im(u + iv)^n = \varrho^n \sin n\theta$, we obtain in the same way $b_n = 0$, all this for $n \geqq 1$. Thus $\Omega(u, v)$ reduces to a constant, so that $\Omega_u \equiv 0$, $\Omega_v \equiv 0$. On account of (29), this gives the equations

$$\left.\begin{aligned} v(E - G) - 2uF &= 0, \\ u(E - G) + 2vF &= 0 \end{aligned}\right\} \text{ for } u^2 + v^2 < 1.$$

From these equations it follows immediately that

$$E - G = 0, \qquad F = 0 \tag{36}$$

for $u^2+v^2 \neq 0$; as however E, F, G are continuous, (36) holds also true for $u=v=0$. So we have $E=G$, $F=0$ in the whole unit circle, which proves our theorem that the minimizing representation (16) is conformal.

12. In the application to the functional $A(g)$ of Mr. Douglas, which we shall consider in the next paragraph, the coördinate functions $x(u, v)$, $y(u, v)$, $z(u, v)$, figuring in the minimizing representation (16), will be *harmonic* functions. This particular situation allows some simplifications which might be sketched here briefly. If $x(u, v)$, $y(u, v)$, $z(u, v)$ are harmonic functions, we can write

$$(37) \quad x(u, v) = \Re f_1(w), \quad y(u, v) = \Re f_2(w), \quad z(u, v) = \Re f_3(w),$$

where f_1, f_2, f_3 are single-valued analytic functions of $w = u+iv$ for $|w|<1$. The Cauchy-Riemann formulas give then

$$(38) \qquad x_u - ix_v = f_1', \quad y_u - iy_v = f_2', \quad z_u - iz_v = f_3'.$$

Consider the analytic function

$$(39) \qquad \Phi(w) = -\int_0^w w(f_1'^2+f_2'^2+f_3'^2)\,dw, \quad |w|<1,$$

and write

$$\Phi(w) = \Omega(u, v)+i\Lambda(u, v).$$

We have then, by (39) and (38),

$$\begin{aligned}\Omega_u+i\Omega_v = -\Phi'(w) &= w(f_1'^2+f_2'^2+f_3'^2) = (u+iv)[(E-G)-2iF] \\ &= u(E-G)+2vF+i[v(E-G)-2uF].\end{aligned}$$

Hence the harmonic function $\Omega(u, v)$ satisfies the relations

$$\Omega_v = v(E-G)-2uF, \qquad \Omega_u = -u(E-G)-2vF.$$

That is to say, in this particular case, the equations (29) may be obtained without any reference to the extremal property of the representation. The equations (29) being established, we show (24) as previously, using the extremal property, and then we can finish the proof by the argument of No. 11.

§ 4. Application to the Functional of Mr. Douglas.

1. Let there be given a Jordan curve Γ in the xyz-space and suppose that Γ admits of some parametric representation g with a finite $A(g)$. To solve the problem of Plateau in this case, Mr. Douglas used the form (2)

of his functional $A(g)$; we are going to show briefly how the demonstration looks if we use the form (3) and the theorem of No. 5, § 3.

The first step is to establish that

there exists a (possibly improper, but not degenerate) *representation*

$$(40)\quad g:\quad x = \xi(\theta), \qquad y = \eta(\theta), \qquad z = \zeta(\theta), \qquad 0 < \theta \leqq 2\pi$$

of Γ minimizing the functional $A(g)$.

The reader may look up the definitions of the terms *improper* and *degenerate* in the paper of Mr. Douglas and verify that the assertion that g is possibly improper but not degenerate implies the following properties.

a) The functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$, figuring in (40), are bounded and their points of discontinuity constitute a denumerable set.

b) The possible discontinuities of these functions are of the following simple type: for any θ, the functions have definite limits from either side, and if these limits are equal, θ is a point of continuity.

c) If the equations (40) take two distinct points P, Q of the unit circle $u = \cos\theta$, $v = \sin\theta$ into the same point of the Jordan curve Γ, then on one of the two arcs PQ of the unit circle the functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ all three reduce to constants.

d) The functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ do not all three reduce to constants identically.

It might be observed that for a general, possibly improper, representation

$$g:\quad x = \xi(\theta), \qquad y = \eta(\theta), \qquad z = \zeta(\theta)$$

the functional $A(g)$ is to be defined, if we use the form (3), as follows. As $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are possibly discontinuous, we cannot talk very well of the harmonic functions with these boundary values. Instead, of course, we consider the three harmonic functions, defined for $u^2+v^2<1$ by the Poisson integral formula, using $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ as boundary functions. Denote these harmonic functions by $x(u, v)$, $y(u, v)$, $z(u, v)$; then, by definition

$$A(g) = \iint \frac{1}{2}(E+G),$$

where

$$E = x_u^2 + y_u^2 + z_u^2, \qquad G = x_v^2 + y_v^2 + z_v^2.$$

The proof for the existence of a representation minimizing the functional $A(g)$ is the same for the form (3) as for the form (2) used by Mr. Douglas, and so the reader is requested to look up the easy details in his paper.

2. Consider now a minimizing representation (40) of Γ. As

$$A(g) = \iint \frac{1}{2}(E+G) = \frac{1}{2}\left[\iint (x_u^2 + x_v^2) + \iint (y_u^2 + y_v^2) + \iint (z_u^2 + z_v^2)\right]$$

is finite by assumption, it follows that the Dirichlet integrals of the three harmonic functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are finite. From this it follows that the functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are *continuous*. Indeed, if for instance $\xi(\theta)$ would have a discontinuity, this would be of the type specified in No. 1, b), and then, by virtue of a lemma of Courant,[15] the harmonic function $x(u, v)$ could not have a finite Dirichlet integral.

3. The boundary functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ being continuous, $x(u, v)$, $y(u, v)$, $z(u, v)$ remain continuous for $u^2 + v^2 \to 1$, and reduce there to $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ respectively, on account of a well known property of the Poisson integral formula. So we have

$$(41) \quad x(u, v) = \xi(\theta), \quad y(u, v) = \eta(\theta), \quad z(u, v) = \zeta(\theta) \text{ for } u^2 + v^2 = 1.$$

Consider now the surface

$$(42) \quad S\colon\ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leqq 1.$$

This is a continuous surface of class C_2 according to the definition given in Nr. 1, § 3. Consider any of the equivalent representations of S:

$$x = \bar{x}(u, v), \quad y = \bar{y}(u, v), \quad z = \bar{z}(u, v), \quad u^2 + v^2 \leqq 1,$$
$$\bar{E} = \bar{x}_u^2 + \bar{y}_u^2 + \bar{z}_u^2, \quad \bar{G} = \bar{x}_v^2 + \bar{y}_v^2 + \bar{z}_v^2.$$

We assert that

$$(43) \qquad \iint \frac{1}{2}(E+G) \leqq \iint \frac{1}{2}(\bar{E}+\bar{G}).$$

Indeed, if $\bar{x}(u, v)$, $\bar{y}(u, v)$, $\bar{z}(u, v)$ happen to be harmonic functions, then (43) expresses simply the extremal property of the representation (40). If $\bar{x}(u, v)$, $\bar{y}(u, v)$, $\bar{z}(u, v)$ are not harmonic, denote by $\tilde{x}(u, v)$, $\tilde{y}(u, v)$, $\tilde{z}(u, v)$ the harmonic functions coinciding with $\bar{x}(u, v)$, $\bar{y}(u, v)$, $\bar{z}(u, v)$ on $u^2 + v^2 = 1$. Then we have, on account of the extremal property of the representation (40),

$$(44) \qquad \iint \frac{1}{2}(E+G) \leqq \iint \frac{1}{2}(\tilde{E}+\tilde{G}).$$

Since a harmonic function with given boundary values minimizes the Dirichlet integral, we have obviously

$$(45) \qquad \iint \frac{1}{2}(\tilde{E}+\tilde{G}) \leqq \iint \frac{1}{2}(\bar{E}+\bar{G}).$$

From (44) and (45) it follows that (43) holds generally.

[15] See for instance loc. cit. [7], Ch. 1, § 3.

The inequality (43) expresses that the representation (42) of S minimizes the integral

$$\iint \frac{1}{2}(E+G)$$

with respect to all the equivalent representations of S. So the theorem of § 3, No. 5 gives that the representation (42) of S satisfies

$$(46) \qquad E = G, \qquad F = 0 \quad \text{for} \quad u^2+v^2<1.$$

4. We are now going to show that the equations (40) carry distinct points of the unit circle $u = \cos\theta$, $v = \sin\theta$ into distinct points of the given Jordan curve Γ. Suppose this be not true. Then, on account of No. 1, c), the functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ all three reduce to constants on a certain arc σ of the unit circle. From (41) it follows then that

$$(47) \quad x(u, v) = \text{const.}, \quad y(u, v) = \text{const.}, \quad z(u, v) = \text{const.}, \quad \text{on} \quad \sigma.$$

From this situation it follows that $x(u, v)$, $y(u, v)$, $z(u, v)$ remain analytic on σ, and that consequently (46) holds also on σ:

$$(48) \qquad E = G, \quad F = 0 \quad \text{on} \quad \sigma.$$

We obtain from (47):

$$x_\theta = 0, \quad y_\theta = 0, \quad z_\theta = 0 \quad \text{on} \quad \sigma,$$

and consequently

$$(49) \qquad x_\theta^2 + y_\theta^2 + z_\theta^2 = 0 \quad \text{on} \quad \sigma.$$

Now $x_\theta = -v x_u + u x_v$ and so on, and it follows from (49), (46) and (48) that

$$x_\theta^2 + y_\theta^2 + z_\theta^2 = E = G = 0 \quad \text{on} \quad \sigma.$$

This shows that

$$x_u = x_v = y_u = y_v = z_u = z_v = 0 \quad \text{on} \quad \sigma,$$

and that consequently the harmonic functions $x(u, v)$, $y(u, v)$, $z(u, v)$ reduce to constants identically. In connection with (41) this conclusion contradicts No. 1, d) however.

5. Summing up, we have proved that the functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$, figuring in the minimizing representation (40), are continuous (see No. 2), and that the equations $x = \xi(\theta)$, $y = \eta(\theta)$, $z = \zeta(\theta)$ carry the unit circle $u = \cos\theta$, $v = \sin\theta$ in a one-to-one way into the given Jordan curve Γ (see No. 4). We have also proved that the harmonic functions $x(u, v)$, $y(u, v)$, $z(u, v)$ with the boundary values $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ satisfy

the relations $E = G$, $F = 0$ for $u^2 + v^2 < 1$ (see No. 3). Hence the surface

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \quad u^2 + v^2 \leqq 1$$

is a minimal surface bounded by the given Jordan curve Γ.[16]

There remains to investigate this minimal surface with respect to *the least area property*. The developments of Mr. Douglas on this subject are, in his own opinion, out of line with the general plan of his work.[17] Indeed, Mr. Douglas considers conformal mapping as an application of his functional and by no means as a necessary tool to study this functional. To show the least area property of the minimal surface obtained by means of his functional, Mr. Douglas uses however the classical theorem on the conformal mapping of polyhedrons which constituted the main argument of my own solution of the problem of Plateau and of the simultaneous problem. I cannot help feeling that the elaborate methods of Mr. Douglas to investigate his functional $A(g)$, if used in connection with the basic arguments of an independent solution of the simultaneous problem, constitute an unnecessary detour; mainly because, as we have seen (§ 2, No. 4), the results of Mr. Douglas concerning $A(g)$ are immediate consequences of the existence of the solution of the simultaneous problem. It would not pay to discuss this point any further, because Mr. Douglas plans to eliminate the use of this polyhedron-theorem anyway.

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[16] The given Jordan curve is supposed to admit of some representation g with a finite $A(g)$. It may be observed that as far as we only consider the problem of Plateau, the immediate generality of the method is not very important. Indeed, as soon as the problem is solved for polygons for instance, the solution for a general Jordan curve may be obtained by a passage to the limit. This theorem has been proved by Mr. Douglas (loc. cit. [1]); particular cases have been proved previously by Garnier (Jordan curves with bounded curvature) and by the author (rectifiable Jordan curves). See R. Garnier, Sur le problème de Plateau, Annales sc. de l'École normale supérieure, vol. 45 (1928) and T. Radó, Some remarks on the problem of Plateau, Proceedings of the National Academy of Sciences, vol. 16 (1930).

[17] See loc. cit. [1], p. 265.

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SEMI-LINEAR INTEGRAL EQUATIONS.[1]

By Cletus O. Oakley.

1. Introduction. By a semi-linear integral equation we shall mean an equation of the form

$$\varrho p(x)\, y(x)+\sigma q(x)\,|y(x)| = f(x)+\lambda\int_0^1 K(x,\xi)\, y(\xi)\, d\xi+\mu\int_0^1 L(x,\xi)\,|y(\xi)|\, d\xi \tag{1}$$

which we shall write as the generic representation of the equations treated in this paper throughout which ϱ, σ, λ, μ are taken to be complex-valued parameters; p, q, f real and continuous functions of the real variable x in the unit interval I: $0 \leqq x \leqq 1$; and K (linear kernel) and L (semi-linear kernel) bounded functions integrable in the sense of Riemann over the unit square S:[2] $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$. Equation (1) we shall refer to as Type (1111), the unit strokes indicating the presence of a y-term, an $|y|$-term, a linear kernel-term and a semi-linear kernel-term, respectively. If any one of these terms is missing, we shall denote the fact by writing "0" in the proper place; thus it is clear that the notation Type (1010) would represent the linear equation of the third kind. Further we shall use, wherever convenient, the symbolic dot-product notation for integration, thus: $\lambda\int_0^1 K(x,\xi)\, y(\xi)\, d\xi = \lambda K\cdot y$.[3]

Questions of the existence and nature of solutions fall naturally into two classes of theorems, those of the one class relating to the regions inside and those of the other to the regions outside certain neighborhoods of the origin in the planes of the various parameters. It will be seen (§ 2) that there exists, within sufficiently small circles (which might be termed Liouville-Neumann series circles), a rather close analogy between the linear equation theory and the semi-linear equation theory. For the regions outside these circles, however, no theory comparable with that of Fredholm for linear equations is developed—in fact, certain theorems which we give (§ 3) suggest that perhaps no such general theory exists.

[1] Received December 1, 1930.

[2] Were continuous kernels K, L considered, we might integrate the one term between one set of limits and the other term between another set as, for example, $\int_\alpha^\beta K(x,\xi)\, y(\xi)\, d\xi$ and $\int_\gamma^\delta L(x,\xi)\,|y(\xi)|\, d\xi$. With integrable kernels, however, the unit interval is no less general and leads to simplification and elegance in formulae.

[3] Products without dots shall represent ordinary multiplication.

Although these so-called theorems of restriction are in a fragmentary state, yet it is in these theorems that the principle interest of this note lies.

2. **Existence theorems for small values of parameters.** Fredholm Type (1111). Case I: $\varrho \neq 0$, $p(x) \neq 0$. We seek a continuous function $y(x)$ which shall satisfy

$$y+\sigma q\,|y| = f+\lambda K\cdot y+\mu L\cdot|y|, \tag{2}$$

and also conditions under which such a solution shall exist. The method, which is that of successive approximations, will be briefly sketched. Set $y_0 = f$; define y_n by the relation

$$y_n = -\sigma q\,|y_{n-1}|+f+\lambda K\cdot y_{n-1}+\mu L\cdot|y_{n-1}|;$$

and set $u_n = y_n - y_{n-1}$, $u_0 = y_0$. We get

$$u_n = -\sigma q\,(|y_{n-1}|-|y_{n-2}|)+\lambda K\cdot(y_{n-1}-y_{n-2})+\mu L\cdot(|y_{n-1}|-|y_{n-2}|).$$

By applying successively the inequalities: $|a+b| \leqq |a|+|b|$, $|a-b| \geqq \big||a|-|b|\big|$, $\left|\int\right| \leqq \int|\ \ |$ to

$$|u_n| = \Big|-\sigma q(|y_{n-1}|-|y_{n-2}|)+\lambda K\cdot(y_{n-1}-y_{n-2})+\mu L\cdot(|y_{n-1}|-|y_{n-2}|)\Big|,$$

we obtain

$$|u_n| \leqq |\sigma|\,|q|\,\big||y_{n-1}|-|y_{n-2}|\big|+|\lambda|\,|K\cdot(y_{n-1}-y_{n-2})|+|\mu|\,\big|L\cdot(|y_{n-1}|-|y_{n-2}|)\big|,$$
$$\leqq |\sigma|\,|q|\,|y_{n-1}-y_{n-2}|+|\lambda|\,|K|\cdot|y_{n-1}-y_{n-2}|+|\mu|\,|L|\cdot|y_{n-1}-y_{n-2}|;$$

and from this we get the recurrence relation

$$|u_n| \leqq |\sigma|\,|q|\,|u_{n-1}|+|\lambda|\,|K|\cdot|u_{n-1}|+|\mu|\,|L|\cdot|u_{n-1}|.$$

Letting $|q| \leqq q_0$, $|f| \leqq f_0 \neq 0$, $|K| \leqq K_0$, $|L| \leqq L_0$, we write

$$|u_n| \leqq f_0\,[|\sigma|\,q_0+|\lambda|\,K_0+|\mu|\,L_0]^n.$$

Hence if $(|\sigma|\,q_0+|\lambda|\,K_0+|\mu|\,L_0)<1$, y exists since $y(x) = \lim_{n=\infty} y_n(x) = \sum_{n=0}^{\infty} u_n(x)$ and, moreover, satisfies (2) as is readily seen. Finally, by methods which are standard in the treatment of the linear case, y is shown to be unique.

Case II: $\sigma \neq 0$, $q(x) \neq 0$. The equation is

$$\varrho p y+|y| = f+\lambda K\cdot y+\mu L\cdot|y|; \tag{3}$$

and the above method yields the existence of a non-negative, continuous solution under the rather stringent conditions: ϱ, λ, μ real, non-negative

and sufficiently small; $p \leqq 0$; $K, L \geqq 0$. This solution $y(x)$ is the same no matter what non-negative first approximating function y_0 is chosen. Further this solution may be so modified as to yield discontinuous solutions $Y(x)$ as follows: $Y(x) = y(x)$ except at zeros x_i of $p(x)$ at which $Y(x_i) = \pm y(x_i)$.

Type (0111) with $\sigma \neq 0$, $q(x) \neq 0$ is similarly treated. Type (0101) with $\sigma \neq 0$, $q(x) \neq 0$ is of interest. In this case if $y(x)$ is a non-negative solution then $Y(x) = S^{-1} y(x)$, where $S = |\ \ |$ is the operation of absolute-value taking and S^{-1} the inverse, is also a solution.

Volterra Type (1111). Case I: $\varrho \neq 0$, $p(x) \neq 0$. The integration is now from 0 to x. By successive approximations we get

$$|u_n| \leqq f_0\left[|\sigma|^n q_0^n + \binom{n}{1}|\sigma|^{n-1} q_0^{n-1} Mx/1! + \binom{n}{2}|\sigma|^{n-2} q_0^{n-2} M^2 x^2/2! + \cdots + M^n x^n/n!\right],$$

where $M = |\lambda| K_0 + |\mu| L_0$. It readily follows that there exists a continuous, unique solution for all f_0 and x provided the inequality $(|\sigma| q_0 + M) < 1$ holds (since $|u_n| \leqq (|\sigma| q_0 + M)^n e^x$); or for all f_0 and M provided $(|\sigma| q_0 + x) < 1$; etc. If $|\lambda| = |\mu|$, and we write $N = K_0 + L_0$, then a unique solution exists for all f_0, λ, x provided $(|\sigma| q_0 + N) < 1$; or for all f_0, N, x provided $(|\sigma| q_0 + |\lambda|) < 1$. The discussion of Case II where $\sigma \neq 0$, $q(x) \neq 0$ follows from that of Fredholm Type (1111) Case II.

Volterra Type (0011). The equation, which might be called a semi-linear equation of the first kind, is

$$f = \lambda K \cdot y + \mu L \cdot |y|. \tag{4}$$

We make $f(0) = 0$; if, further we assume conditions for the differentiability with regard to x of equation (4),[4] we may write

$$f'(x) = \lambda K'_x \cdot y + K(x, x) y(x) + \mu L'_x \cdot |y| + L(x, x)|y(x)|$$

which is a Volterra Type (1111).[5]

It is obvious that mixed Fredholm and Volterra equations, say of the form

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[4] It might be pointed out that an integration by parts of (4) leads to

$$f = K(x, x)\varphi(x) - K'_\xi \cdot \varphi(\xi) + L(x, x)\psi(x) - L'_\xi \cdot \psi(\xi)$$

where $\varphi(x) = \int_0^x y(\xi)\, d\xi$, $\psi(x) = \int_0^x |y(\xi)|\, d\xi$. Hence, coupled with the integral equation is the semi-linear differential equation $|\varphi'| = \psi'$.

[5] An example of Type (0001) might be mentioned in passing. Let $\vartheta = 2n - 1$ and $\eta = (-1)^n$ in the interval $[(n-1)\pi, n\pi]$. Then a solution of $\vartheta + \eta \cos x = \int_0^x |y(\xi)|\, d\xi$ is $y(x) = S^{-1} \sin x$.

$$\varrho p y + \sigma q |y| = f + \lambda \int_0^1 K(x, \xi)\, y(\xi)\, d\xi + \mu \int_0^x L(x, \xi)\, |y(\xi)|\, d\xi,$$

might be considered.

3. **Certain theorems of restriction.** We begin the discussion of the existence of the solutions outside small circles by considering the following special example of Type (1001), namely

$$y = f + \mu \int_0^1 g(x)\, h(\xi)\, |y(\xi)|\, d\xi. \tag{5}$$

If a solution is to exist at all it must be of the form $y = f + \mu g b$ where b is a constant:

$$b = \int_0^1 h(\xi)\, |y(\xi)|\, d\xi = \int_0^1 h(\xi)\, |f(\xi) + \mu g(\xi)\, b|\, d\xi.$$

Let us suppose that $h(\xi)$ does not change sign throughout the interval; then

$$|b| = \int_0^1 |h(\xi) f(\xi) + \mu g(\xi)\, h(\xi)\, b|\, d\xi,$$

from which

$$|b| \geqq \left| \int_0^1 [h(\xi) f(\xi) + \mu g(\xi)\, h(\xi)\, b]\, d\xi \right|.$$

If, now, $h(\xi)$ and $f(\xi)$ are non-absolutely orthogonal functions $\left(\text{i. e. } \int_0^1 h(\xi) f(\xi)\, d\xi = 0 \text{ but } \int_0^1 |h(\xi) f(\xi)|\, d\xi \neq 0\right)$ we get

$$|b| \geqq |\mu|\, |b| \left| \int_0^1 g(\xi)\, h(\xi)\, d\xi \right|.$$

Hence[6]

$$|\mu| \leqq 1 \Big/ \left| \int_0^1 g(\xi)\, h(\xi)\, d\xi \right| \tag{6}$$

necessarily and there is no possibility of a solution existing for values of μ outside this known circle which is finite if g is not orthogonal to h. This is in marked contrast to the situation in the linear theory. If $hf \equiv 0$, then $y = f$ is a solution for all μ. Moreover, if $f \equiv 0$ (homogeneous equation), there is a continuous spectrum of characteristic values, namely, all values of μ for which $|\mu| = 1 \Big/ \int_0^1 |g(\xi)\, h(\xi)|\, d\xi$.

Extension is immediate to the more general kernel $L(x, \xi)$ which satisfies one orthogonality condition and the condition of positivity: $1 \leqq L \leqq L_0$ in which case necessarily $|\mu| \leqq L_0$ in order that a solution exist. The theorem can also be generalized in the direction of equations of different

[6] The condition of non-absolute orthogonality precludes b from being zero.

type. We express it for an equation of Type (1011) with kernels of finite rank. The equation is

$$(7)\qquad y = f + \lambda \int_0^1 \sum_{i=1}^{n} g_i(x)\, h_i(\xi)\, y(\xi)\, d\xi + \mu \int_0^1 \sum_{i=1}^{m} \varphi_i(x)\, \psi_i(\xi)\, |y(\xi)|\, d\xi .$$

The hypotheses are:—I. ψ_i has a plus characteristic sign[7] and is orthogonal to $f, g_1, g_2, \cdots, g_n$; II. φ_j is such that each c_{ij}, where $c_{ij} = \int_0^1 \psi_i(\xi)\varphi_j(\xi)d\xi$, is positive; we call by c the smallest c_{ij}. We have then the

THEOREM. *There is no solution of* (7) *for* $|\mu| > 1/mc$.

For a solution must be of the form

$$y = f + \lambda \sum_{i=1}^{n} a_i\, g_i(x) + \mu \sum_{i=1}^{m} b_i\, \varphi_i(x)$$

where

$$a_i = \int_0^1 h_i(\xi)\, y(\xi)\, d\xi ,$$

$$b_i = \int_0^1 \psi_i(\xi)\, |y(\xi)|\, d\xi = \int_0^1 \psi_i(\xi) \left| f(\xi) + \lambda \sum_{j=1}^{n} a_j\, g_j(\xi) + \mu \sum_{j=1}^{m} b_j\, \varphi_j(\xi) \right| d\xi .$$

Making use of condition I we have

$$\begin{aligned} b_i &\geqq \left| \int_0^1 [\psi_i f + \lambda\, \psi_i \textstyle\sum a_j g_j + \mu\, \psi_i \textstyle\sum b_j \varphi_j]\, d\xi \right| , \\ &\geqq |\mu| \left| \int_0^1 \psi_i \textstyle\sum b_j\, \varphi_j\, d\xi \right| ; \end{aligned}$$

and by II this reduces to

$$\begin{aligned} |\mu| &\leqq b_i \Big/ \left| \sum_{j=1}^{m} c_{ij}\, b_j \right| , \\ &\leqq b_i \Big/ c \sum_{j=1}^{m} b_j . \end{aligned}$$

By addition of the m such inequalities it follows that

$$m\, |\mu| \leqq \textstyle\sum b_i / c \sum b_i ,$$

whence, finally

$$|\mu| \leqq 1/mc .$$

Hence the theorem.

A further specialization of (5) leads to the following interesting results. If $g \equiv f$, the inequality (6) gives us no specific information about the

[7] A function $\zeta(x)$ is said to have a characteristic sign in a given interval if, in that interval, it neither changes sign nor vanishes identically on any sub-interval.

region of existence and the nature of the solution. A solution will be of the form $y = f(1+\mu b)$ where

$$b = |1+\mu b| \int_0^1 h(\xi)\,|f(\xi)|\,d\xi.$$

Let

$$1/B = \int_0^1 h(\xi)\,|f(\xi)|\,d\xi > 0.$$

Then, if $\mu = \mu_0 + i\mu_1$, we have

$$Bb = |1+\mu b|$$

and b satisfies the quadratic equation

$$(\mu_0^2 + \mu_1^2 - B^2)\,b^2 + 2\mu_0 b + 1 = 0.$$

Whence

(8) $$b = [-\mu_0 \pm (B^2 - \mu_1^2)^{1/2}]/[\mu_0^2 + \mu_1^2 - B^2].$$

There is an immediate restriction necessary upon μ_1, namely: $|\mu_1| \leqq B$, since b is real (positive). Consequently the region of existence is narrowed to a strip in the μ-plane about the axis of reals. It can be verified with no great trouble that in the right-half plane: no solution exists outside and on the boundary of the circle $|\mu| = B$; within B there is one and only one solution where in (8) the minus sign is taken; that in the left-half plane: within B there is one and only one solution where in (8) the minus sign is taken; on the boundary of B there is one and only one solution and $b = -1/(2\mu_0)$; on the boundary of the strip $|\mu_1| \leqq B$ there is one and but one solution and $b = -1/\mu_0$; outside the strip there is no solution; and, finally, inside the strip and outside the circle B there are two and just two solutions where in (8) either sign can be taken. In every case the solution is $y = f(1+\mu b)$. We might express the things essential here in the following

THEOREM. *The region of existence of a solution of* (5) *with* $g \equiv f$ *is non-circular (portion of a strip) and the solution itself is not unique.*

A similar situation holds for the corresponding equation of Type (1011) where, as is to be expected, the size of the strip is a function of λ. As a matter of fact in this case it turns out that

$$b = (-\mu_0 \pm [B^2\{(1-\lambda_0 A)^2 + \lambda_1^2 A^2\} - \mu_1^2]^{1/2})/[\mu_0^2 + \mu_1^2 - B^2\{(1-\lambda_0 A)^2 + \lambda_1^2 A^2\}],$$

where $A = \int hf$, $\lambda = \lambda_0 + i\lambda_1$, $\mu = \mu_0 + i\mu_1$, $1/B = \int \psi|f| \neq 0$. And the solution is $y = f(1+\lambda a + \mu b)$ where $a = A(1+\mu b)/(1-\lambda A)$. For the Type (1111) equation and under certain simple orthogonality conditions a translation of the strip occurs in the μ-plane, the center of the circle "B" being shifted to (σ_0, σ_1).

For this extension it is necessary to know the form of the solution $y(x)$ of the equation Type (1111). We separate the real and imaginary parts and calculate the form of $y = u + iv$ by solving simultaneously the two equations

$$u + \sigma_0 q (u^2 + v^2)^{1/2} = f + \lambda_0 k_0 - \lambda_1 k_1 + \mu_0 l = A,$$
$$v + \sigma_1 q (u^2 + v^2)^{1/2} = \lambda_0 k_1 + \lambda_1 k_0 + \mu_1 l = B,$$

(where $y = f + \lambda k(x) + \mu l(x)$, $k = k_0 + ik_1$, $l = l_0 + il_1$, etc.) for u and v. This gives

$$u = [-\{B\sigma_0\sigma_1 q^2 + A(1 - \sigma_1^2 q^2)\} \pm \sigma_0 q\{A^2 + B^2 - q^2(\sigma_1 A - \sigma_0 B)^2\}^{1/2}] / [(\sigma_0^2 + \sigma_1^2)q^2 - 1],$$
$$v = [-\{A\sigma_1\sigma_0 q^2 + B(1 - \sigma_0^2 q^2)\} \pm \sigma_1 q\{A^2 + B^2 - q^2(\sigma_1 A - \sigma_0 B)^2\}^{1/2}] / [(\sigma_0^2 + \sigma_1^2)q^2 - 1],$$

where, for $|\sigma q| < 1$, the plus sign before the radical is to be taken.

BROWN UNIVERSITY,
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LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH DIRICHLET SERIES AS COEFFICIENTS.*

By S. Borofsky.

1. **Introduction.** It is well known from the work of Fuchs,[1] Frobenius[2] and others that it is necessary and sufficient in order that a differential equation

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)\, y = 0, \tag{1}$$

where each P is developable in a Laurent series in a neighborhood of $x = 0$, have a fundamental system of solutions of the form

$$y_i = x^{r_i}\, [\varphi_{i0}(x) + (\log x)\, \varphi_{i1}(x) + \cdots + (\log x)^{n_i}\, \varphi_{in_i}(x)], \quad (i = 1, 2, \cdots, n),$$

where each φ is analytic in a neighborhood of $x = 0$, that P_i be of the form $Q_i(x)/x^i$, where $Q_i(x)$ is analytic in a neighborhood of $x = 0$.

In this paper we consider the equation (1) where P_i, $(i = 1, 2, \cdots, n)$, is developable in a left half-plane in an absolutely convergent series

$$\cdots + b_{i,-\nu}\, e^{-\beta_\nu x} + \cdots + b_{i,-1}\, e^{-\beta_1 x} + b_{i0} + b_{i1}\, e^{\alpha_1 x} + \cdots + b_{i\nu}\, e^{\alpha_\nu x} + \cdots \tag{2}$$

where

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\nu < \cdots, \qquad \lim \alpha_\nu = \infty,$$
$$0 < \beta_1 < \beta_2 < \cdots < \beta_\nu < \cdots, \qquad \lim \beta_\nu = \infty,$$

and we obtain the necessary and sufficient conditions that the equation have a fundamental system of solutions of the form

$$y_i = e^{r_i x}\, [\varphi_{i0}(x) + x\,\varphi_{i1}(x) + \cdots + x^{n_i}\, \varphi_{in_i}(x)], \quad (i = 1, 2, \cdots, n), \tag{3}$$

where each φ is a Dirichlet series with non-negative exponents[3] absolutely convergent in a left half-plane. Our result is contained in the following theorem.

* Received March 4, 1931.

[1] Journal für Mathematik, Bd. 66 (1866), S. 121.

[2] Ibid. Bd. 76 (1873), S. 214.

[3] For convenience, throughout this paper we work with positive exponents rather than the customary negative ones.

THEOREM. *It is necessary and sufficient in order that the equation* (1), *where* $P_i(x)$, $(i = 1, 2, \cdots, n)$, *is representable in a left half-plane by an absolutely convergent series* (2), *have an independent system of solutions of the form* (3), *that* $b_{i,-\nu} = 0$ *for every* i *and every* $\nu > 0$.

If we let $\beta_\nu = \nu$, $\alpha_\nu = \nu$, $(\nu = 1, 2, \cdots)$, and make the change of variable $z = e^x$, this theorem gives us the result of Fuchs.

In § 2—4 we treat the question of sufficiency, first determining the formal expansions of the solutions and then showing that they are convergent and that the solutions obtained are independent. We obtain the formal expansions following in a general way the method of Frobenius. In § 5 we prove the necessity of the condition. The final paragraphs consist of two notes, one on the independence of the solutions and one a new proof of a known result on the quotient of two Dirichlet series.

2. **The formal expansions.** Let $P_i(x)$ be representable in a left half-plane by an absolutely convergent Dirichlet series

$$b_{i0} + b_{i1}\, e^{\alpha_1 x} + \cdots + b_{i\nu}\, e^{\alpha_\nu x} + \cdots,$$

where

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\nu < \cdots, \qquad \lim \alpha_\nu = \infty.$$

Let

$$\lambda_1, \lambda_2, \cdots, \lambda_\nu, \cdots$$

be all the distinct quantities, ordered according to increasing magnitude, obtained by forming all the expressions of the form

$$\delta_1\, \alpha_{p_1} + \delta_2\, \alpha_{p_2} + \cdots + \delta_m\, \alpha_{p_m}, \qquad (m = 1, 2, \cdots),$$

where $\alpha_{p_1}, \alpha_{p_2}, \cdots, \alpha_{p_m}$ are any m α's and $\delta_1, \delta_2, \cdots, \delta_m$ any set of non-negative integers. Let $\lambda_0 = 0$.

Let

$$P_i(x) = a_{i0} + a_{i1}\, e^{\lambda_1 x} + \cdots + a_{i\nu}\, e^{\lambda_\nu x} + \cdots, \qquad (i = 1, 2, \cdots, n),$$

where some of the a's may be zero.

If we assume the existence of a solution of the form

$$e^{rx}(g_0 + g_1\, e^{\lambda_1 x} + \cdots + g_\nu\, e^{\lambda_\nu x} + \cdots), \qquad g_0 \neq 0,$$

absolutely convergent in a left half-plane, the g's and r can be determined by replacing y in (1) by this series and equating to zero the coefficients of $e^{(r+\lambda_\nu)x}$ $(\nu = 0, 1, \cdots)$. Since the sum of any two λ's is itself a λ, we need not concern ourselves explicitly with the terms $e^{(r+\lambda_\nu+\lambda_\mu)x}$.

Let

$$f(r) = r^n + a_{10}\, r^{n-1} + \cdots + a_{n-1,0}\, r + a_{n0},$$
$$f_\nu(r) = \qquad a_{1\nu} r^{n-1} + \cdots + a_{n-1,\nu} r + a_{n\nu}, \qquad (\nu = 1, 2, \cdots),$$
$$f(x, r) = f(r) + f_1(r)\, e^{\lambda_1 x} + \cdots + f_\nu(r)\, e^{\lambda_\nu x} + \cdots$$
$$D = \frac{d^n}{dx^n} + P_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{d}{dx} + P_n(x).$$

Then

$$D(g_\nu e^{(r+\lambda_\nu)x}) = g_\nu\, e^{(r+\lambda_\nu)x} f(x, r+\lambda_\nu), \qquad (\nu = 0, 1, 2, \cdots)$$

Equating to zero the coefficient of $e^{(r+\lambda_0)x}$ we obtain

$$g_0 f(r) = 0.$$

Consider any $\lambda_\nu > 0$. Let $0 = i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1} = \nu$ be all the values of p for which $\lambda_\nu - \lambda_p$ is equal to a λ. Let

(4) $$\lambda_\nu = \lambda_{i_s} + \lambda_{j_s} \qquad (s = 0, 1, \cdots, k+1).$$

Obviously, $\lambda_{j_0} = \lambda_\nu$, $\lambda_{j_{k+1}} = \lambda_0$.

Equating to zero the coefficient of $e^{(r+\lambda_\nu)x}$ we obtain

(5) $$g_0 f_\nu(r+\lambda_0) + g_{i_1} f_{j_1}(r+\lambda_{i_1}) + \cdots + g_{i_k} f_{j_k}(r+\lambda_{i_k}) + g_\nu f(r+\lambda_\nu) = 0.$$

If we choose r so that $f(r) = 0$, g_0 may be chosen arbitrarily and the remaining g's will be uniquely determined provided $f(r+\lambda_\nu) \neq 0$, $(\nu = 1, 2, \cdots)$, for this value of r.

To dispose of all possible cases at once we proceed as follows. Let the distinct roots of $f(r) = 0$ be $r_1, r_2, \cdots, r_t$ of multiplicities $\mu_1, \mu_2, \cdots, \mu_t$ respectively. Let these roots be so named that if $i < j$ then $\Re(r_i) \geqq \Re(r_j)$.[4]

Some of the distinct roots of $f(r) = 0$ may differ from others by λ's. Let η be so chosen that λ_η is at least as great as the greatest of those differences which are equal to a λ.

Let $\Re$ be a domain of the complex variable r consisting of neighborhoods of $r_1, r_2, \cdots, r_t$ so defined that no neighborhood contains a point $r_i - \lambda_\tau$, $\tau > \eta$, $(i = 1, 2, \cdots, t)$. Let $\bar{g}(r)$ be a function of r analytic in $\Re$ and different from zero for $r = r_1, r_2, \cdots, r_t$.

Let g_0 be a function of r defined by

(6) $$g_0 \equiv g_0(r) = f(r+\lambda_1) f(r+\lambda_2) \cdots f(r+\lambda_\eta)\, \bar{g}(r).$$

[4] $\Re(a)$ is the real part of a and $\Im(a)$ is the coefficient of $\sqrt{-1}$.

From the equations (5) we obtain

$$g_\nu(r) = \frac{g_0(r)\,\overline{\overline{g}}_\nu(r)}{f(r+\lambda_1)\cdots f(r+\lambda_\nu)}, \qquad (\nu = 1, 2, \cdots) \tag{7}$$

where $\overline{\overline{g}}_\nu(r)$ is a polynomial in r.

Obviously $g_\nu(r)$, $(\nu = 1, 2, \cdots, \eta)$, is finite for every r in $\mathfrak{R}$. For $\nu > \eta$, $f(r+\lambda_\nu) \neq 0$ in $\mathfrak{R}$, for, in the contrary case, we would have a point $\bar{r}$ in $\mathfrak{R}$ and a λ_τ with $\tau > \eta$ for which $\bar{r} + \lambda_\tau = r_i$ for some i. This is impossible because of the definition of $\mathfrak{R}$. Hence $g_\nu(r)$ is finite for every ν for every value of r in $\mathfrak{R}$.

The equations (6), (7) determine a series

$$g(x, r) = e^{rx}[g_0(r) + g_1(r)e^{\lambda_1 x} + \cdots + g_\nu(r)e^{\lambda_\nu x} + \cdots]. \tag{8}$$

In the following paragraph we show that this series converges uniformly for r in $\mathfrak{R}$ and x in a sufficiently small left half-plane.

From (8) we now obtain the solutions of (1). The function defined by the series in (8) is a solution, for every fixed value of r, of the equation

$$D(y) = g_0(r)f(r)e^{rx} = f(r)f(r+\lambda_1)\cdots f(r+\lambda_\eta)\overline{g}(r)e^{rx}. \tag{9}$$

Consider any root of $f(r) = 0$, say r_i. Suppose there exists another root r_j distinct from it and such that

$$r_j - r_i = \lambda_\alpha, \tag{10}$$

where λ_α is some $\lambda \neq 0$. Certainly $\alpha \leqq \eta$. Then $f(r+\lambda_\alpha)$ is divisible by $(r - r_i)^{\mu_j}$. If there exists no r_j for which we have a relation (10), then $f(r+\lambda_1)\cdots f(r+\lambda_\eta)$ is not divisible by any power of $(r - r_i)$.

In general, let

$$f(r+\lambda_1)\cdots f(r+\lambda_\eta)\,\overline{g}(r) = (r-r_1)^{\alpha_1}\cdots(r-r_t)^{\alpha_t}\,\overline{h}(r), \tag{11}$$

where $\overline{h}(r)$ is not divisible by any power of $r - r_i$, $(i = 1, 2, \cdots, t)$. Some of the α's may be zero. Certainly $\alpha_1 = 0$ since no relation such as (10) can exist for $i = 1$, because $\mathfrak{R}(r_1) \geqq \mathfrak{R}(r_j)$ for every j.

Because the function $g(x, r)$ satisfies (9), we have

$$D(g(x, r)) = \prod_{i=1}^{t}(r - r_i)^{\mu_i+\alpha_i}\,\overline{h}(r)e^{rx}.$$

Since

$$\frac{\partial^i D(g(x, r))}{\partial r^i} = D\left(\frac{\partial^i g(x, r)}{\partial r^i}\right), \qquad (i = 0, 1, 2, \cdots),^5$$

[5] The derivative of order zero is the function itself.

therefore, we have as solutions of (1)

$$(12)\quad g(x, r_i), \quad \frac{\partial g(x, r_i)}{\partial r}, \quad \cdots, \quad \frac{\partial^{\mu_i+\alpha_i-1} g(x, r_i)}{\partial r^{\mu_i+\alpha_i-1}}, \quad (i = 1, 2, \cdots, t).$$

Not all of these solutions are, in general, independent. Of these solutions we choose

$$(13)\quad \frac{\partial^{\alpha_i} g(x, r_i)}{\partial r^{\alpha_i}}, \quad \frac{\partial^{\alpha_i+1} g(x, r_i)}{\partial r^{\alpha_i+1}}, \quad \cdots, \quad \frac{\partial^{\mu_i+\alpha_i-1} g(x, r_i)}{\partial r^{\mu_i+\alpha_i-1}}, \quad (i = 1, 2, \cdots, t).$$

These are $\sum_{i=1}^{t} \mu_i = n$ in number. In § 4 we show that these are linearly independent.

Let

$$g(x, r) = e^{rx} \varphi(x, r),$$
$$\varphi^{(i)}(x, r) = \frac{\partial^i \varphi(x, r)}{\partial r^i}.$$

Then

$$(14)\quad \frac{\partial^p g(x, r)}{\partial r^p}$$
$$= e^{rx}[\varphi^{(p)}(x,r) + {}^pC_1 x \varphi^{(p-1)}(x,r) + \cdots + {}^pC_i x^i \varphi^{(p-i)}(x,r) + \cdots + x^p \varphi(x,r)].$$

Thus, the solutions (13) are of the form (3).

3. **Convergence proof.** We show that there exists a constant H such that

$$(15)\quad |g_1(r)| + \cdots + |g_\nu(r)| < e^{H\lambda_\nu}, \quad (\nu = 1, 2, \cdots),$$

for r in $\mathfrak{R}$.

Let

$$f(r) = |r|^{n-1} + \cdots + |r| + 1$$

and let Γ, G, R be constants so chosen that for every r in $\mathfrak{R}$

$$(16)\quad |g_0(r)| < G,$$
$$\frac{1}{|f(r+\lambda_\nu)|} < \frac{\Gamma}{\lambda_\nu^n}, \quad (\nu > \eta)$$
$$(17)\quad f(r) < R.$$

Let λ be so chosen that

$$\lambda \geqq \frac{1}{\lambda_1^i}, \quad (i = 1, 2, \cdots, n),$$

and let $A = \lambda\Gamma$.

We have then for $\nu > \eta$

$$(18)\quad \frac{1}{|f(r+\lambda_\nu)|} < \frac{\Gamma}{\lambda_\nu^n} \leqq \frac{\Gamma}{\lambda_1^n} \leqq \lambda\Gamma = A$$

for every r in $\mathfrak{R}$.

Let

$$h_i(r) = |r|^{n-i} + (n-i)|r|^{n-i-1} + \frac{(n-i)(n-i-1)}{2!}|r|^{n-i-2} + \cdots$$
$$\cdots + \frac{(n-i)(n-i-1)\cdots 1}{(n-i)!}, \qquad (i = 1, 2, \cdots, n)$$

and let

$$F(x, r) = h_1(r)\sum_{j=1}^{\infty}|a_{1j}|\, e^{-\lambda_j x} + \cdots + h_n(r)\sum_{j=1}^{\infty}|a_{nj}|\, e^{-\lambda_j x}.$$

Let B be a constant so chosen that

$$(19) \quad |a_{i1}| + |a_{i2}| + \cdots + |a_{i\nu}| < e^{B\lambda_\nu} \quad (i = 1, 2, \cdots, n;\ \nu = 1, 2, \cdots).$$[6]

Let H be a constant greater than B such that for every r in $\mathfrak{R}$

$$(20) \qquad \lambda_1 H > \log 3\,[|g_1(r)| + \cdots + |g_\eta(r)|],$$

$$(21) \qquad \lambda_1 H > \lambda_1 B + \log 3AGR$$

and such that for r in $\mathfrak{R}$

$$(22) \qquad F(H, r) < \frac{1}{3A}.$$

We show that (15) holds with the H just selected.

Because of (20), (15) obviously holds for $\nu = 1, 2, \cdots, \eta$. We proceed, therefore, by induction on ν, supposing (15) true for $\nu = 1, 2, \cdots, \gamma - 1$ and proving it true for $\nu = \gamma$.

For $\nu > \eta$ we have, because of (5),

$$|g_\nu(r)| \leqq \frac{|g_0(r)|\cdot|f_\nu(r)| + |g_{i_1}(r)|\cdot|f_{j_1}(r+\lambda_{i_1})| + \cdots + |g_{i_k}(r)|\cdot|f_{j_k}(r+\lambda_{i_k})|}{|f(r+\lambda_\nu)|}.$$

After expanding by Taylor's theorem we obtain, making use of (16) and (18),

$$|g_\nu(r)| < \lambda \Gamma G |f_\nu(r)|$$
$$+ \sum_{\tau=1}^{k}\frac{\Gamma|g_{i_\tau}(r)|}{\lambda_\nu^n}\left|f_{j_\tau}(r) + f'_{j_\tau}(r)\,\lambda_{i_\tau} + \cdots + f_{j_\tau}^{(n-1)}(r)\,\frac{\lambda_{i_\tau}^{n-1}}{(n-1)!}\right|$$

where superscripts indicate differentiation with respect to r.

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[6] From the proof of Theorem 5, E. Landau, Handbuch der Primzahlen, pp. 732-4; it can be seen that when $L = \limsup(\log|a_1 + \cdots + a_n|)/\lambda_n$, then

a) if $\sum a_n e^{\lambda_n x}$ converges for $x = x_0$ with $\mathfrak{R}(x_0) < 0$, we have $L \leqq \mathfrak{R}(-x_0)$,

b) the series converges for any value of x for which $\mathfrak{R}(x) < 0$, $\mathfrak{R}(-x) > L$.

Since $\lambda_{i_\tau} < \lambda_\nu$ and $\lambda_\nu \geqq \lambda_1$, therefore

$$\frac{1}{\lambda_\nu^n} < \lambda, \quad \frac{\lambda_{i_\tau}}{\lambda_\nu^n} < \lambda, \quad \cdots, \quad \frac{\lambda_{i_\tau}^{n-1}}{\lambda_\nu^n} < \lambda.$$

Hence,

$$|g_\nu(r)| < AG|f_\nu(r)|$$
$$+ \sum_{\tau=1}^{k} \lambda \Gamma |g_{i_\tau}(r)| \left[|f_{j_\tau}(r)| + |f'_{j_\tau}(r)| + \cdots + \frac{1}{(n-1)!} |f_{j_\tau}^{(n-1)}(r)| \right].$$

Let

$$G_\sigma(r) = |f_\sigma(r)| + |f'_\sigma(r)| + \cdots + \frac{1}{(n-1)!} |f_\sigma^{(n-1)}(r)|, \quad (\sigma = 1, 2, \cdots).$$

Then

$$(23) \quad |g_\nu(r)| < AG|f_\nu(r)| + A(|g_{i_1}(r)|\, G_{j_1}(r) + \cdots + |g_{i_k}(r)|\, G_{j_k}(r)).$$

For every $\nu > \eta$ we obtain an expression corresponding to (23). We add up the left hand sides and the corresponding right hand sides of these expressions for $\nu = \eta + 1, \cdots, \gamma$.

Let G_j be any G which appears in the sum of the right hand sides. Suppose it appears in the expressions for which $\nu = \delta_1, \delta_2, \cdots, \delta_p$ with the corresponding coefficients $|g_{t_1}|, \cdots, |g_{t_p}|$. Then, the coefficient of G_j in the sum is, except for the factor A,

$$|g_{t_1}| + \cdots + |g_{t_p}|.$$

Since

$$\lambda_{\delta_1} = \lambda_j + \lambda_{t_1}, \cdots, \lambda_{\delta_p} = \lambda_j + \lambda_{t_p},$$

therefore $t_1, t_2, \cdots, t_p$ are all distinct and less than γ. Hence,

$$|g_{t_1}| + \cdots + |g_{t_p}| \leqq |g_1| + \cdots + |g_i|,$$

where i is some integer less than γ and $\lambda_i + \lambda_j \leqq \lambda_\gamma$.

By the hypothesis of the induction

$$|g_1| + \cdots + |g_i| < e^{H\lambda_i}.$$

This does not exceed $e^{H(\lambda_\gamma - \lambda_j)}$, which is equal to $e^{H\lambda_\nu} e^{-H\lambda_j}$.

Thus, after adding up the expressions (23), we have that

$$(24) \quad \begin{aligned} &|g_{\eta+1}(r)| + \cdots + |g_\gamma(r)| < AG(|f_{\eta+1}(r)| + \cdots + |f_\gamma(r)|) \\ &\qquad + Ae^{H\lambda_\gamma}(G_1(r)\, e^{-\lambda_1 H} + G_2(r)\, e^{-\lambda_2 H} + \cdots). \end{aligned}$$

Since, by the definition of $G_\nu(r)$,

$$G_\nu(r) \leqq |a_{1\nu}|\,h_1(r) + \cdots + |a_{n-1,\nu}|\,h_{n-1}(r) + |a_{n\nu}|\,h_n(r),$$

therefore

$$\sum_{\nu=1}^{\infty} G_\nu(r)\, e^{-\lambda_\nu H} \leqq F(H, r).$$

We have also

$$|f_{\eta+1}(r)| + \cdots + |f_\gamma(r)| \leqq |f_1(r)| + \cdots + |f_\gamma(r)|$$
$$\leqq |r|^{n-1} \sum_{i=1}^{\gamma} |a_{1i}| + |r|^{n-2} \sum_{i=1}^{\gamma} |a_{2i}| + \cdots + \sum_{i=1}^{\gamma} |a_{ni}|,$$

which, because of (19), is less than

$$e^{B\lambda_\gamma}(|r|^{n-1} + \cdots + |r| + 1) = \mathbf{f}(r)\, e^{B\lambda_\nu}.$$

From the inequality (24) we have, therefore,

$$\begin{aligned} |g_1(r)| + \cdots + |g_\gamma(r)| &< |g_1(r)| + \cdots + |g_\eta(r)| \\ &\quad + AG\mathbf{f}(r)\, e^{B\lambda_\gamma} + A e^{H\lambda_\gamma} F(H, r). \end{aligned} \tag{25}$$

By (20) we have that

$$|g_1(r)| + \cdots + |g_\eta(r)| < \frac{1}{3} e^{\lambda_1 H} \leqq \frac{1}{3} e^{\lambda_\gamma H}$$

and by (17) and (21) that

$$\mathbf{f}(r) < R$$

and

$$AGR < \frac{1}{3} e^{\lambda_1 (H-B)} \leqq \frac{1}{3} e^{\lambda_\gamma (H-B)}.$$

By (22) we see that $F(H, r) < \frac{1}{3} A$.

Thus, by (25), we have

$$|g_1(r)| + \cdots + |g_\gamma(r)| < e^{\lambda_\gamma H}.$$

(15) has, therefore, been shown to hold for every ν.

It follows from (15) that the series (8) converges uniformly for r in $\mathfrak{R}$ and x in some left half-plane.[7]

4. **Independence proof.** The proof of the linear independence of the solutions (13) is based upon the following lemma.

LEMMA. *Let*

$$F(x) \equiv \sum_{i=0}^{m} (c_{i0}\, e^{\gamma_0 x} + c_{i1}\, e^{\gamma_1 x} + \cdots)\, x^i \tag{26}$$

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[7] This follows from the proof of Theorem 5, E. Landau, loc. cit.

where the coefficient of x^i, $(i = 0, 1, \cdots, m)$ *is a series, absolutely convergent in a left half-plane, with exponents which are complex numbers such that there is only a finite number of exponents which have the same real part and such that* $\lim \Re(\gamma_i) = \infty$. *Then, if* $F(x) \equiv 0$, *it is necessary that* $c_{ij} = 0$ *for every value of* i *and* j.

We suppose the exponents so ordered that no exponent precedes another which has a smaller real part and that of two exponents with the same real part the one with the smaller coefficient of $\sqrt{-1}$ precedes the other.

Without loss of generality we assume that $\Re(\gamma_0) = 0$, $\Im(\gamma_0) \geqq 0$ and that at least one of c_{i0}, $(i = 1, 2, \cdots, m)$ is different from zero.

Let $\gamma_0, \gamma_1, \cdots, \gamma_q$ be those exponents which have zero real parts. Let $\gamma_i = \tau_i \sqrt{-1}$, $(i = 0, 1, \cdots, q)$. Then $0 \leqq \tau_0 < \tau_1 < \cdots < \tau_q$.

Let $c_{\alpha\beta}$ be the last non-zero term in

$$c_{00}, \cdots, c_{0q}, \cdots, c_{m0}, \cdots, c_{mq}.$$

Let

$$F_1(x) = \sum_{i=0}^{\alpha-1} (c_{i0}\, e^{\gamma_0 x} + \cdots + c_{iq}\, e^{\gamma_q x}) x^i + (c_{\alpha 0}\, e^{\gamma_0 x} + \cdots + c_{\alpha\beta}\, e^{\gamma_\beta x}) x^\alpha,$$

$$F_2(x) = \sum_{i=0}^{m} (c_{i,q+1}\, e^{\gamma_{q+1} x} + \cdots) x^i.$$

Then

$$F(x) = F_1(x) + F_2(x). \tag{27}$$

We have

$$\frac{F_1(x)}{e^{\gamma_\beta x} x^\alpha} = \sum_{i=0}^{\alpha-1} (c_{i0}\, e^{\gamma_0 x} + \cdots + c_{iq}\, e^{\gamma_q x}) e^{-\gamma_\beta x} x^{i-\alpha}$$
$$+ c_{\alpha 0}\, e^{(\gamma_0 - \gamma_\beta) x} + \cdots + c_{\alpha,\beta-1}\, e^{(\gamma_{\beta-1} - \gamma_\beta) x} + c_{\alpha\beta}.$$

Let $x = u + v\sqrt{-1}$. Then, since $|e^{(\gamma_i - \gamma_\beta) x}| = e^{(\tau_\beta - \tau_i) v}$, $(i = 0, 1, \cdots, \beta-1)$, and $\tau_\beta - \tau_i > 0$, therefore, v can be fixed at a value v_0 such that

$$|c_{\alpha 0}\, e^{(\gamma_0 - \gamma_\beta) x} + \cdots + c_{\alpha,\beta-1}\, e^{(\gamma_{\beta-1} - \gamma_\beta) x} + c_{\alpha\beta}| > \frac{1}{2} |c_{\alpha\beta}|$$

whatever be u.

There exists a value u_0 such that for $x = u + v_0 \sqrt{-1}$, $u < u_0$, we have

$$\left| \sum_{i=0}^{\alpha-1} (c_{i0}\, e^{\gamma_0 x} + \cdots + c_{iq}\, e^{\gamma_q x}) e^{-\gamma_\beta x} x^{i-\alpha} \right| < \frac{1}{4} |c_{\alpha\beta}|.$$

Hence, for $x = u + v_0 \sqrt{-1}$, $u < u_0$,

$$\left| \frac{F_1(x)}{e^{\gamma_\beta x} x^\alpha} \right| > \frac{1}{4} |c_{\alpha\beta}|. \tag{28}$$

Let $x_0 = u_1 + v_0\sqrt{-1}$, be a value of x for which the series in (26) all converge absolutely. We have

$$|F_2(x)| \leqq \sum_{i=q+1}^{\infty} |c_{0i} + c_{1i}x + \cdots + c_{mi}x^m| \cdot |e^{\gamma_i(x-x_0)}| \cdot |e^{\gamma_i x_0}|.$$

For $x = u + v_0\sqrt{-1}$, $u < u_1$, we have

$$\begin{aligned} |F_2(x)| &\leqq \sum_{i=q+1}^{\infty} (|c_{0i}e^{\gamma_i x_0}| + |c_{1i}e^{\gamma_i x_0}| \cdot |x| + \cdots + |c_{mi}e^{\gamma_i x_0}| \cdot |x|^m) e^{\Re[\gamma_i(x-x_0)]} \\ &\leqq (A_0 + A_1|x| + \cdots + A_m|x|^m) e^{\Re[\gamma_{q+1}(u-u_1)]}, \end{aligned}$$

where

$$A_j = \sum_{i=q+1}^{\infty} |c_{ji}e^{\gamma_i x_0}|.$$

Since $\Re(\gamma_{q+1}) > 0$, therefore, $|F_2(x)| \to 0$ as $u \to -\infty$.

Let u_2 be a constant such that

$$u_2 < u_0, \qquad u_2 < u_1$$

$$(29) \qquad |F_2(x)| < \frac{1}{8}|c_{\alpha\beta}| \quad \text{for } u < u_2,\ v = v_0,$$

$$(30) \qquad |e^{\gamma_\beta x} x^\alpha| > 1 \qquad \text{for } u < u_2,\ v = v_0.$$

Then, because of (27), (28), (29)

$$|F(x)| > \frac{1}{8}|c_{\alpha\beta}| \quad \text{for } u < u_2,\ v = v_0.$$

Since $|c_{\alpha\beta}| \neq 0$, we have a contradiction which proves the lemma.

We return now to the solutions (13). Since, because of (6) and (11)

$$g_0(r_i) = g_0'(r_i) = \cdots = g_0^{(\alpha_i-1)}(r_i) = 0, \quad g_0^{(\alpha_i)}(r_i) \neq 0, \quad (i = 1, 2, \cdots, t),$$

and since the constant term in $\varphi^{(j)}(x, r)$ in (14) is $g_0^{(j)}(r)$, therefore, the solutions (13) can be written in the form

$$(31) \qquad \begin{aligned} &e^{r_i x}[\varphi_{i\alpha_i}(x) + x\varphi_{i,\alpha_i-1}(x) + \cdots + x^{\alpha_i}\varphi_{i0}(x)] \\ &e^{r_i x}[\varphi_{i,\alpha_i+1}(x) + c_i^{(1)} x\varphi_{i\alpha_i}(x) + \cdots + x^{\alpha_i+1}\varphi_{i0}(x)] \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &e^{r_i x}[\varphi_{i,\mu_i+\alpha_i-1}(x) + \cdots + c_i^{(\mu_i-1)} x^{\mu_i-1}\varphi_{i\alpha_i}(x) + \cdots + x^{\mu_i+\alpha_i-1}\varphi_{i0}(x)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \cdots, t), \end{aligned}$$

where the c's are non-zero constants and the φ's are such that the constant term in $\varphi_{i\alpha_i}(x)$ is not zero while the constant terms in $\varphi_{i0}(x), \cdots, \varphi_{i,\alpha_i-1}(x)$ are zero.

Let the solutions (31) be

$$y_{i1},\quad y_{i2},\cdots,y_{i\mu_i},\qquad (i=1,2,\cdots,t).$$

Suppose that these are linearly dependent. We have then a relation of the form

$$\sum(c_{i1}\,y_{i1}+\cdots+c_{i\mu_i}\,y_{i\mu_i})\equiv 0 \tag{32}$$

where not all the c's are zero. We suppose that in this summation i ranges over all values for which not all of $c_{i1},\cdots,c_{i\mu_i}$ are zeros.

In $c_{i1},\cdots,c_{i\mu_i}$ let $c_{i\sigma_i}$ be the last non-vanishing term. For convenience, let this be called c.

In

$$c_{i1}\,y_{i1}+\cdots+c_{i\sigma_i}\,y_{i\sigma_i} \tag{33}$$

the constant term in the coefficient of $e^{r_i x}x^{\sigma_i-1}$ is, because of (31), the constant term in $c_i c_i^{(\sigma_i-1)}\varphi_{i\alpha_i}(x)$. This constant is of the form $c_i\,d_i$ where $d_i \neq 0$. The constant term in the coefficient of $e^{r_i x}x^{\sigma}$ in (33) for $\sigma>\sigma_i-1$ is zero.

Let $\bar{\sigma}$ be one of the greatest of the σ_i. For definiteness, suppose that $\sigma_1,\sigma_2,\cdots,\sigma_\tau$ are the σ_i which are equal to $\bar{\sigma}$. In (32), in the coefficient of $x^{\bar{\sigma}-1}$ there appear the terms $c_1\,d_1\,e^{r_1x},\cdots,c_\tau\,d_\tau\,e^{r_\tau x}$ which are cancelled by no other terms. It follows from the lemma, therefore, that $c_1=c_2=\cdots=c_\tau=0$. This contradiction proves the independence of the solutions.

5. **Necessity proof.** We consider an equation (1) in which each P is representable in a left half-plane by an absolutely convergent series of the form (2). We suppose that this equation has an independent system of solutions of the form (3). We show that the terms with negative exponents do not appear in the P's.

Let the solutions (3) be denoted by $y_1,\cdots,y_n$. Since the equation (1) is equivalent to the equation

$$\begin{vmatrix} y & y_1 & \cdots & y_n \\ \dfrac{dy}{dx} & \dfrac{dy_1}{dx} & \cdots & \dfrac{dy_n}{dx} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{d^n y}{dx^n} & \dfrac{d^n y_1}{dx^n} & \cdots & \dfrac{d^n y_n}{dx^n} \end{vmatrix} = 0,$$

we see that each P is of the form

$$\frac{\varphi_0(x)+x\varphi_1(x)+\cdots+x^q\varphi_q(x)}{\psi_0(x)+x\psi_1(x)+\cdots+x^p\psi_p(x)} \tag{34}$$

where each φ and each ψ is a Dirichlet series with non-negative exponents absolutely convergent in a left half-plane.

Let

$$P_i(x) = Q_i(x) + R_i(x) \tag{35}$$

where $R_i(x)$ contains the terms with negative exponents and $Q_i(x)$ the terms with non-negative exponents. We show first that $R_i(x)$ contains only a finite number of terms.

Because of (34) and (35), $R_i(x)$ is of the form

$$R_i(x) = \frac{\chi_0(x) + x\chi_1(x) + \cdots + x^m \chi_m(x)}{\psi_0(x) + x\psi_1(x) + \cdots + x^p \psi_p(x)}. \tag{36}$$

The χ's and ψ's are series with non-negative exponents. Since $R_i(x)$ contains only negative exponents and converges in a left half-plane, it converges over the entire plane.

Let

$$\psi_i(x) = \alpha_{i1} e^{\gamma_1 x} + \alpha_{i2} e^{\gamma_2 x} + \cdots, \qquad (i = 0, 1, \cdots, p)$$

where $0 \leqq \gamma_1 < \gamma_2 < \cdots$, $\lim \gamma_\sigma = \infty$ and at least one of $\alpha_{01}, \alpha_{11}, \cdots, \alpha_{p1}$ is not zero.

Let the denominator in (36) be denoted by $F(x)$. Then

$$\begin{aligned} F(x) &= (\alpha_{01} + \alpha_{11} x + \cdots + \alpha_{p1} x^p) e^{\gamma_1 x} \\ &\qquad + \sum_{i=2}^{\infty} (\alpha_{0i} + \alpha_{1i} x + \cdots + \alpha_{pi} x^p) e^{\gamma_i x}. \end{aligned} \tag{37}$$

Let α_{k1} be the last non-vanishing term in $\alpha_{01}, \alpha_{11}, \cdots, \alpha_{p1}$.

There exists a constant A_0 such that for $\Re(x) < A_0$

$$| \alpha_{01} + \alpha_{11} x + \cdots + \alpha_{k1} x^k | > \frac{1}{2} | x |^k | \alpha_{k1} |. \tag{38}$$

There exist two constants A_1 and A_2 such that for $\Re(x) < A_2$

$$\sum_{i=2}^{\infty} | \alpha_{0i} e^{\gamma_i x} | + | x | \sum_{i=2}^{\infty} | \alpha_{1i} e^{\gamma_i x} | + \cdots + | x |^p \sum_{i=2}^{\infty} | \alpha_{pi} e^{\gamma_i x} | \tag{39}$$

is less than

$$A_1 e^{\gamma_2 \Re(x)} (1 + |x| + \cdots + |x|^p).$$

There is a constant A_3 such that for $\Re(x) < A_3$ the last expression is less than

$$e^{\gamma_2 \Re(x)} e^{(\gamma_1 - \gamma_2) \Re(x)} \frac{1}{4} | \alpha_{k1} |. \tag{40}$$

Since

$$\left|\sum_{i=2}^{\infty}(\alpha_{0i}+\alpha_{1i}x+\cdots+\alpha_{pi}x^p)\,e^{\gamma_i x}\right|$$

does not exceed the expression (39), therefore, because of (37), (38), (40), in a sufficiently small left half-plane

$$(41)\qquad |F(x)|>\frac{1}{2}|\alpha_{k1}|\,e^{\gamma_1\Re(x)}-\frac{1}{4}|\alpha_{k1}|\,e^{\gamma_1\Re(x)}=A_4 e^{\gamma_1\Re(x)}.$$

There exist two constants A_5 and A_6 such that for $\Re(x)<A_6$ the numerator in (36) is less in absolute value than

$$A_5(1+|x|+\cdots+|x|^m).$$

This, in turn, in a left half-plane is less than $A_7\,e^{-\gamma_1\Re(x)}$.

It follows from this and (41) that in some left half-plane

$$|R_i(x)|<A_8\,e^{-\gamma_1\Re(x)}.$$

It follows, then, from a theorem of J. F. Ritt,[8] that $R_i(x)$ contains only a finite number of terms.

Thus, each P in (1) is a single Dirichlet series with only a finite number of terms with negative exponents.

Let

$$y=e^{rx}[\varphi_0(x)+x\varphi_1(x)+\cdots+x^p\varphi_p(x)]$$

be one of the solutions of (1). We show that $e^{rx}\varphi_p(x)$ is also a solution of (1).

We have

$$D(y)=e^{rx}[V_0(x)+xV_1(x)+\cdots+x^pV_p(x)]$$

where each V is a Dirichlet series absolutely convergent in a left half-plane. Since y is a solution of (1), therefore, by the lemma of the preceding paragraph,

$$(42)\qquad V_0\equiv V_1\equiv\cdots\equiv V_p\equiv 0.$$

Let

$$u(t,x)=e^{rx}[\varphi_0(x)+(x+t)\varphi_1(x)+\cdots+(x+t)^p\varphi_p(x)].$$

Then,

$$D[u(t,x)]=e^{rx}[V_0(x)+(x+t)V_1(x)+\cdots+(x+t)^pV_p(x)].$$

Hence, because of (42), $u(t,x)$ is a solution of (1) whatever be t.

[8] "On Certain Points in the Theory of Dirichlet Series", Amer. Journal of Math. vol. 50, 1928, p. 77.

As

$$\frac{\partial^i}{\partial t^i} D(u(t, x)) = D\left[\frac{\partial^i}{\partial t^i} u(t, x)\right] \equiv 0,$$

therefore, $\frac{\partial^p}{\partial t^p} u(0, x)$ is a solution. Hence, $e^{rx} \varphi_p(x)$ is a solution and our statement is proved.

We now prove the necessity of the condition stated in the theorem by induction on n.

For $n = 1$ the equation is of the form

$$\frac{dy}{dx} + P(x) y = 0.$$

From the preceding, we know that this has a solution of the form $e^{rx} \varphi(x)$ with the constant term in $\varphi(x)$ different from zero. As

$$P(x) = -\frac{y'}{y} = r + \frac{\varphi'(x)}{\varphi(x)},$$

the theorem is true for $n = 1$, since, as is shown in § 7, φ'/φ is expressible as a Dirichlet series with non-negative exponents.

We suppose the theorem true for equations of orders $1, 2, \cdots, n-1$ and we consider an equation of order n. From the preceding, we know that this equation has a solution $y_1 = e^{rx} \varphi(x)$ with the constant term in $\varphi(x)$ different from zero.

Let $y = y_1 \int z\, dx$. The equation becomes

$$\frac{d^{n-1} z}{d x^{n-1}} + Q_1(x) \frac{d^{n-2} z}{d x^{n-2}} + \cdots + Q_{n-1}(x) z = 0$$

where the Q's are of the same form as the P's. The solutions of this equation are of the form (3). Hence, by the hypothesis of the induction, the Q's are Dirichlet series with non-negative exponents. It follows, from the expressions of the P's in terms of the Q's, that $P_1(x), \cdots, P_{n-1}(x)$ are Dirichlet series with non-negative exponents. Since

$$P_n(x) = -\frac{y_1^{(n)}}{y_1} - P_1(x) \frac{y_1^{(n-1)}}{y_1} - \cdots - P_{n-1}(x) \frac{y_1'}{y_1},$$

the same is true of $P_n(x)$.

6. **Remark on linear dependence.** In § 4, we showed that the solutions (13) are linearly independent. These solutions are, in the general case, a subset of the solutions (12). In the present paragraph, we take

a special case and give a direct demonstration of the fact that the solutions (12) are not independent.

Suppose $r_1 - r_2 = \lambda_m$. We show that $g(x, r_2)$ differs from $g(x, r_1)$ only by a constant factor.

Since r_1 is a μ_1-fold root of $f(r) = 0$, therefore, r_2 is a μ_1-fold root of $f(r+\lambda_m) = 0$. Since $\Re(r_2) \geqq \Re(r_j)$, $j > 2$, there is no other root which is the sum of r_2 and a λ. Therefore, α_2 in (11) is μ_1.

Because of (6) and (7),

$$g_0(r) = (r - r_2)^{\mu_1} \bar{h}_0(r),$$

$$g_\nu(r) = \frac{(r-r_2)^{\mu_1} \bar{h}_\nu(r)}{f(r+\lambda_1)\cdots f(r+\lambda_\nu)}, \qquad (\nu = 1, 2, \cdots)$$

where $\bar{h}_\nu(r)$, $(\nu = 0, 1, \cdots)$ does not vanish for $r = r_2$.

Since $r - r_2$ does not appear as a factor in $f(r+\lambda_1)\cdots f(r+\lambda_{m-1})$, we see that $g_\nu(r_2) = 0$ for $\nu = 0, 1, \cdots, m-1$. We say, also, that $g_\nu(r_2) = 0$ if λ_ν is not the sum of some λ and λ_m. For, suppose that μ is the first integer greater than m for which λ_μ is not the sum of some λ and λ_m and for which $g_\mu(r_2) \neq 0$. Consider the equation (5) which determines $g_\mu(r_2)$. In that equation, $i_1 \neq m$ since λ_μ is not the sum of a λ and λ_m. Also, λ_{i_1} cannot be the sum of a λ and λ_m, for in that case we would have

$$\lambda_\mu = \lambda_{i_1} + \lambda_{j_1} = (\lambda_{t_1} + \lambda_m) + \lambda_{j_1} = \lambda_m + (\lambda_{t_1} + \lambda_{j_1}) = \lambda_m + \lambda_{s_1}.$$

From the hypothesis then, we must have $g_{i_1}(r_2) = 0$. In the same way we must have $g_{i_2}(r_2) = \cdots = g_{i_k}(r_2) = 0$ in (5). Since $g_0(r_2) = 0$, therefore, $g_\mu(r_2) = 0$. Hence, we have a contradiction.

We may write, therefore,

$$g(x, r_2) = e^{(r_2+\lambda_m)x} \sum_{\nu=m}^{\infty} g_\nu(r_2) e^{(\lambda_\nu - \lambda_m)x}$$

$$= e^{r_1 x} \sum_{\nu=0}^{\infty} \bar{g}_\nu(r_2) e^{\lambda_\nu x},$$

where $\bar{g}_\nu(r_2)$ is merely the $g_\mu(r_2)$ for which $\lambda_\mu - \lambda_m = \lambda_\nu$.

Consider now any λ_ν. As in (4), let

$$\lambda_\nu = \lambda_0 + \lambda_\nu = \lambda_{i_1} + \lambda_{j_1} = \cdots = \lambda_{i_k} + \lambda_{j_k} = \lambda_\nu + \lambda_0. \tag{43}$$

Let $\lambda_t = \lambda_\nu + \lambda_m$ and again, as in (4), let

$$\lambda_t = \lambda_0 + (\lambda_\nu + \lambda_m) = \lambda_{p_1} + \lambda_{q_1} = \cdots = \lambda_{p_l} + \lambda_{q_l} = (\lambda_\nu + \lambda_m) + \lambda_0. \tag{44}$$

If λ_{p_i} is not the sum of a λ and λ_m, then $g_{p_i}(r_2) = 0$. Hence, in the determination of $g_t(r_2)$ by means of (5), we may disregard this case. We suppose, therefore, that every λ_{p_i} in (44) is the sum of a λ and λ_m. Let

$$\lambda_{p_i} = \lambda_{s_i} + \lambda_m. \qquad (i = 1, 2, \cdots, l).$$

Then

$$\lambda_t = (\lambda_{s_i} + \lambda_m) + \lambda_{q_i}. \tag{45}$$

It follows that

$$\lambda_\nu = \lambda_{s_i} + \lambda_{q_i}, \qquad (i = 1, 2, \cdots, l). \tag{46}$$

For every equality (46) we have an equality (44), and conversely. Hence, the system (46) is identical with the system (44). Since $\lambda_{q_i} \neq \lambda_0$, therefore $\lambda_{s_i} \neq \lambda_\nu$. λ_{s_i} may, however be λ_0. Hence, we may suppose $l = k + 1$ and

$$\begin{aligned} s_1 &= 0, \quad s_2 = i_1, \quad \cdots, \quad s_l = i_k, \\ q_1 &= \nu, \quad q_2 = j_1, \quad \cdots, \quad q_l = j_k. \end{aligned} \tag{47}$$

The equation which determines $g_t(r_2)$ is then

$$\begin{aligned} g_0(r_2)f_t(r_2) + g_{p_1}(r_2)f_\nu(r_2 + \lambda_{p_1}) + g_{p_2}(r_2)f_{j_1}(r_2 + \lambda_{p_2}) + \cdots \\ \cdots + g_{p_k}(r_2)f_{j_{k-1}}(r_2 + \lambda_{p_k}) + g_{p_{k+1}}(r_2)f_{j_k}(r_2 + \lambda_{p_{k+1}}) + g_t(r_2)f(r_2 + \lambda_t) = 0. \end{aligned}$$

Since

$$g_{p_1}(r_2) = \bar{g}_{s_1}(r_2), \quad \cdots, \quad g_{p_{k+1}}(r_2) = \bar{g}_{s_l}(r_2), \quad g_t(r_2) = \bar{g}_\nu(r_2), \quad g_0(r_2) = 0$$

the preceding equation can be written, because of (47),

$$\begin{aligned} \bar{g}_0(r_2)f_\nu(r_2 + \lambda_{s_1} + \lambda_m) + \bar{g}_{i_1}(r_2)f_{j_1}(r_2 + \lambda_{s_2} + \lambda_m) + \cdots \\ \cdots + \bar{g}_{i_k}(r_2)f_{j_k}(r_2 + \lambda_{s_{k+1}} + \lambda_m) + \bar{g}_\nu(r_2)f(r_2 + \lambda_\nu + \lambda_m) = 0. \end{aligned}$$

Since $r_1 = r_2 + \lambda_m$ the last equation can be written, because of (47),

$$\begin{aligned} \bar{g}_0(r_2)f_\nu(r_1 + \lambda_0) + \bar{g}_{i_1}(r_2)f_{j_1}(r_1 + \lambda_{i_1}) + \cdots + \bar{g}_{i_k}(r_2)f_{j_k}(r_1 + \lambda_{i_k}) \\ + \bar{g}_\nu(r_2)f(r_1 + \lambda_\nu) = 0. \end{aligned}$$

Thus, to determine $\bar{g}_\nu(r_2)$, $(\nu = 1, 2, \cdots)$, we have an equation which is identical with the equation which determines the $g_\nu(r_1)$. Therefore,

$$g(x, r_2) = cg(x, r_1).$$

7. Quotient of two Dirichlet series. We show that the quotient of two Dirichlet series

$$\frac{b_0 e^{\beta_0 x} + b_1 e^{\beta_1 x} + \cdots}{g_0 e^{\alpha_0 x} + g_1 e^{\alpha_1 x} + \cdots} \tag{48}$$

where

$$\beta_0 < \beta_1 < \cdots < \beta_\nu < \cdots, \qquad \lim \beta_\nu = \infty,$$
$$\alpha_0 < \alpha_1 < \cdots < \alpha_\nu < \cdots, \qquad \lim \alpha_\nu = \infty,$$

each absolutely convergent in a left half-plane is a series

$$c_0 e^{\gamma_0 x} + c_1 e^{\gamma_1 x} + \cdots$$
$$\gamma_0 < \gamma_1 < \cdots < \gamma_\nu < \cdots, \qquad \lim \gamma_\nu = \infty,$$

absolutely convergent in a left half-plane.

This result is a very special case of a general theorem by J. F. Ritt[9] concerning a function defined by equating to zero a polynomial in this function with coefficients which are absolutely convergent Dirichlet series. The proof which we give for this case, however, is radically different from that of Ritt. We present it here because it gives another illustration of the method used in § 3.

Since the product of two Dirichlet series absolutely convergent in a half-plane is another series absolutely convergent in the same half-plane,[10] it suffices to take the numerator in (48) to be unity and $\alpha_0 = 0$, $g_0 = 1$.

As in § 2, form all linear combinations with positive integral coefficients of $\alpha_1, \alpha_2, \cdots, \alpha_\nu, \cdots$. Let the distinct quantities thus obtained, ordered according to increasing magnitude, be $\lambda_1, \lambda_2, \cdots$. Let $\lambda_0 = 0$.

Let $\gamma_i = \lambda_i$, $(i = 0, 1, \cdots)$.

Let the denominator of (48) be written

$$1 + d_1 e^{\lambda_1 x} + d_2 e^{\lambda_2 x} + \cdots, \tag{49}$$

where some of the d's may be zeros.

Let A, c be constants so chosen that

$$\sum_{\nu=1}^{\infty} |a_\nu| = A \tag{50}$$

where $a_\nu = d_\nu e^{-\lambda_\nu c}$.

Let $x = z - c$. The series (49) becomes

$$1 + a_1 e^{\lambda_1 z} + a_2 e^{\lambda_2 z} + \cdots. \tag{51}$$

[9] Algebraic Combinations of Exponentials, Trans. Amer. Math. Soc., vol. 31, p. 654.

[10] E. Landau, loc. cit., p. 752.

We assume the existence of a series

$$c_0 + c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \cdots \tag{52}$$

absolutely convergent in a left half-plane, and equal to the reciprocal of the function defined by the series (49). Obviously, $c_0 = 1$. The remaining c's are determined by equating to zero the coefficients of $e^{\lambda_\nu z}$, $(\nu = 1, 2, \cdots)$, in the formal expansion of the product

$$(1 + a_1 e^{\lambda_1 z} + \cdots)(1 + c_1 e^{\lambda_1 z} + \cdots). \tag{53}$$

Since the sum of any two λ's is itself a λ, we need not concern ourselves explicitly with the coefficients of $e^{(\lambda_\mu + \lambda_\nu)x}$.

Consider any λ_ν. Let $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = \nu$ be the values of p for which $\lambda_\nu - \lambda_p$ is equal to a λ. Let

$$\lambda_\nu = \lambda_{i_s} + \lambda_{j_s}, \qquad (s = 0, 1, \cdots, k+1).$$

Equating to zero the coefficient of $e^{\lambda_\nu z}$ in (53), we have

$$c_\nu + a_{i_1} c_{j_1} + \cdots + a_{i_k} c_{j_k} + a_\nu = 0. \tag{54}$$

This equation determines c_ν.

We show now that the series $\sum c_\nu e^{\lambda_\nu z}$, thus determined formally, converges absolutely in a left half-plane. To do this we show that there exists a constant G such that

$$|c_1| + \cdots + |c_\nu| < e^{\lambda_\nu G}, \qquad (\nu = 1, 2, \cdots). \tag{55}$$

Since $|c_1| = |a_1|$, G can be chosen that (55) is true for $\nu = 1$. Let G be chosen so that in addition $2A \leqq e^{\lambda_1 G}$. With this choice of G we prove (55) by induction on ν, supposing it to be true for $\nu = 1, 2, \cdots, \mu - 1$ and proving it true for $\nu = \mu$.

From (54) we have that

$$|c_\nu| \leqq |a_\nu| + |a_{i_1}| \cdot |c_{j_1}| + \cdots + |a_{i_k}| \cdot |c_{j_k}|. \tag{56}$$

For every ν from 1 to μ we have an expression corresponding to (56). We add up the left hand sides and the corresponding right hand sides of these expressions for $\nu = 1, 2, \cdots, \mu$.

Let c_j be a c which appears in the sum of the right hand sides. Suppose that it appears in the expressions for which $\nu = \delta_1, \delta_2, \cdots, \delta_r$. Let $\lambda_{\delta_i} = \lambda_{\tau_i} + \lambda_j$, $(i = 1, 2, \cdots, r)$.

The coefficient of $|c_j|$ in the sum is

$$|a_{\tau_1}|+|a_{\tau_2}|+\cdots+|a_{\tau_r}|.$$

Since $\lambda_{\tau_1},\cdots,\lambda_{\tau_r}$ are all distinct and less than λ_μ, therefore, $\tau_1,\cdots,\tau_r$ are all distinct and less than μ. Hence

$$|a_{\tau_1}|+\cdots+|a_{\tau_r}|<|a_1|+\cdots+|a_\mu|<A.$$

We have, therefore,

$$|c_1|+\cdots+|c_\mu|<|a_1|+\cdots+|a_\mu|+A(|c_1|+\cdots+|c_p|),$$

where p is some integer such that for some $i\leqq\mu$ we have $\lambda_i-\lambda_p$ equal to a λ different from zero. Obviously, $\lambda_\mu-\lambda_p\geqq\lambda_1$.

Because of (50) and the hypothesis of the induction, we have

$$\begin{aligned}|c_1|+\cdots+|c_\mu|&<A+Ae^{\lambda_p G}\\&\leqq\frac{1}{2}e^{\lambda_1 G}+\frac{1}{2}e^{\lambda_1 G}e^{(\lambda_\mu-\lambda_1)G}\\&\leqq\frac{1}{2}e^{\lambda_\mu G}+\frac{1}{2}e^{\lambda_\mu G}=e^{\lambda_\mu G}.\end{aligned}$$

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NECESSARY AND SUFFICIENT CONDITIONS IN THE THEORY OF FOURIER TRANSFORMS.[1]

By Andrew C. Berry.[2]

In a recent note,[3] R. Salem gives necessary and sufficient conditions that $\{a_n, b_n\}$ be the Fourier coefficients of a function integrable in the sense of Lebesgue over $(0, 2\pi)$. The present paper solves the corresponding problem in the theory of Fourier transforms by giving necessary and sufficient conditions that a function $F(x)$ be the Fourier transform of a function which is absolutely integrable over $(-\infty, \infty)$. Furthermore, the associated problem of determining necessary and sufficient conditions that $F(x)$ be the Fourier transform of a function whose pth power, $1 < p \leqq 2$, is integrable over $(-\infty, \infty)$ is also solved. To simplify the presentation we first recall the now nearly classical results in the theory of convergence in mean and in the theory of the Fourier transform.

Definition 1. A complex function $f(x)$ of the single real variable x, $-\infty < x < \infty$, shall be said to be in L_ϱ for some ϱ such that $1 \leqq \varrho < \infty$ if $f(x)$ is measurable and if there exists, as a finite number,

$$l_\varrho(f) = \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^\varrho \, dx\right\}^{1/\varrho}.$$

Definition 2. A function $f(x)$ shall be said to be in L_∞ if it is measurable and if there exists, as a finite number,

$$l_\infty(f) = \underset{-\infty < x < \infty}{\text{upper measurable bound}} |f(x)|,$$

a number such that, for each $\varepsilon > 0$,

$$\text{meas}\{|f(x)| > l_\infty(f) + \varepsilon\} = 0, \qquad \text{meas}\{|f(x)| > l_\infty(f) - \varepsilon\} > 0.$$

We note that each L_ϱ is a linear metric space, the distance between elements f, g being defined as $l_\varrho(f - g)$ and this distance satisfying the usual requirements

$$\begin{cases} l_\varrho(f) = 0, \text{ if and only if } f(x) = 0 \text{ almost everywhere,} \\ l_\varrho(af) = |a| l_\varrho(f), \text{ if } a \text{ is a complex number,} \\ l_\varrho(f+g) \leqq l_\varrho(f) + l_\varrho(g), \quad \text{(Minkowski's inequality for } L_\varrho). \end{cases}$$

[1] Received May 23, 1931.

[2] National Research Fellow.

[3] R. Salem, Comptes Rendus, 192 (1931), 144–146.

DEFINITION 3. A one-parameter family $f_\lambda(x)$ shall be said to be convergent (ϱ), say as $\lambda \longrightarrow \infty$, for some ϱ such that $1 \leqq \varrho \leqq \infty$, if each $f_\lambda(x)$ is in L_ϱ and if

$$\lim_{\lambda,\mu\to\infty} l_\varrho(f_\lambda - f_\mu) = 0.$$

DEFINITION 4. A one-parameter family $f_\lambda(x)$ shall be said to converge (ϱ) to $f(x)$, say as $\lambda \longrightarrow \infty$, if $f(x)$ and all $f_\lambda(x)$ are in L_ϱ and if

$$\lim_{\lambda\to\infty} l_\varrho(f - f_\lambda) = 0.$$

We shall write

$$f_\lambda \underset{\varrho}{\longrightarrow} f.$$

Minkowski's inequality has as one consequence the fact that if $f_\lambda \underset{\varrho}{\longrightarrow} f$ and if $f_\lambda \underset{\varrho}{\longrightarrow} g$, then $f(x) = g(x)$ almost everywhere, and as a second consequence the fact that if $f_\lambda \underset{\varrho}{\longrightarrow} f$, then $f_\lambda(x)$ is convergent (ϱ). The Riesz-Fischer completeness theorem[4] states that if $f_\lambda(x)$ is convergent (ϱ), then there exists $f(x)$ in L_ϱ such that $f_\lambda \underset{\varrho}{\longrightarrow} f$.

A necessary and sufficient condition[5] that $\varphi(x)$ be an indefinite integral of a function in some L_ϱ, where $1 < \varrho < \infty$, is that

$$\sum_{k=1}^{n-1} \frac{|\varphi(x_{k+1}) - \varphi(x_k)|^\varrho}{(x_{k+1} - x_k)^{\varrho-1}} \leqq C_\varphi,$$

where C_φ is independent of the finite sub-division $-\infty < x_1 < x_2 < \cdots < x_n < \infty$.

A necessary and sufficient condition that $\varphi(x)$ be an indefinite integral of a function in L_1 is that $\varphi(x)$ be absolutely continuous and be of bounded variation over $(-\infty, \infty)$.

DEFINITION 5. Hereafter p shall denote a number such that

$$1 \leqq p \leqq 2,$$

and p' the corresponding number

$$p' = \begin{cases} \infty, & p = 1, \\ \dfrac{p}{p-1}, & 1 < p \leqq 2, \end{cases}$$

which satisfies, therefore, the inequality

$$2 \leqq p' \leqq \infty.$$

[4] F. Riesz, Comptes Rendus, 144 (1907), 615–619; E. Fischer, Comptes Rendus, 144 (1907), 1022–1024; H. Weyl, Math. Annalen, 67 (1909), 225–245; J. von Neumann, Göttinger Nachrichten, (1927), 1–57. That the theorem holds also when $0 < \varrho < 1$ does not interest us here. For L_∞ we note that convergence (∞) means uniform convergence except on a set of zero measure.

[5] F. Riesz, Über Systeme integrierbarer Funktionen, Math. Annalen, 69 (1910), 449–497.

The Hölder inequality states that if $f(x)$ is in some L_p and if $g(x)$ is in $L_{p'}$ then

$$\left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \right| \leqq l_p(f)\, l_{p'}(g).$$

A consequence of this inequality is the fact that if $f_\lambda \xrightarrow[p]{} f$ and if $g_\lambda \xrightarrow[p']{} g$, then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\lambda(x)\, g_\lambda(x)\, dx \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

The Fourier transform theorem[6]) states that if $f(x)$ is in some L_p it possesses in $L_{p'}$ a Fourier transform $F(x)$ defined by the relation

$$\frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} f(y)\, e^{-ixy}\, dy \xrightarrow[p']{} F(x), \quad \text{as } \lambda \longrightarrow \infty.$$

Secondly,

$$\begin{cases} l_2\,(F) = l_2\,(f), & p = 2. \\ l_{p'}(F) \leqq l_p(f), & 1 \leqq p < 2. \end{cases}$$

Thirdly, if $f(x)$ and $g(x)$ are both in L_p and if $F(x)$ and $G(x)$ are their respective Fourier transforms, then we have the adjoint relation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, g(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, G(x)\, dx.$$

Fourthly,

$$F(x) = \lim_{a \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{\sin ay}{ay} e^{-ixy}\, dy, \quad 1 \leqq p \leqq 2,$$

$$f(x) = \lim_{a \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) \frac{\sin ay}{ay} e^{ixy}\, dy, \quad 1 < p \leqq 2,$$

$$f(x) = \lim_{a \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) \left(\frac{\sin ay}{ay}\right)^2 e^{ixy}\, dy, \quad p = 1,$$

the limit in each case existing almost everywhere, $-\infty < x < \infty$. Finally, we have the identity theorem[7] a particular case of which is that if $f(x)$ is in L_2 and has the Fourier transform $F(x)$, then $F(x)$ has the Fourier transform $f(-x)$.

The following elementary calculations furnish the necessary tools for the remainder of the paper. The trigonometric identity

[6] E. C. Titchmarsh, A contribution to the theory of Fourier transforms, Proc. London Math. Soc. (2), 23 (1924), 279–289. For the case $p = 1$ which is not discussed here see A. C. Berry, The Fourier transform theorem in n dimensions, which is to appear in the London Math. Soc. Proc.

[7] In addition to the papers quoted in footnote 6, see A. C. Berry, The Fourier transform identity theorem, Annals of Math., 32 (1931), 227–232.

$$2 \sin a \sin b \cos x$$

$$= \sin^2 \frac{a+b+x}{2} + \sin^2 \frac{a+b-x}{2} - \sin^2 \frac{a-b+x}{2} - \sin^2 \frac{a-b-x}{2}$$

and the fact that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a x}{x} \right)^2 dx = |a|, \quad a \text{ real},$$

yield the fundamental relation

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin a y \sin b y}{y^2} e^{ixy} \, dy$$

$$= \begin{cases} 2a, & 0 \leqq |x| \leqq b-a, \\ a+b-|x|, & b-a \leqq |x| \leqq b+a, \\ 0, & \text{otherwise.} \end{cases} \qquad (0 < a < b).$$

From this it follows firstly that the function

$$g_{\xi,\eta}^{(a)}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\sin a x}{a x} \cdot \frac{e^{i\eta x} - e^{i\xi x}}{i x}, \qquad 0 < 2a < (\eta - \xi),$$

has for its Fourier transform the function

$$G_{\xi,\eta}^{(a)}(x) = \begin{cases} \dfrac{a-\xi+x}{2a}, & \xi - a \leqq x \leqq \xi + a, \\ 1 \quad , & \xi + a \leqq x \leqq \eta - a, \\ \dfrac{a+\eta-x}{2a}, & \eta - a \leqq x \leqq \eta + a, \\ 0 \quad , & \text{otherwise.} \end{cases}$$

and secondly that the function

$$g_{\xi,\eta}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{i\eta x} - e^{i\xi x}}{i x}, \qquad \xi < \eta,$$

has the Fourier transform

$$G_{\xi,\eta}(x) = \begin{cases} 1, & \xi < x < \eta, \\ 0, & \text{otherwise,} \end{cases}$$

Now, let (ξ_1, η_1), (ξ_2, η_2), $\cdots$, (ξ_n, η_n) be non-overlapping intervals, and let $\beta_1, \beta_2, \cdots, \beta_n$ be a given set of numbers with $|\beta_k| = 1$. Let

$$0 < 2a < \min_{k=1,2,\cdots,n} (\eta_k - \xi_k).$$

Then we see that

$$\begin{cases} l_\infty \left\{ \sum_{k=1}^{n} \beta_k \, G^{(a)}_{\xi_k, \eta_k} \right\} \leqq 2, \\ l_1 \left\{ \sum_{k=1}^{n} \beta_k \, G^{(a)}_{\xi_k, \eta_k} \right\} \leqq \left\{ 2\, n\, a + \sum_{k=1}^{n} (\eta_k - \xi_k) \right\} l_\infty \left\{ \sum_{k=1}^{n} \beta_k \, G^{(a)}_{\xi_k, \eta_k} \right\}. \end{cases}$$

We have now collected sufficient material to establish the two theorems of this paper.

THEOREM 1. *A necessary and sufficient condition that $F(x)$ be the Fourier transform of a function in L_p for some p such that $1 < p \leqq 2$ is that*
(i) *$F(x)$ be in $L_{p'}$, and*

$$\text{(ii)} \qquad \sum_{k=1}^{n-1} \frac{\left| \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(y) \frac{e^{ix_{k+1}y} - e^{ix_k y}}{iy} \, dy \right|^p}{(x_{k+1} - x_k)^{p-1}} \leqq C_F,$$

where C_F is independent of the sub-division $-\infty < x_1 < x_2 < \cdots < x_n < \infty$.

Proof of necessity. Let $f(x)$ be in L_p and let $F(x)$ be its Fourier transform. We know that (i) is satisfied. By the adjoint relation and Hölder's inequality, we have

$$\left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) \frac{e^{ix_{k+1}y} - e^{ix_k y}}{iy} \, dy \right| = \left| \int_{x_k}^{x_{k+1}} f(t)\, dt \right|$$
$$\leqq \left\{ \int_{x_k}^{x_{k+1}} |f(t)|^p \, dt \right\}^{1/p} (x_{k+1} - x_k)^{\frac{p-1}{p}}.$$

Whence (ii) is satisfied with

$$C_F = \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

Proof of sufficiency. Let $F(x)$ be in $L_{p'}$ and let

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) \frac{e^{ixy} - 1}{iy} \, dy.$$

Condition (ii) states that

$$\varphi(x) = \int_0^x f(t)\, dt,$$

where $f(x)$ is in L_p. Let $H(x)$ be the Fourier transform of $f(x)$; let

$$\varphi_H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(y) \frac{e^{ixy} - 1}{iy} \, dy.$$

By the adjoint relation,

$$\varphi_H(x) = \int_0^x f(t)\, dt = \varphi(x).$$

But, the two functions

$$F(x)\frac{2\sin x}{x}, \qquad H(x)\frac{2\sin x}{x}$$

are in L_2 and have the respective Fourier transforms

$$\varphi(x+1)-\varphi(x-1), \qquad \varphi_H(x+1)-\varphi_H(x-1).$$

These being identical, it follows that $F(x)=H(x)$ almost everywhere and so that $F(x)$ is the Fourier transform of $f(x)$.

THEOREM 2. *A necessary and sufficient condition that $F(x)$ be the Fourier transform of a function in L_1 is that*

(i) *$F(x)$ be in L_∞,*

(ii)
$$\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)\,g(x)\,dx\right| \leqq C_F\,l_\infty(G)$$

for each $g(x)$ in L_1, $G(x)$ being the Fourier transform of $g(x)$, and

(iii) *to each $\varepsilon>0$ there correspond $\delta=\delta(\varepsilon, F)$ such that*

$$\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)\,g(x)\,dx\right| \leqq \varepsilon\,l_\infty(G)$$

whenever $g(x)$ of L_1 is such that its Fourier transform $G(x)$ satisfies the condition

$$l_1(G)\leqq \delta\, l_\infty(G).$$

Proof of necessity. Let $f(x)$ be in L_1 and let $F(x)$ be its Fourier transform which, of course, is in L_∞. Then,

$$\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)\,g(x)\,dx\right| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,G(x)\,dx\right| \leqq l_1(f)\,l_\infty(G)$$

and so (ii) is satisfied with $C_F=l_1(f)$. As to (iii), let $\varepsilon>0$ be given. Determine $\delta=\delta(\varepsilon,F)$ so that whenever E is a set of measure not exceeding $\sqrt{2\pi\delta}$, then

$$\frac{1}{\sqrt{2\pi}}\int_E|f(x)|\,dx+\sqrt{\delta}\,l_1(f)<\varepsilon.$$

That this is the desired determination of $\delta(\varepsilon, F)$ follows from the fact that when we require $l_1(G)\leqq\delta\, l_\infty(G)$, then the set

$$E_\delta:\ |G(x)|\geqq\sqrt{\delta}\,l_\infty(G)$$

is of measure not exceeding $\sqrt{2\pi\delta}$, and so

$$\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)\,g(x)\,dx\right| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,G(x)\,dx\right|$$

$$\leqq l_\infty(G)\frac{1}{\sqrt{2\pi}}\int_{E_\delta}|f(x)|\,dx + \sqrt{\delta}\,l_\infty(G)\frac{1}{\sqrt{2\pi}}\int_{CE_\delta}|f(x)|\,dx$$

$$\leqq l_\infty(G)\left\{\frac{1}{\sqrt{2\pi}}\int_{E_\delta}|f(x)|\,dx + \sqrt{\delta}\,l_1(f)\right\} \leqq \varepsilon\, l_\infty(G).$$

Proof of sufficiency. 1) Let $F(x)$ satisfy the stated conditions. Then there exists

$$\varphi_a(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(y)\frac{\sin ay}{ay}\,\frac{e^{ixy}-1}{iy}\,dy.$$

Let $\xi < \eta$. Then

$$\varphi_a(\eta) - \varphi_a(\xi) = \int_{-\infty}^{\infty}F(y)\,g_{\xi,\eta}^{(a)}(y)\,dy.$$

Let

$$0 < 2a < (\eta - \xi), \qquad 0 < 2b < (\eta - \xi).$$

Noting that

$$\begin{cases} l_\infty(G_{\xi,\eta}^{(a)} - G_{\xi,\eta}^{(b)}) \leqq 1, \\ l_1\,(G_{\xi,\eta}^{(a)} - G_{\xi,\eta}^{(b)}) \leqq 2\max(a,b)\,l_\infty(G_{\xi,\eta}^{(a)} - G_{\xi,\eta}^{(b)}), \end{cases}$$

and applying (iii) we find that

$$\lim_{a,b\to 0}\left|\{\varphi_a(\eta) - \varphi_a(\xi)\} - \{\varphi_b(\eta) - \varphi_b(\xi)\}\right| = 0,$$

and so by the Cauchy criterion that there exists

$$\lim_{a\to 0}\{\varphi_a(\eta) - \varphi_a(\xi)\}.$$

Since $\varphi(0) = 0$, we have shown that for each x, $-\infty < x < \infty$, there exists

$$\varphi(x) = \lim_{a\to 0}\varphi_a(x) = \lim_{a\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(y)\frac{\sin ay}{ay}\,\frac{e^{ixy}-1}{iy}\,dy.$$

2) Next, we shall show that $\varphi(x)$ is of bounded variation over $(-\infty, \infty)$. Let $-\infty < x_1 < x_2 < \cdots < x_n < \infty$ be an arbitrary sub-division. Let

$$\beta_k = \operatorname{sgn}\{\varphi(x_{k+1}) - \varphi(x_k)\}.$$

By (ii)

$$\left|\sum_{k=1}^{n-1}\beta_k\{\varphi(x_{k+1}) - \varphi(x_k)\}\right| \leqq C_F\,l_\infty\left\{\sum_{k=1}^{n-1}\beta_k\,G_{x_k,x_{k+1}}^{(a)}\right\} \leqq 2C_F.$$

Taking the limit as $a \longrightarrow 0$ we obtain the desired result that

$$\sum_{k=1}^{n-1} |\varphi(x_{k+1}) - \varphi(x_k)| \leqq 2 C_F.$$

3) Our next task is to prove that $\varphi(x)$ is absolutely continuous. Let $\varepsilon > 0$ be given. Determine $\delta(\varepsilon, F)$ by (iii). We shall show that if $(\xi_1, \eta_1), (\xi_2, \eta_2), \cdots, (\xi_n, \eta_n)$ are arbitrary non-overlapping intervals such that

$$\sum_{k=1}^{n} (\eta_k - \xi_k) \leqq \frac{\delta}{2}$$

then

$$\sum_{k=1}^{n} |\varphi(\eta_k) - \varphi(\xi_k)| \leqq 2\varepsilon$$

and so prove our point. Let

$$\beta_k = \operatorname{sgn}\{\varphi(\eta_k) - \varphi(\xi_k)\},$$

and let

$$0 < 2a < \min_{k=1,2,\cdots,n} (\eta_k - \xi_k), \quad 0 < a < \frac{\delta}{4n}.$$

Then

$$\begin{cases} l_\infty \left\{ \sum_{k=1}^{n} \beta_k G^{(a)}_{\xi_k, \eta_k} \right\} \leqq 2, \\ l_1 \left\{ \sum_{k=1}^{n} \beta_k G^{(a)}_{\xi_k, \eta_k} \right\} \leqq \delta\, l_\infty \left\{ \sum_{k=1}^{n} \beta_k G^{(a)}_{\xi_k, \eta_k} \right\}, \end{cases}$$

and so

$$\left| \sum_{k=1}^{n} \beta_k \{\varphi_a(\eta_k) - \varphi_a(\xi_k)\} \right| \leqq 2\varepsilon.$$

Allowing $a \to 0$, we find

$$\sum_{k=1}^{n} |\varphi(\eta_k) - \varphi(\xi_k)| \leqq 2\varepsilon.$$

4) Collecting our data on $\varphi(x)$ we see that

$$\varphi(x) = \int_0^x f(t)\, dt$$

where $f(x)$ is in L_1. Let $H(x)$ be the Fourier transform of $f(x)$. Then, if we set

$$\varphi_{a,H}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(y) \frac{\sin ay}{ay} \frac{e^{ixy} - 1}{iy}\, dy$$

and note that

$$g_{\xi,\eta}^{(a)} \underset{\infty}{\longrightarrow} g_{\xi,\eta} \quad \text{as} \quad a \longrightarrow 0,$$

we see that

$$\varphi_H(x) = \lim_{a\to 0} \varphi_{a,H}(x) = \lim_{a\to 0} \int_{-\infty}^{\infty} f(t)\, g_{0,x}^{(a)}(t)\, dt$$
$$= \int_{-\infty}^{\infty} f(t)\, g_{0,x}(t)\, dt = \int_0^x f(t)\, dt = \varphi(x).$$

Now, the functions

$$F(x)\frac{2\sin x}{x}, \qquad H(x)\frac{2\sin x}{x}$$

are both in L_2 and have the respective Fourier transforms

$$\varphi(x+1) - \varphi(x-1), \qquad \varphi_H(x+1) - \varphi_H(x-1).$$

Hence, $F(x) = H(x)$ almost everywhere, and so $F(x)$ is the Fourier transform of $f(x)$.

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